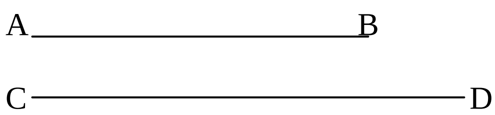
Problems

1) Explain why $a^0 = 1$, for any rational number a. What is 0^0? Explain your solution. Define a^b for a and b real numbers and a>0 and sketch how you would prove that a^b is well defined
2) Consider the two line segments AB and CD. Is the number of points on line segment CD smaller than/equal to/greater than the number of points in line segment AB? Prove your claim.



3) Consider two infinite sets {1, 2, 3, 4, …} and { 1^2, 2^2, 3^2, 4^2, …}. Is the number of elements in these sets the same? If not, which set contains more elements and why?
4) Why is a negative integer multiplied by a positive integer negative? Why is a negative integer multiplied by another negative one equal to a positive integer?
5) How do you explain to someone why 2 + (-7) = -5 ?
6) How can one justify why in solving percent problems, one cross-multiplies? Ex: 2/15= x/100
7) Explain why the rules for transforming periodic decimals into fractions works (e.g., $0.\bar{3} = \frac{3}{9} = \frac{1}{3}$).
8) In solving the equation $3x^2 = 9x$, how can one justify dividing the two sides by x?
9) How can you interpret graphically the relationship between a rational function and the result of the long division of the polynomial functions in the numerator and the denominator?
10) Starting from the vector representation of a complex number in the plane, provide a geometrical interpretation to the operations among complex numbers
11) Pascal's triangle can be used to determine the coefficients in the expansion of expressions in the form $(a + b)^n$:

$n = 0$					1				
$n = 1$				1		1			
$n = 2$			1		2		1		
$n = 3$		1		3		3		1	
$n = 4$	1		4		6		4		1

The numbers at the edges of the triangle are all equal to one, and the interior numbers are each equal to the sum of the two numbers above it. Explain why this construction of the triangle produces the coefficients of the binomial expansion of $(a + b)^n$.
12) How would you explain why a - (b - c) and a – b + c are equivalent?
13) Many students find properties of logarithms confusing and hard to remember. Why are the following properties of logarithms true, and how would you explain them to a high school algebra class?
a. $\log_b(x) + \log_b(y) = \log_b(xy)$

b. $\log_b(x) - \log_b(y) = \log_b(x/y)$
c. $\log_b(x^m) = m \log_b(x)$

14) Explain why $\sin^2 x + \cos^2 x = 1$.

15) Write a proof of the Pythagorean Theorem that uses dissection and a proof of the Pythagorean theorem that uses similarity

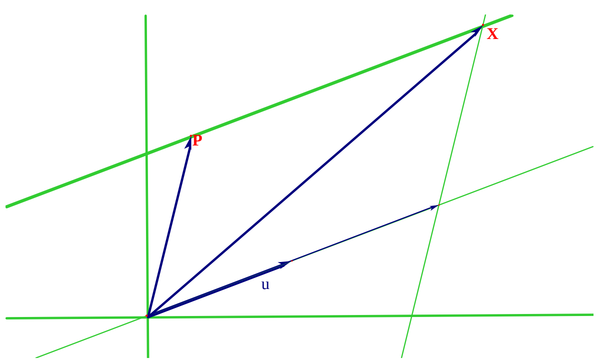
16) Prove that each of the following segments in a triangle meet at a point :
   a. angle bisectors,
   b. medians,
   c. altitudes, and
   d. perpendicular bisectors

17) Define inscribed and semi-inscribed angles in a circle, state and prove relationships between those angles and their corresponding arcs in a circle.

18) How many points of intersection do the angle bisectors of a kite make? Prove it.

19) Joe would like to chart the actual mileage per gallon of his car and for that purpose he has gathered the following records for the past three months. Three months ago he filled the tank and set the mileage counter at 0. He has since kept all the gas receipts and jotted in them the mileage at the time of purchase. Can he use that information to find out about his car's miles per gallon?

20) Show where the slope-intercept equation of the line and the general equation of a line comes from, starting from the vector equation of a line below.



The line directed by vector **u** that contains point **P** is the set of points $\{\mathbf{X} \,/\, \mathbf{X} = \mathbf{P} + \lambda\mathbf{u}$, for all $\lambda \in \mathbb{R}\}$

21) A parabola can be defined as the locus of points that are equidistant from a given point called the focus, and a given line called the directrix. The equation of a parabola is $(x - h)^2 = 4p(y - k)$, where (h, k) is the vertex of the parabola and p is the distance from the vertex to the focus. Explain how to derive this equation from the locus definition of the parabola.

22) The number of combinations of n objects taken r at a time is written

$$C(n,r) = \frac{n!}{(n-r)!r!}$$

Use the following problem to explain why this formula works.

From a list of 5 books, how many books can be taken 3 at a time?

23) If two events, A and B, are independent, then the probability of both events occurring is found as follows:

P(A and B) = P(A)P(B)

Create an example of two independent events. Identify the probability of each event happening. Explain why the probability of both events happening is the product of the probabilities of each event.

24) A series is the sum of the terms of a sequence of numbers. Let S be the sum of the infinite series $(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots)$. Express this sum in mathematical notation using a summation sign and write an explanation of how the mathematical notation summarizes the sum of the series.

One illustration of how S converges to 1 can be found below.

$$S = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$$
$$S = \frac{1}{2} + \frac{1}{2}(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots)$$
$$S = \frac{1}{2} + \frac{1}{2}S$$
$$S - \frac{1}{2}S = \frac{1}{2}$$
$$\frac{1}{2}S = \frac{1}{2}$$
$$S = 1$$

Now look at the sum of the following series. The same method was used to show that it converges to -1, yet this does not make sense. Explain where the explanation below is flawed.

$$S = 1 + 2 + 4 + 8 + 16 + \ldots$$
$$S = 1 + 2(1 + 2 + 4 + 8 + 16 + \ldots)$$
$$S = 1 + 2S$$
$$S - 2S = 1$$
$$-S = 1$$
$$S = -1$$