A new inversion method for dissipating electromagnetic problem *

Elena Cherkaev[†]

Abstract

The paper suggests a new technique for solution of the inverse problem for Maxwell's equations in a dissipating medium. Being non-self-adjoint, the problem generally requires computing an adjoint operator on each step of gradient descent minimization procedure. The suggested approach introduces a conjugation operator and develops a representation of the system of Maxwell's equations based on iteratively updated eigenfunction expansion. Used in numerical scheme, the method effectively avoids calculation of the adjoint operator and reduces to the gradient minimization in a small number of descent directions.

In: Mathematical and numerical aspects of wave propagation (Santiago de Compostela, 2000), A. Bermudez et al., eds., SIAM, Philadelphia, PA, 2000, 637–642.

*This work was supported by DOE/OBES grant DE-FG03-93ER14313 and ONR grant N0000149310141.

[†]Department of Mathematics, University of Utah, 155 South 1400 East, JWB 233, Salt Lake City, UT, 84112, e-mail: elena@math.utah.edu

1 Introduction

The paper deals with the inverse electromagnetic problem for a dissipating medium. The problem is to reconstruct the dielectric permittivity ϵ and the electric conductivity σ from measurements of the electric and/or magnetic fields generated in the medium by applied electromagnetic sources. Problems of this type arise in geophysical surface and borehole electromagnetic imaging [1,3,7], such as monitoring conducting contaminant plumes or flooding oil fields, and in engineering and biomedical applications, as in interaction of electromagnetic radiation with biological tissues.
Because the medium is assumed to be conducting, the corresponding Maxwell's problem is not self-adjoint. Recently the not self-adjoint time harmonic Maxwell's system received a great deal of attention, for forward modeling, as well as for solution of inverse problem ([13,15,7]). Though elegant solutions are possible in particular geometries ([15]), general inversion techniques developed for solution of the problem are based on gradient descent least square minimization procedure. The spatially changing electromagnetic properties of the medium are recovered in an iterative process of gradient minimization of the misfit functional. This functional is a least square norm of the error of representation of the measured data using an iteratively updated model of electromagnetic properties of the medium. The gradient of the functional is used on each iteration step for updating the model properties of the medium. An efficient way to compute the gradient of the functional is to exploit the adjoint operator; that is why the method is often referred as adjoint or backpropagation method ([12]). For self-adjoint operators this does not require any additional effort. For not self-adjoint problems, the adjoint problem needs to be solved, and this doubles the computation time. Being not self-adjoint, the problem generally requires computation of the solution of the adjoint problem on each step of gradient descent minimization (i.e. [13,7]).
This approach usually exploits all available data. Meanwhile, antenna optimization for transmission through atmosphere ([9,11]) as well as optimization of electromagnetic fields in media ([10,2,4,14]), shows that instead of all available signals it is often preferable to use only one or several signals, but optimally designed ones. Optimally designed electromagnetic signal can significantly increase amount of information about electromagnetic target, or create desired fields and increase energy deposition within a specified region. Using gradient descent minimization procedure, we developed in [5] an efficient computational algorithm for electrical tomography based on the optimally chosen applied electric currents. The optimal currents are given by iteratively updated eigenfunction expansion of Neumann-to-Dirichlet operator. For a conducting medium, the corresponding eigenvalues are fast decaying, hence only data due to several dominant eigenfunctions are of importance for the reconstruction procedure.
Solution of this optimization problem leads to diagonalization of the misfit operator which describes the least square difference in measured and model based calculated fields. In application to numerical reconstruction, this diagonalization effectively reduces gradient minimization scheme to minimization in a small number of descent directions.
At the present paper we apply this approach to inverse electromagnetic problem in

dissipating medium. However, this is not a self-adjoint problem. To avoid doubling the computation time, we show that the Maxwell's operator is J-self-adjoint operator. This means that the Maxwell's system of equations admits a self-adjoint extension using some conjugation operator J ([8]). We introduce the conjugation operator and represent Maxwell's system of equations using a real, symmetric, self-adjoint operator. This permits calculation of the gradient of the least square misfit functional with respect to the unknown properties from one forward solution. Similarly to the inversion scheme for the inverse conductivity problem in [5], the approach is based on iteratively updated eigenfunction expansion. This corresponds to the construction of the optimal finite dimensional approximation of the infinite dimensional operator on every iteration step. Used in numerical gradient technique in combination with the self-adjoint conjugated representation, the method effectively avoids calculation of the adjoint operator.

2 J-self-adjoint extension of the Maxwell's operator

Let a medium in a domain Ω be characterized by the magnetic permeability $\bar{\mu} + i\bar{\nu}$, the dielectric permittivity $\bar{\epsilon}$, and the electric conductivity $\bar{\sigma}$. The matrices $\bar{\mu}, \bar{\nu}$, $\bar{\epsilon}$, and $\bar{\sigma}$ are real valued, with $\bar{\sigma}$ and $\bar{\nu}$ being positive definite. Let J_m and J_e be applied magnetic and electric currents. With the assumed time dependence of the form $\exp(-i\omega t)$, the Maxwell's equations for time-harmonic excitations are

$$i\bar{\epsilon}\omega E - \bar{\sigma}E + \nabla \times H = J_e, \qquad -\nabla \times E + i\bar{\mu}\omega H - \bar{\nu}H = J_m. \tag{1}$$

We assume that EM fields are generated by currents that differ from zero only in a bounded domain. Denoting $\mathcal{M}$ the Maxwell's operator, we have: $\mathcal{M}$: $\{L^2(\Omega)^3 \times L^2(\Omega)^3\} \rightarrow \{L^2(\Omega)^3 \times L^2(\Omega)^3\}$.

The gradient descent inversion scheme usually exploited to solve the inverse problem is based on a gradient technique, which uses the adjoint operator. To avoid additional computations for solving the adjoint problem, we introduce a J-self-adjoint extension of the Maxwell's operator $\mathcal{M}$.

First, we write the equations (1) as a system of equations for real and imaginary parts of fields and currents.

$$\begin{bmatrix} -\bar{\sigma} & -\omega\bar{\epsilon} & \nabla\times & 0_3 \\ \omega\bar{\epsilon} & -\bar{\sigma} & 0_3 & \nabla\times \\ -\nabla\times & 0_3 & -\nu & -\omega\bar{\mu} \\ 0_3 & -\nabla\times & \omega\bar{\mu} & -\nu \end{bmatrix} \begin{bmatrix} E_r \\ E_i \\ H_r \\ H_i \end{bmatrix} = \begin{bmatrix} J_e^r \\ J_e^i \\ J_m^r \\ J_m^i \end{bmatrix}. \tag{2}$$

Here 0_3 is the 3x3 zero matrix and the indexes r and i refer to the real and imaginary parts of the functions. Let the vector of the fields in the left hand side of (2) be w and the vector of the currents be f. Then, (2) is: $\mathcal{M}w = f$.

We introduce an isometry operator J and a self-adjoint operator $\mathcal{L}$ as

$$J = \begin{bmatrix} 0_3 & -I_3 & 0_3 & 0_3 \\ -I_3 & 0_3 & 0_3 & 0_3 \\ 0_3 & 0_3 & 0_3 & I_3 \\ 0_3 & 0_3 & I_3 & 0_3 \end{bmatrix}, \qquad \mathcal{L} = \begin{bmatrix} \omega\bar{\epsilon} & \bar{\sigma} & 0_3 & \nabla\times \\ \bar{\sigma} & -\omega\bar{\epsilon} & \nabla\times & 0_3 \\ 0_3 & \nabla\times & -\omega\bar{\mu} & -\nu \\ \nabla\times & 0_3 & -\nu & \omega\bar{\mu} \end{bmatrix}. \tag{3}$$

Here I_3 is the 3x3 unit matrix.

Proposition 1 The operator J in (3) is a conjugation operator with the following properties

$$J = J^*, \quad JJ^* = I, \quad J^{-1} = J. \tag{4}$$

Proposition 2 If the matrices $\bar{\mu}$, $\bar{\nu}$, $\bar{\epsilon}$ and $\bar{\sigma}$ are symmetric, the Maxwell's operator is J-self-adjoint operator, with a self-adjoint extension $\mathcal{L}$ given by (3):

$$\mathcal{L} = \mathcal{M}J, \qquad J^*\mathcal{M}J = \mathcal{M}^*. \tag{5}$$

The self adjoint operator $\mathcal{L}$ maps a set of 4-component field vector functions into the set of 4-component source vector functions. From (5),

$$\mathcal{M} = \mathcal{L}J, \qquad \text{and} \qquad (\mathcal{M}w, f) \;=\; (\mathcal{L}Jw, f) \;=\; (\mathcal{L}u, f) \tag{6}$$

Using the operator $\mathcal{L}$ the system (1) can be written in a form $\mathcal{L}u = f$, with u and f being vector functions of fields and sources, $u = Jw$. The vector functions u and f are 2-nd rank tensor functions: $u \in \mathcal{U}, f \in \mathcal{U}$, $\mathcal{U} = \{u_k(x)\}$, $u_k = \{u_k^x, u_k^y, u_k^z\}, k = 1, ..., 4$. Each component of a 4-component vector is a 3D spatially changing vector function from $L^2(\Omega)^3$.
Symmetric tensors of the properties correspond to a reciprocal medium [6]. For an isotropic medium with scalar properties, $\bar{\mu} = \mu I$, $\bar{\nu} = \nu I$, $\bar{\epsilon} = \epsilon I$, and $\bar{\sigma} = \sigma I$, with $\mu(x), \nu(x), \epsilon(x)$, and $\sigma(x)$ being scalar functions. In this case, the matrices of electromagnetic properties are symmetric, and $\mathcal{L}$ is a self-adjoint operator. We assume further scalar properties of the medium.
Solution w of (2) for known functions $\mu, \nu, \epsilon, \sigma$ and a prescribed right hand side vector f is given by $w = \mathcal{G}f$, where $\mathcal{G}$ is the Green's operator. In the present case, this is the integral operator with 3D Green's dyadic function [6], that can be calculated numerically. From (6), $u = J\mathcal{G}f$.

3 The inverse problem. Optimal finite dimensional approximation

Let us assume $\mu = \mu_0$ is given, $\nu = 0$, the problem is to determine the functions $\sigma(x)$ and $\epsilon(x)$.
In order to determine $(\epsilon(x), \sigma(x)) = \gamma(x)$ inside the body, currents f are applied and the generated fields are compared with the fields due to a known model medium $\gamma_0(x) = (\epsilon_0(x), \sigma_0(x))$. The least square difference between the real measurements of the fields $E(x), H(x)$ and the response to the same currents of the known model $E_0(x), H_0(x)$ is minimized with respect to γ. The fields $E_0(x), H_0(x)$ are obtained using the operator $\mathcal{M}$ defined by (2) with known γ_0.
We reformulate this inverse problem as an inverse problem for the operator $\mathcal{L}$. Clearly, the solution of these two inverse problems is the same. Let the J transformed difference between $E(x), H(x)$ and $E_0(x), H_0(x)$ be $u_1 = (u - u_0) = \mathcal{R}_\gamma f$. The operator $\mathcal{R}_\gamma$ maps the applied currents to the J transformed measured field difference. The inverse problem is to determine γ from the knowledge of $\mathcal{R}_\gamma f$.

The eigenfunctions $\{f_i\}$ of the operator $\mathcal{R}_\gamma$ are the optimal vectors of the currents, which maximize the detectability of the difference in the properties $\gamma - \gamma_0$. They are proportional to the field difference: $\mathcal{R}_\gamma f_i = \mu_i f_i$. As we deal with conducting medium, the eigenvalues are fast decaying, hence only a finite number K of the eigenvalues μ_i exceed a reasonable noise level η. Data due to the corresponding currents are the only informative measurements. This gives a basis for designing an optimal finite dimensional approximation of the infinite dimensional operator $\mathcal{R}_\gamma$.

Proposition 3 The optimal K-dimensional approximation of the operator $\mathcal{R}_\gamma$ is given by its projection $\mathcal{R}_\gamma^F$ on a subspace F spanned by first K eigenfuctions.

Indeed, it follows from min-max principle, that this projection minimizes the length of the $||\mathcal{R}_\gamma - \mathcal{R}_\gamma^F||$. Introduction of K-dimensional approximation $\mathcal{R}_\gamma^F$ corresponds to cutting off the eigenvalues $\mu_k, k > K$ by letting $\mu_k = 0, k > K$. Thus we obtain an optimal K-dimensional approximation of the infinite dimensional inversion operator.

Remark As the reconstructed solution is in the range of the K-dimensional approximation, the choice of the projection subspace determines the reconstructed projection of the solution.

The solution of the inverse problem - vector $\delta\gamma = \gamma - \gamma_0$, is spanned by the functions of the type $\psi_{ij} = u_0^i \cdot B u_0^j$, where u_0^j is a solution of the model problem for the applied current f_j. The eigenvalues μ_k are products $\mu_k = \langle \delta\gamma, \psi_{kk} \rangle$ of $\delta\gamma$ with the functions ψ_{kk}. Hence, setting $\mu_k = 0$ for $k > K$, is equivalent to reconstructing a function $\delta\gamma$ which is orthogonal to a subspace spanned by all ψ_{ij}, except $i = j, i \leq K$. This approach gives a solution of the inversion problem based on min-max formulation,

$$\min_\gamma \max_F ||\mathcal{R}_\gamma^F|| \tag{7}$$

where $F \subset \mathcal{U}$ is a K- dimensional set of source functions f.
In the case of one dimensional approximation, the misfit functional $\mathcal{J}$ is a norm of the field difference u_1 due to the first eigencurrent f_1. To solve the inverse problem, we minimize $\mathcal{J}$ with respect to γ: $\mathcal{J} \to \min_\gamma$, $\quad \mathcal{J} = ||\mathcal{R}_\gamma f||$. The inverse problem becomes a minimax problem: find γ which minimizes the operator norm of the difference, $\min_\gamma \max_f ||\mathcal{R}_\gamma f||$.

4 Gradient scheme of solution

In order to calculate the gradient of the misfit functional $\mathcal{J}$ we need the Frechet derivative $\partial\mathcal{R}_\gamma/\partial\gamma$. We linearize the problem near the known coefficients γ_0. For a small perturbation $\delta\gamma$ of the background properties γ_0, the operator $\mathcal{L}$ can be represented as $\mathcal{L} = \mathcal{L}_0 + \varepsilon\mathcal{L}_1$ where $\mathcal{L}_0$ is the operator with γ_0. The linearized equations are:

$$\mathcal{L}_0\, u_0 = f, \qquad \mathcal{L}_0\, u_1 = -\mathcal{L}_1 u_0, \tag{8}$$

with $\mathcal{L}_1$ depending on

$$\delta\gamma = \gamma - \gamma_0 = (\delta\epsilon, \delta\sigma). \tag{9}$$

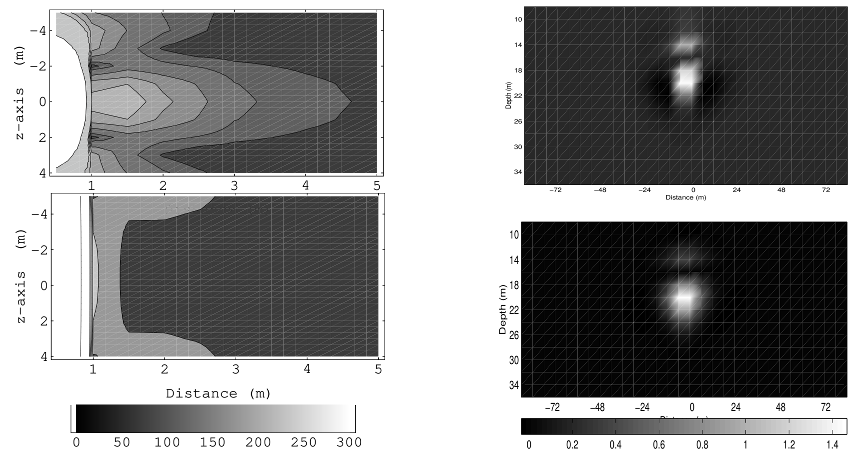

FIG. 1. *Left figure: The amplitude of H for the optimal applied current (top) and current of uniform intensity (bottom) for the model of a resistive layer embedded in more conductive medium. Right figure: Reconstruction (one iteration) of the medium with an embedded block with two different properties: ϵ (top) and σ (bottom).*

Having represented $\mathcal{L}_1$ as a sum

$$\mathcal{L}_1 = A_\sigma \delta\sigma + A_\epsilon \delta\epsilon = \begin{bmatrix} \omega\delta\epsilon & \delta\sigma & 0 & 0 \\ \delta\sigma & -\omega\delta\epsilon & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \delta\sigma \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \delta\epsilon \begin{bmatrix} \omega & 0 & 0 & 0 \\ 0 & -\omega & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(10)

we obtain the 2D vector of derivatives of u_1 with respect to γ:

$$u_1 = -\tilde{\mathcal{G}}_0 \mathcal{L}_1 u_0 = -\tilde{\mathcal{G}}_0 \mathcal{L}_1 \tilde{\mathcal{G}}_0 f = (\tilde{\mathcal{G}}_0 A_\sigma \tilde{\mathcal{G}}_0 f)(\delta\sigma) + (\tilde{\mathcal{G}}_0 A_\epsilon \tilde{\mathcal{G}}_0 f)(\delta\epsilon) = A\delta\gamma \tag{11}$$

Here $\tilde{\mathcal{G}}_0 = J\mathcal{G}_0$ is J-transformed Green's operator for the model problem. Varying the misfit functional with respect to γ, and using the linearized equations we calculate the gradient of the functional with respect to $\delta\gamma$: $\delta\mathcal{J} = \langle u_1, \delta\mathcal{R}f \rangle$, and $\delta\mathcal{R}f = A\delta\gamma$. Then, $\delta\mathcal{J} = \langle u_1, A\delta\gamma \rangle = \langle A^* u_1, \delta\gamma \rangle = \langle \Gamma, \delta\gamma \rangle$, where Γ is the gradient $\Gamma = (\Gamma_\sigma, \Gamma_\epsilon)$, that can be readily calculated from (10-11).
Steepest descent search gives $\delta\gamma = -\tau\Gamma$. An iterative process for solving the inverse problem can be constructed as $\gamma_{n+1} = \gamma_n + \delta\gamma$, where γ_n is an approximation of the solution on n-th iteration step, τ is the step in the direction minimizing the functional, and Γ is the gradient calculated at the point $\gamma = \gamma_n$. Because the introduced operator $\mathcal{L}$ is self-adjoint this approach leads to efficient computation of the gradient Γ during one forward run.

5 Numerical simulations

The suggested method of solution of the inverse electromagnetic problem for a dissipating medium is applied to numerically simulated models. To illustrate it, we bring here results of numerical simulations for two different problems.

First problem is optimization of applied electromagnetic inductive sources for detection of a resistive layer using measurements in a borehole. Left figure on Figure 1 shows the magnetic field due to the optimal applied current (top) and current of uniform intensity (bottom) for the model of a resistive layer embedded in a conductive medium. The currents are applied in the vertical borehole on the left. The top left figure shows how the field due to first eigenfunction of $\mathcal{R}_\gamma$ is concentrating in the resistive layer outside borehole.
The second problem is the inverse problem for two properties (conductivity and dielectric permittivity). The right figure shows results of reconstruction (after one iteration) of ϵ and σ using described approach. We use steepest gradient descent technique to minimize the misfit functional $\mathcal{J}$. Fields generated in the known background model are used to calculate the gradient Γ of the misfit functional. Because the introduced operator $\mathcal{L}$ is self-adjoint, we efficiently compute the gradient Γ during one forward run. No solution of adjoint problem is needed.

References

[1] D. Alumbaugh and H.F. Morrison, Monitoring subsurface changes over time with cross-well electromagnetic tomography, Geophysical Prospecting, 1995, 43, 873–902.
[2] Bohm, M., Kremer, J., and Louis, A. K., Efficient algorithm for computing optimal control of antennas in hyperthermia, Surveys Math. Indust., 1993, 3, 4, 233–251.
[3] L. Borcea, J.G. Berryman, and G. Papanicolaou, High contrast impedance tomography, Inverse problems, 1996, 12, 835–858.
[4] E. Cherkaeva and A.C. Tripp, Optimal survey design using focused resistivity arrays, IEEE, Trans. Geoscience and Remote Sensing, 1996, 34, 2, 358–366.
[5] E. Cherkaeva and A. C. Tripp, Source optimization in the inverse geoelectrical problem, in: Inverse Problems in Geophysical Applications, eds. H. W. Engl, A. Louis, and W. Rundell, SIAM, Philadelphia, 1997, 240–256.
[6] W. C. Chew, Waves and fields in inhomogeneous media, Van Nostrand Reinhold, New York, 1990.
[7] O. Dorn, H. Bertete-Aguirre, J.G. Berryman, and G. Papanicolaou, A nonlinear inversion method for 3D electromagnetic imaging using adjoint fields, Inverse Problems, in print.
[8] I.M. Glazman, Direct methods of qualitative spectral analysis of singular differential operators, 1965.
[9] R.F. Harrington, Field computation by moment methods, New York, Macmillan, 1968.
[10] D. Isaacson and M. Cheney, Current problems in impedance imaging, in Inverse Problems in PDE, ed. D.Colton, R.Ewing, and W.Rundell, 1990.
[11] A. Kirsch and P. Wilde, The optimization of directivity and signal-to-noise ratio of an arbitrary antenna array, Math. Meth. Appl. Sci., 1988, 10, 153–164.
[12] F. Natterer, The mathematics of computerized tomography, 1986.
[13] G.A. Newman and D. Alumbaugh, Three-dimensional massively parallel electromagnetic inversion I. Inversion, Geophys. J. Intern. 1997, 128, 345–354.
[14] R. L. Ochs, Jr., and C. R. Vogel, Energy optimization in electromagnetic wave propagation, Invariant Imbedding and Inverse Problems, Corones, J.P., et al., eds,

1992, 51–63.
[15] E. Somersalo, D. Isaacson, and M. Cheney, A linearized inverse boundary value problem for Maxwell's equations, J. Comp. Appl. Math., 1992, 42, 123–136.
[16] Z. Xiong, Electromagnetic modeling of three-dimensional structures by the method of system iterations using integral equations, Geophysics, 1992, 57, 1556–1561.