Connected sums

And Products
\[ \mathbb{P}^2 \# \mathbb{P}^2 = ??? \]

Take two copies of \( \mathbb{P}^2 \setminus \mathbb{B}^2 \) and glue along their boundaries. Make sure to remember what was the boundary of the disk you cut out from \( \mathbb{P}^2 \).
Solution

$P^2 \setminus B^2$

$P^2 \setminus B^2$
Solution

Klein bottle $K^2$
How are these two spaces the same?
Big Theorem

Every surface is a connected sum of tori and/or projective planes.
## All surfaces

<table>
<thead>
<tr>
<th># of pp</th>
<th># of tori</th>
<th>$0$</th>
<th>$1$</th>
<th>$2$</th>
<th>$3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td>$S^2$</td>
<td>$P^2$</td>
<td>$P^2 # P^2$</td>
<td>$P^2#P^2#P^2$</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>$T^2$</td>
<td>$T^2 # P^2$</td>
<td>$T^2#P^2#P^2$</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>$T^2 # T^2$</td>
<td>$T^2#T^2#P^2$</td>
<td>....</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>$T^2#T^2#T^2$</td>
<td></td>
<td>....</td>
<td></td>
</tr>
</tbody>
</table>
Exercise

- Find a surface on the list that is topologically equivalent to
  - $K^2 \# P^2$
  - $K^2 \# T^2$
  - $K^2 \# K^2$
Question one should ask

- Does this list contain duplicates?
  - Yes.
Investigation

I would like us to consider these two spaces

$\mathbb{T}^2 \# \mathbb{P}^2$ and $\mathbb{K}^2 \# \mathbb{P}^2$
Study the pictures

A torus with a disk removed $(T^2 - D^2)$

A Klein bottle with a disk removed $(K^3 - D^2)$

The connected sum of a torus and a Möbius strip $(T^2 \# \text{Möbius})$

The connected sum of a Klein bottle and a Möbius strip $(K^3 \# \text{Möbius})$
Exercise

Show that

\[ T^2 \# P^2 = P^2 \# P^2 \# P^2 \]
Exercise

• Why can our table from few slides ago be replaced by this table:

\[
\begin{array}{ccc}
S^2 \\
T^2 & P^2 \\
T^2 \# T^2 & P^2 \# P^2 \\
T^2 \# T^2 \# T^2 & P^2 \# P^2 \# P^2
\end{array}
\]

and so on
Exercise

- Match the entries from Column A with the same entries in Column B:

<table>
<thead>
<tr>
<th>Column A</th>
<th>Column B</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S^2 # T^2$</td>
<td>$P^2 # P^2$</td>
</tr>
<tr>
<td>$S^2$</td>
<td>$K^2 # P^2$</td>
</tr>
<tr>
<td>$S^2 # S^2 # S^2$</td>
<td>$P^2 # P^2 # P^2 # K^2$</td>
</tr>
<tr>
<td>$P^2 # T^2$</td>
<td></td>
</tr>
<tr>
<td>$K^2 # T^2 # P^2$</td>
<td></td>
</tr>
</tbody>
</table>
Exercise

• Which of the surfaces on the list are orientable and which are nonorientable?
  • The connected sums of tori only are orientable
  • The connected sums of pps only are nonorientable
Exercise

• How would you generalize the connected sum to 3-manifolds?

• Take out an open ball from both manifolds and glue them back up along their spherical boundary.
Question

• If you make a connected sum of two spaces, what is the dimension of the resulting space equal to?

  • Same as the dimension of the spaces we started with.

• Could we form new manifolds from old, but so that the dimension changes?
Products

- Stack some circles!

Cylinder
-- interval of circles
Or maybe

• Stack some intervals!

Cylinder
-- circle of intervals
Cylinder

- Is a product of a circle and an interval

\[ S^1 \times I \]

(circle cross interval)
In coordinates:

- Any point P on the cylinder can be given as (x,y), where x is a point on the circle and y is a point on the interval.
Exercise

• Show on the gluing diagram of the cylinder its product structure.
  • Mark how it is an interval of circles
  • Mark how it is a circle of intervals