Section 3.2.

#6. Consider two subspaces V & W of $\mathbb{R}^n$

a) Is $V \cap W$ necessarily a subspace of $\mathbb{R}^n$?

b) Is $V \cup W$ necessarily a subspace of $\mathbb{R}^n$?

Solution

a) Yes. Let $\vec{x}, \vec{y} \in V \cap W$ and $a, b \in \mathbb{R}$.

$$\left.\begin{array}{l} \text{Since } V \leq \mathbb{R}^n \text{ \& } \vec{x}, \vec{y} \in V \Rightarrow a\vec{x} + b\vec{y} \in V \\ \text{Since } W \leq \mathbb{R}^n \text{ \& } \vec{x}, \vec{y} \in W \Rightarrow a\vec{x} + b\vec{y} \in W \end{array}\right\} \Rightarrow$$

$\Rightarrow a\vec{x} + b\vec{y} \in V \cap W \Rightarrow V \cap W \leq \mathbb{R}^n$.

b) No. Consider $V = \text{Span}(\vec{e_1})$ & $W = \text{Span}(\vec{e_2})$. Then $\vec{e_1} + \vec{e_2} \notin V \cup W$, i.e. $V \cup W$ is not closed under addition

#24. Find redundant column vector, write it as a lin. combination of the preceding columns. Write nontrivial relation among columns and thus find a nonzero vector in $\ker A$.

$$A = \begin{bmatrix} 1 & 3 & 6 \\ 1 & 2 & 5 \\ 1 & 1 & 4 \end{bmatrix}$$

Solution: Column 2 is not a multiple of the first, so two of them are linearly independent. We will try to see if there are $a, b \in \mathbb{R}$ st

$$a\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix} \Leftrightarrow a\vec{v_1} + b\vec{v_2} = \vec{v_3}$$

$$\left[\begin{array}{cc|c} 1 & 3 & 6 \\ 1 & 2 & 5 \\ 1 & 1 & 4 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 3 & 6 \\ 0 & -1 & -1 \\ 0 & -2 & -2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 3 & 6 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right] \Leftrightarrow \begin{array}{l} a = 3 \\ b = 1 \end{array}$$

$$\begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow 3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix} = \vec{0} \Rightarrow \begin{bmatrix} 1 & 3 & 6 \\ 1 & 2 & 5 \\ 1 & 1 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = \vec{0}$$

$$\Rightarrow \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \in \ker A.$$

#34 Consider a 5x4 matrix

$$A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \end{bmatrix}$$

We are told that $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ is in the kernel of A. Write $\vec{v}_4$ as a linear combination of $\vec{v}_1, \vec{v}_2$ & $\vec{v}_3$.

Solution:

$$A \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \vec{0} \quad \Leftrightarrow \quad \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \vec{0} \quad \Rightarrow$$

$$\Rightarrow \vec{v}_1 + 2\vec{v}_2 + 3\vec{v}_3 + 4\vec{v}_4 = \vec{0} \Leftrightarrow 4\vec{v}_4 = -\vec{v}_1 - 2\vec{v}_2 - 3\vec{v}_3 \Leftrightarrow$$

$$\Leftrightarrow \quad \vec{v}_4 = -\frac{1}{4}\vec{v}_1 - \frac{1}{2}\vec{v}_2 - \frac{3}{4}\vec{v}_3$$

#36: Consider a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^p$ and some linearly dependent vectors $\vec{v}_1, \ldots, \vec{v}_m \in \mathbb{R}^n$. Are the vectors $T(\vec{v}_1), \ldots, T(\vec{v}_m)$ necessarily linearly dependent? How can you tell?

Solution:

Yes. Since $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ are linearly dependent we can find real numbers $c_1, c_2, \ldots, c_m$ which are not all zeros such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_m\vec{v}_m = \vec{0}$$

then

$$0 = T(c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_m\vec{v}_m) =$$

$$= c_1 T(\vec{v}_1) + c_2 T(\vec{v}_2) + \ldots + c_m T(\vec{v}_m)$$

and not all c_i are zero, i.e. we got a nontrivial relation among vectors $T(\vec{v}_1), \ldots, T(\vec{v}_m) \Rightarrow T(\vec{v}_1), \ldots, T(\vec{v}_m)$ are lin. dependent.

#37. Consider a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^p$ and some linearly independent vectors $\vec{v}_1, \ldots, \vec{v}_m \in \mathbb{R}^n$. Are the vectors $T(\vec{v}_1), \ldots, T(\vec{v}_m)$ necessarily lin. indep.? How can you tell?

Solution

No. It is possible that some nontrivial linear combination of vectors $\vec{v}_1, \ldots, \vec{v}_m$ ($c_1\vec{v}_1 + \ldots + c_m\vec{v}_m$, some $c_i \neq 0$) is in the kernel of T:

$$T(c_1\vec{v}_1 + \ldots + c_m\vec{v}_m) = \vec{0}$$

$$\shortparallel$$

$$c_1 T(\vec{v}_1) + \ldots + c_m T(\vec{v}_m) = \vec{0} \quad \text{and} \quad \text{some } c_i \neq 0 \Rightarrow$$

$\Rightarrow T(\vec{v}_1), \ldots, T(\vec{v}_m)$ are not lin. independent.

#50. Consider two subspaces V and W of $\mathbb{R}^n$. Let $V+W$ be the set of all vectors in $\mathbb{R}^n$ of the form $\vec{v}+\vec{w}$, where $\vec{v}\in V$ and $\vec{w}\in W$. Is $V+W$ necessarily a subspace of $\mathbb{R}^n$? If V and W are two distinct lines in $\mathbb{R}^3$ what is $V+W$? Draw a sketch.

Solution:

$V+W$ is always a subspace of $\mathbb{R}^n$. Let $\vec{x},\vec{y}\in V+W$ and $a,b\in\mathbb{R}$. We claim $a\vec{x}+b\vec{y}$ is also in $V+W$.

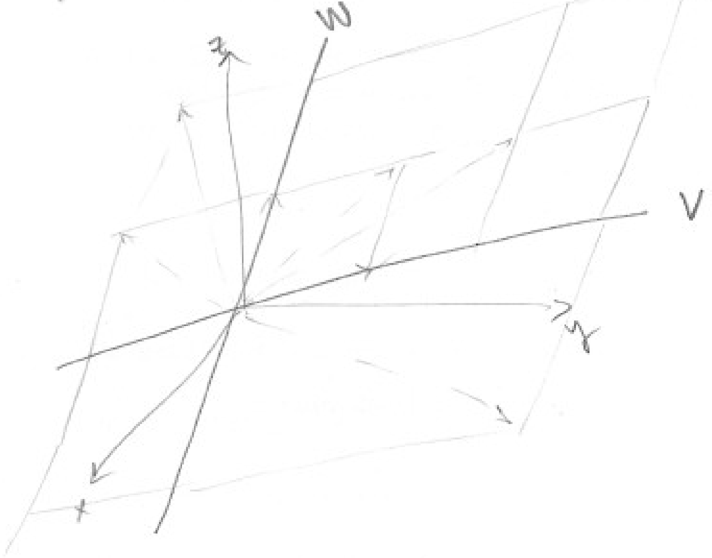
$\vec{x}\in V+W \to \vec{x}=\vec{v}_1+\vec{w}_1$ where $\vec{v}_1\in V$ & $\vec{w}_1\in W$

$\vec{y}\in V+W \to \vec{y}=\vec{v}_2+\vec{w}_2$ where $\vec{v}_2\in V$ & $\vec{w}_2\in W$.

Then

$$a\vec{x}+b\vec{y}=a(\vec{v}_1+\vec{w}_1)+b(\vec{v}_2+\vec{w}_2)=\underbrace{(a\vec{v}_1+b\vec{v}_2)}_{\substack{\in V \\ \text{since } V \text{ is a} \\ \text{subspace}}}+\underbrace{(a\vec{w}_1+b\vec{w}_2)}_{\substack{\in W \\ \text{since } W \text{ is a} \\ \text{subspace}}}\in V+W.$$

If V & W are two distinct lines in $\mathbb{R}^3$, then $V+W$ is a plane



Homework Set #4

Section 3.3:

#22. Find rref(A), basis for the image & kernel of A.

$$A = \begin{bmatrix} 2 & 4 & 8 \\ 4 & 5 & 1 \\ 7 & 9 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 & 8 \\ 4 & 5 & 1 \\ 7 & 9 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 \\ 4 & 5 & 1 \\ 7 & 9 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & -3 & -15 \\ 0 & 5 & -25 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 5 \\ 0 & 1 & 5 \end{bmatrix} \sim$$

$$\sim \begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix} = \text{rref}(A)$$

$$\begin{matrix} \shortparallel & \shortparallel & \shortparallel \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{matrix}$$

$\Rightarrow \vec{v}_3 = -6\vec{v}_1 + 5\vec{v}_2 \Rightarrow$

$\Rightarrow$ Same relationship holds for column vectors of A

$\Rightarrow$ Im A is spanned by 1st & 2nd column vector, in fact they form the basis of Im A

$$\text{basis of Im}A = \left\{ \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 9 \end{bmatrix} \right\}$$

From rref A $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \ker A$ if $\left.\begin{matrix} x_1 - 6x_3 = 0 \\ x_2 + 5x_3 = 0 \end{matrix}\right\} \Rightarrow \begin{matrix} x_1 = 6x_3 \\ x_2 = -5x_3 \end{matrix}$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6x_3 \\ -5x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 6 \\ -5 \\ 1 \end{bmatrix}$$

$$\text{basis for Ker } A = \left\{ \begin{bmatrix} 6 \\ -5 \\ 1 \end{bmatrix} \right\}$$

#26: Consider the matrices

$$C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix},$$

$$T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad X = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}; \quad Y = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

a) which of the matrices have the same kernel as C

b) Which of the matrices have the same image as C.

c) Which of these matrices has an image that is different from the images of all other matrices in the list?

a) Reminder: $A = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3]$ is a 3x3 matrix & $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \ker A$ iff
$x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = 0$

For C: $\vec{v}_2 = \vec{v}_3$, so we have $x_1\vec{v}_1 + (x_2+x_3)\vec{v}_2 = 0$ & $\vec{v}_1$ & $\vec{v}_2$ are lin. independent so $x_1 = 0$ & $x_2 = -x_3$ $\Rightarrow$
$\Rightarrow$ basis for $\ker C = \left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

For H: $\vec{v}_1 = \vec{v}_3$ $\Rightarrow$ $(x_1 + x_3)\vec{v}_1 + x_2\vec{v}_2 = 0$ & $\vec{v}_1$ & $\vec{v}_2$ are lin. independent
$\Rightarrow$ $x_1 + x_3 = 0$ & $x_2 = 0$ $\Rightarrow$ basis for $\ker H = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

L has $\vec{v}_2 = \vec{v}_3$ $\Rightarrow$ $\ker L = \ker C$

T, X & Y have $\vec{v}_1 = \vec{v}_3$ $\Rightarrow$ $\ker T = \ker X = \ker Y = \ker H$.

$\Rightarrow$ The only matrix with kernel equal to that of C is L.

b) We see that first two columns of C are lin. independent, hence they form a basis for Im C. So if $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ is in Im C, then
$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a+b \\ a \\ a+b \end{bmatrix}$, for some $a, b \in \mathbb{R}$ $\Rightarrow$

$\Rightarrow$ We see that 1^{st} & 3^{rd} coordinate of $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ are equal that is $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \in \operatorname{Im} C \iff y_1 = y_3$.

H: Similar considerations show that

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \in \operatorname{Im} H \iff \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ a+b \\ a \end{bmatrix} \iff y_1 = y_3$$

$$L: \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \in \operatorname{Im} L \iff \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ a \\ a+b \end{bmatrix} \iff y_1 = y_2$$

$$T: \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \in \operatorname{Im} T \iff \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a+b \\ b \\ b \end{bmatrix} \iff y_2 = y_3$$

$$X: \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \in \operatorname{Im} X \iff \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \\ a \end{bmatrix} \iff y_1 = y_3$$

$$Y: \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \in \operatorname{Im} Y \iff \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ b \end{bmatrix} \iff y_2 = y_3$$

$\Rightarrow \operatorname{Im} C = \operatorname{Im} H = \operatorname{Im} X$ & $\operatorname{Im} T = \operatorname{Im} Y$

c) From b) we see that the answer is L.

31. Let V be the subspace of $\mathbb{R}^4$ defined by the equation

$$x_1 - x_2 + 2x_3 + 4x_4 = 0$$

Find a linear transformation T from $\mathbb{R}^3$ to $\mathbb{R}^4$ st $\ker T = \{\vec{0}\}$ and $\operatorname{Im} T = V$. Describe T by its matrix A.

Solution: $T: \mathbb{R}^3 \to \mathbb{R}^4 \Rightarrow A$ is a 4×3 matrix. Its column vectors should be basis vectors of V since column vectors span the $\operatorname{Im} A$. We need to find basis for $V \iff$ we need to solve

$$x_1 - x_2 + 2x_3 + 4x_4 = 0$$

$$x_1 = x_2 - 2x_3 - 4x_4$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in V \iff \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 - 2x_3 - 4x_4 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} =$$

$$= x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad A = \begin{bmatrix} 1 & -2 & -4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

#33. A subspace V of $\mathbb{R}^n$ is called a hyperplane if V is defined by a homogeneous equation

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n = 0$$

where at least one of the coefficients c_i is nonzero. What is the dimension of a hyperplane in $\mathbb{R}^n$? What is a hyperplane in $\mathbb{R}^3$ and what in $\mathbb{R}^2$?

Solution:

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n = 0 \quad \text{and}, \quad c_i \neq 0$$

$$c_ix_i = -c_1x_1 - \cdots - c_{i-1}x_{i-1} - c_{i+1}x_{i+1} - \cdots - c_nx_n$$

$$x_i = -\frac{c_1}{c_i}x_1 - \cdots - \frac{c_{i-1}}{c_i}x_{i-1} - \frac{c_{i+1}}{c_i}x_{i+1} - \cdots - \frac{c_n}{c_i}x_n$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} \in V \iff \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ -\frac{c_1}{c_i}x_1 - \cdots - \frac{c_n}{c_i}x_n \\ \vdots \\ x_n \end{bmatrix} =$$

$$= x_1 \underbrace{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -c_1/c_i \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{=\vec{a}_1} + x_2 \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ -c_2/c_i \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{=\vec{a}_2} + \cdots + x_{i-1} \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ -c_{i-1}/c_i \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{=\vec{a}_{i-1}} + x_{i+1} \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ -c_{i+1}/c_i \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{=\vec{a}_{i+1}} + \cdots + x_n \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -c_n/c_i \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{=\vec{a}_n}$$

all these vectors $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_{i-1}, \vec{a}_{i+1}, \ldots, \vec{a}_n$ are lin. independent, they span V and there are $n-1$ of them

$\Rightarrow \quad \dim V = n-1$

Hyperplanes in $\mathbb{R}^3$ are all planes through the origin.
Hyperplanes in $\mathbb{R}^2$ are all lines through the origin.

#38. a) Consider a lin transf. $T: \mathbb{R}^5 \to \mathbb{R}^3$. What are the possible values of $\dim(\ker T)$

b) Consider a lin. trans. $T: \mathbb{R}^4 \to \mathbb{R}^7$. What are the possible values of $\dim(\operatorname{Im} T)$?

Solution:

we know: If $T: \mathbb{R}^m \to \mathbb{R}^n$ is a lin transf. then

$$\dim(\ker T) + \dim(\operatorname{Im} T) = m$$

$$\text{\&} \quad \ker T \subseteq \mathbb{R}^m \quad \text{\&} \quad \operatorname{Im} T \subseteq \mathbb{R}^n$$

a)

$$\left.\begin{array}{l} \dim(\ker T) + \dim(\operatorname{Im} T) = 5 \\ \text{\&} \ \operatorname{Im} T \subseteq \mathbb{R}^3 \Rightarrow 0 \leq \dim(\operatorname{Im} T) \leq 3 \end{array}\right\} \Rightarrow$$

$\Rightarrow 5 = \dim \ker T + \dim \operatorname{Im} T \leq \dim \ker T + 3 \Rightarrow$

$\Rightarrow \quad \dim \ker T \geq 2$.

b)

$$\left.\begin{array}{l} \dim(\ker T) + \dim(\operatorname{Im} T) = 4 \\ \text{\&} \ \ker T \subseteq \mathbb{R}^4 \Rightarrow 0 \leq \dim \ker T \leq 4 \end{array}\right\} \Rightarrow$$

$\Rightarrow 4 = \dim \ker T + \dim \operatorname{Im} T \geq 0 + \dim \operatorname{Im} T \Rightarrow \dim \operatorname{Im} T \leq 4$

$4 = \dim \ker T + \dim \operatorname{Im} T \leq 4 + \dim \operatorname{Im} T \Rightarrow \dim \operatorname{Im} T \geq 0$

$\Rightarrow \quad 0 \leq \dim \operatorname{Im} T \leq 4$.

#47. Consider two subspaces V, W of $\mathbb{R}^n$. Show that

$$\dim(V) + \dim(W) = \dim(V+W) + \dim(V \cap W).$$

Solution:

$V \cap W$ is a subspace, so let $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_k$ be its basis.

Since $\vec{u}_1, \dots, \vec{u}_k$ are lin. independent vectors in V we can find vectors $\vec{v}_1, \dots, \vec{v}_\ell$ st $\{\vec{u}_1, \dots, \vec{u}_k, \vec{v}_1, \dots, \vec{v}_\ell\}$ is a basis for V.

Similarly, we can find $\vec{w}_1, \dots, \vec{w}_r$ st $\{\vec{u}_1, \dots, \vec{u}_k, \vec{w}_1, \dots, \vec{w}_r\}$ is a basis for W.

We would then have

$$\dim(V \cap W) = k$$
$$\dim V = k + \ell$$
$$\dim W = k + r$$

If we show that $\{\vec{u}_1, \dots, \vec{u}_k, \vec{v}_1, \dots, \vec{v}_\ell, \vec{w}_1, \dots, \vec{w}_r\}$ is a basis for $V+W$ then $\dim(V+W) = k + \ell + r$, and we would have

$$\dim V + \dim W = k + \ell + k + r = k + (\ell + k + r) =$$
$$= \dim(V \cap W) + \dim(V+W).$$

So we need to show that $\{\vec{u}_1, \dots, \vec{u}_k, \vec{v}_1, \dots, \vec{v}_\ell, \vec{w}_1, \dots, \vec{w}_r\}$ is a basis for $V+W$, i.e. that this set of vectors span $V+W$ and the vectors are lin. independent.

i) Span. Let $\vec{x} \in V+W$. Then $\vec{x} = \vec{v} + \vec{w}$ for some $\vec{v} \in V$ & $\vec{w} \in W$.

$\vec{v} \in V \Rightarrow \vec{v} = a_1\vec{u}_1 + \dots + a_k\vec{u}_k + a_{k+1}\vec{v}_1 + \dots + a_{k+\ell}\vec{v}_\ell$, for some a_i

$\vec{w} \in W \Rightarrow \vec{w} = b_1\vec{u}_1 + \dots + b_k\vec{u}_k + b_{k+1}\vec{w}_1 + \dots + b_{k+r}\vec{w}_r$, for some b_i

$$\Rightarrow \vec{x} = a_1\vec{u}_1 + \dots + a_k\vec{u}_k + a_{k+1}\vec{v}_1 + \dots + a_{k+\ell}\vec{v}_\ell + b_1\vec{u}_1 + \dots + b_{k+r}\vec{w}_r =$$
$$= (a_1+b_1)\vec{u}_1 + \dots + (a_k+b_k)\vec{u}_k + a_{k+1}\vec{v}_1 + \dots + a_{k+\ell}\vec{v}_\ell + b_{k+1}\vec{w}_1 + \dots + b_{k+r}\vec{w}_r$$

$\Rightarrow \vec{u}_1, \dots, \vec{u}_k, \vec{v}_1, \dots, \vec{v}_\ell, \vec{w}_1, \dots, \vec{w}_r$ span $V+W$

2) lin. independence: Suppose

$$a_1\vec{u}_1 + \cdots + a_k\vec{u}_k + b_1\vec{v}_1 + \cdots + b_\ell \vec{v}_\ell + c_1\vec{w}_1 + \cdots + c_r\vec{w}_r = \vec{0} \qquad (*)$$

$$\Rightarrow \quad \underbrace{a_1\vec{u}_1 + \cdots + a_k\vec{u}_k + b_1\vec{v}_1 + \cdots + b_\ell\vec{v}_\ell}_{\in V} = \underbrace{-c_1\vec{w}_1 - \cdots - c_r\vec{w}_r}_{\in W}$$

So, we have a vector that lies both in V & W, i.e it lies in $V \cap W$

$$\|$$

$$a_1\vec{u}_1 + \cdots + a_k\vec{u}_k + b_1\vec{v}_1 + \cdots + b_\ell\vec{v}_\ell \in V \cap W$$

Since $\{\vec{u}_1, \ldots, \vec{u}_k\}$ is a basis for $V \cap W$, this vector's representation in terms of the basis is unique, which implies that

$$b_1 = b_2 = \cdots = b_\ell = 0$$

Putting this into equation $(*)$ we get

$$a_1\vec{u}_1 + \cdots + a_k\vec{u}_k + c_1\vec{w}_1 + \cdots + c_r\vec{w}_r = 0$$

but $\{\vec{u}_1, \ldots, \vec{u}_k, \vec{w}_1, \ldots, \vec{w}_r\}$ is a basis for W, hence

$$a_1 = \cdots = a_k = c_1 = \cdots = c_r = 0$$