Source optimization in the inverse geoelectrical problem *

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Abstract

In electrical tomography, to determine the conductivity of a body or of a region of the earth, currents are injected on the surface and surface voltage responses are measured. Then the data are inverted to a conductivity distribution which matches the measurements. Usually all available data are used in the belief that pertinent information in the complete data set will be utilized. However, only part of the data contains useful information. This paper addresses means of specifying the optimal data so that pertinent data or data combinations can be interpreted while less pertinent or irrelevant data or data combinations are ignored.

It is suggested to reconstruct only those projections of the unknown conductivity distribution which can be reliably restored from the data with a given noise level. Information about these projections is contained in the eigenvalues of the currents to voltage mapping. Reconstruction of only these projections simplifies the inversion procedure. A numerical example is given for the geoenvironmental problem of monitoring a contaminated area.

1 Introduction

An attractive and popular method of estimating the conductivity structure of the earth is to use minimization of a least squares functional of the data misfit in terms of the earth conductivity distribution [7, 19]. However, as implemented, such techniques use data which are specified prior to implementation of the imaging and which may not be optimal in resolving the earth structure of interest. The general philosophy is to use all the specified data in the trust that pertinent information is contained in the complete data set. Most inversion schemes that have been developed for geoelectrical imaging deal with predetermined data [14, 20]. However, not all data contain equally important information about the conductivity

*This work was supported by DOE/OBES under grant number DE-FG03-93ER14313.

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distribution. An approach based on using optimal data was developed in [15, 16, 3].

This paper addresses means of specifying optimal data in the context of gradient imaging methods, so that pertinent data or data combinations can be interpreted while less pertinent or irrelevant data or data combinations are ignored. We suggest to reconstruct only those projections of the unknown conductivity distribution which can be reliably restored from the data with a given noise level. Information about these projections together with information about the optimal data is contained in the eigenvalues of the currents to voltages mapping. The eigenvalues of this operator form a very rapidly decreasing sequence [9]. It is clear that only data due to the currents with eigenvalues which are greater than the data noise level contain valuable information about the conductivity distribution. Numerical computation of the eigenvalues for the models with a localized inclusion show that only two to five eigenvalues are greater than the reasonable noise in measurements. Reconstruction of only the projections corresponding to these data simplifies the inversion procedure and allows us to save computational efforts.

2 Formulation of the problem

Let us consider a body Ω of conductivity γ and assume that arbitrary currents can be applied on its boundary $\partial\Omega$ and the generated voltages measured also on the boundary. The forward problem is described by the conductivity equation:

$$\nabla \cdot \gamma \nabla w = 0 \text{ in } \Omega, \quad \gamma \frac{\partial w}{\partial n} = f \text{ on } \partial\Omega, \tag{1}$$

where the current f satisfies the integral restriction:

$$\int_{\partial\Omega} f \, dx = 0. \tag{2}$$

The solution of equation (1) is unique up to a constant component which is determined if a zero value of the function is prescribed; so we assume that $\int_{\partial\Omega} w \, dx = 0$. We denote as $\tilde{L}_2(\partial\Omega)$ a subspace of functions from $L_2(\partial\Omega)$ satisfying to (2).

Boundary voltages are given by a linear operator $R_\gamma(f) = w$, which is the Neumann to Dirichlet operator, $\tilde{L}_2(\partial\Omega) \to \tilde{L}_2(\partial\Omega)$. It maps currents applied on the boundary to measured voltages. This operator is linear with respect to current f and it is nonlinear with respect to γ.

Let γ^* be the unknown conductivity of the real medium. The inverse conductivity problem is to determine the function γ^* from measuring the boundary voltage responses or from knowledge of the Neumann to Dirichlet map $R_{\gamma^*}(f)$. Solution of this problem is unique in the class of smooth or piecewise analytic functions [6, 13, 17] provided the Neumann to Dirichlet map R_{γ^*} is known.

Hence solving this inverse problem is equivalent to solving a system of an infinite number of equations with respect to γ^*:

$$\{\ R_{\gamma^*}(f_k)\ =\ w_k\ \},\quad k=1,2,...,\tag{3}$$

where $\{f_k\}$ is a system of applied currents, and $\{w_k\}$ are the corresponding voltages measured on the surface.

As the Neumann to Dirichlet map is linear, solving the inverse problem can be accomplished by solving (3) for a system of functions $\{f_i\}$ which forms a basis in $\tilde{L}_2$.

Let γ be a calculated approximation of the function of conductivity γ^*. The voltage responses generated on the boundary of the body of conductivity γ are $R_\gamma(f)$. Let us consider a current to voltage difference operator $V_{\gamma^*-\gamma}$:

$$V_{\gamma^*-\gamma}(f) = R_{\gamma^*}(f) - R_\gamma(f),\tag{4}$$

which corresponds to the difference in voltage responses generated by the media γ^* and γ due to the same injected current f.

Inverse problem now can be formulated as a problem of minimization of a norm of the operator $V_{\gamma^*-\gamma}$ with respect to γ:

$$\parallel V_{\gamma^*-\gamma} \parallel \to \min_{\gamma}.\tag{5}$$

It means that we want to find a function γ such that if used in the equation (1), generates responses $R_\gamma f$ which are as close as possible to the measured responses $R_{\gamma^*} f$ for any applied current f.

It follows from the uniqueness results cited above that solution of the problem (5) provides the true solution of the inverse problem in the class of smooth or piecewise analytic functions, because in this case the Neumann-to-Dirichlet map R_γ for the constructed function γ coincides with the Neumann-to-Dirichlet map R_{γ^*} corresponding to the true conductivity function γ^*.

In the next section we describe eigenfunctions and eigenvalues of the voltage difference operator (4) and the corresponding spectral boundary value problem. A variant of this spectral problem, valid in the case when the operator (4) is positive-definite, was introduced in [11, 9] for evaluating the best optimal current. In the present form, which is valid also when eigenvalues of $V_{\gamma^*-\gamma}$ are of arbitrary sign, this problem was introduced in [2]. We will see below in (28) that the sign of the first eigenvalue $\mathcal{M}_1$ of the operator (4) depends on the difference of the conductivities $\gamma^* - \gamma$ and can be arbitrary.

The difference between the measured and calculated responses is described by the misfit functional

$$\Sigma_{f_\alpha} \parallel R_{\gamma^*}(f_\alpha) - R_\gamma(f_\alpha) \parallel^2,\tag{6}$$

where f_α are applied currents. Usually this least squares functional is considered for some chosen functions f_α [5, 18] instead of (5).

When the system of functions $\{f_\alpha\}$ forms a basis in $L_2(\partial\Omega)$ this functional is a particular case of (5). This corresponds to the euclidean norm of the operator in (5) and takes all possible data into consideration. However, as we show in section 4, in the presence of data noise, solution of the inverse problem with functional (6) turns out to be equivalent to a solution for a least squares functional with only a few dominant eigenfunctions of the voltage difference operator taken as applied currents. Using this observation we will show in sections 5 and 6, that having chosen some approximation γ, the function $\delta\gamma = \gamma^* - \gamma$ is orthogonal to some directions in $L_2(\Omega)$. Therefore, avoiding these directions in constructing a numerical scheme, we can significantly save computational efforts.

3 Eigenfunctions of the current to voltage difference operator

For the eigenvalues $\mathcal{M}_i$ and eigenfunctions f_i of the operator $V_{\gamma^*-\gamma}$ we have:

$$\mathcal{M}_i\, f_i = V_{\gamma^*-\gamma}\, f_i. \tag{7}$$

The eigenvalues M_i of the operator $H_{\gamma^*-\gamma}$, where $H_{\gamma^*-\gamma} = (V^*_{\gamma^*-\gamma} V_{\gamma^*-\gamma})^{1/2}$ and $V^*_{\gamma^*-\gamma}$ is the adjoint operator, satisfy the variational formulation:

$$M_i = \max_{f:\, f \perp Sp[f_1,\dots,f_{i-1}]} \frac{\langle H_{\gamma^*-\gamma} f,\ f\rangle}{\langle f,\ f\rangle}, \tag{8}$$

where the scalar product is in $L_2(\partial\Omega)$. The eigenvalues M_i are equal to the spectral values for $V_{\gamma^*-\gamma}$, $M_i = |\,\mathcal{M}_i\,|$, and the norm of the operator $V_{\gamma^*-\gamma}$ is M_1:

$$\|\, V_{\gamma^*-\gamma}\, \|_1 = M_1. \tag{9}$$

The inverse conductivity problem can now be formulated as a problem of minimization of the norm $\|\cdot\|_1$ of the operator $V_{\gamma^*-\gamma}$ with respect to γ

$$J_1 = \|\, V_{\gamma^*-\gamma}\, \|_1 = \max_{f:\, \|f\|=1} \|\, R_{\gamma^*}(f) - R_\gamma(f)\, \|_{L_2(\partial\Omega)} \to \min_\gamma . \tag{10}$$

A closely related minimax variational functional is used for a different class of minimizing functions in [1] for formulation of the problem of detection of an inclusion with a restriction on the total volume fraction of the inclusion material.

When M_1, which depends on $\gamma^* - \gamma$, is less than ϵ, where ϵ is a noise level in measurements, we cannot distinguish the true solution γ^* from the solution given by the function γ. This approach was exploited in application to the resolution problem of electrical tomography in [11, 4].

It follows from (7)-(8) that the eigenfunctions of the operator $V_{\gamma^*-\gamma}$ satisfy the following system of Euler equations:

$$\text{(11)} \qquad \nabla \cdot \gamma^* \nabla w_i = 0 \ \text{ in } \ \Omega, \ \ \gamma^* \frac{\partial w_i}{\partial n} = \frac{1}{\mathcal{M}_i} (w_i - u_i) \ \text{ on } \ \partial\Omega,$$

$$\text{(12)} \qquad \nabla \cdot \gamma \nabla u_i = 0 \ \text{ in } \ \Omega, \ \ \gamma \frac{\partial u_i}{\partial n} = \frac{1}{\mathcal{M}_i} (w_i - u_i) \ \text{ on } \ \partial\Omega,$$

where γ^* is the unknown conductivity distribution and w_i is the corresponding potential in the real medium; γ is the calculated approximation of the function of conductivity distribution and u_i is the corresponding potential; and the function $f_i = \frac{1}{\mathcal{M}_i} (w_i - u_i)$ is the eigenfunction of $V_{\gamma^*-\gamma}$.

The eigenfunctions f_i form an orthogonal system, and any eigenfunction f_k is proportional to the potential difference $R_{\gamma^*}(f_k) - R_\gamma(f_k)$ measured on the boundary $\partial\Omega$. The coefficient of proportionality $\mathcal{M}_k$ is such that $\| R_{\gamma^*}(f_k) - R_\gamma(f_k) \|$ is equal to $\| \mathcal{M}_k f_k \|$, which results in $| \mathcal{M}_k | = M_k$.

In [2] it is shown that for a localized inclusion the eigenfunctions f_i concentrate the energy of the scattering current in the region of the inclusion, hence maximizing the voltage response from the inclusion on the surface.

4 On equivalency of solutions for inexact measurements

We will show that for inexact measurements it is enough to use only a few dominant eigenfunctions of the Neumann-to-Dirichlet difference map $R_{\gamma^*}(f) - R_\gamma(f)$ whose corresponding eigenvalues are greater than the noise level in the data.

Generally the voltage difference operator $V_{\gamma^*-\gamma}$ has eigenvalues $M_0 > \ ... \ > M_q > ...$ which decrease very rapidly. In [9] it was shown that they decrease exponentially. Numerical computations of the eigenvalues for the models with a localized inclusion show that only two to five eigenvalues are significantly different from zero.

models of localized inclusions used in geoelectrical exploration.

Let ϵ be a data noise level, and suppose that only the first q eigenvalues are greater than this noise, such that

$$\text{(13)} \qquad M_1 > \ ... \ > M_q > \quad \epsilon \quad > M_{q+1} > ...$$

We can define the ϵ-null space $\mathcal{N}_\epsilon$ of the operator $V_{\gamma^*-\gamma}$ as a subspace of functions $f \in L_2(\partial\Omega)$ such that, subject to the norm of current f equaling one, $\| V_{\gamma^*-\gamma}(f) \| < \epsilon$. From the inversion point of view, functions from the ϵ-null space $\mathcal{N}_\epsilon$ do not provide any valuable information on the conductivity distribution, because the generated difference in responses is less than the noise in the measurements.

It follows that only the first q eigenfunctions 'effectively' span the range of the difference operator $V_{\gamma^*-\gamma}$, and all other eigenfunctions belong to the ϵ-null space of this operator.

Hence different solutions of the inverse problem corresponding to a voltage difference which is less than the noise level are equivalent solutions, insofar as they cannot be distinguished from analysis of boundary measurements.

Let us suppose that we have two sets of voltage differences representing the values of the operator $V_{\gamma^*-\gamma}$ for the functions $f_1, f_2, ...$:

$$\{v_1, v_2, ..., v_q, v_{q+1}, ...\} \ \text{and} \ \{v_1, v_2, ..., v_q, \ 0, \ 0, ...\}. \tag{14}$$

Then if $\gamma + \sigma_1 = \gamma^*$ is a true solution of the inverse problem:

$$\left\{ \begin{array}{ccc} R_{\gamma+\sigma_1}(f_1) - R_\gamma(f_1) & = & v_1 \\ ... & & ... \\ R_{\gamma+\sigma_1}(f_q) - R_\gamma(f_q) & = & v_q \\ R_{\gamma+\sigma_1}(f_{q+1}) - R_\gamma(f_{q+1}) & = & v_{q+1} \\ ... & & v_{q+2} \\ ... & & ... \end{array} \right\} \tag{15}$$

and $\gamma + \sigma_2$ satisfies a "cut" system:

$$\left\{ \begin{array}{ccc} R_{\gamma+\sigma_2}(f_1) - R_\gamma(f_1) & = & v_1 \\ ... & & ... \\ R_{\gamma+\sigma_2}(f_q) - R_\gamma(f_q) & = & v_q \\ R_{\gamma+\sigma_2}(f_{q+1}) - R_\gamma(f_{q+1}) & = & 0 \\ ... & = & 0 \end{array} \right\}, \tag{16}$$

then $\gamma + \sigma_1$ and $\gamma + \sigma_2$ are equivalent solutions.

Proof of this immediately follows from the fact that since the eigenfunctions f_i form an orthonormal system in L_2, then for any applied current the voltage difference on the boundary $\partial\Omega$ generated by the functions $\gamma + \sigma_1$ and $\gamma + \sigma_2$ is less than the error of measurements ϵ:

$$\max_{f:\, \|f\|=1} \| R_{\gamma+\sigma_1}(f) - R_{\gamma+\sigma_2}(f) \|_{L^2(\partial\Omega)} = M_{q+1} < \epsilon. \tag{17}$$

This means that no current applied on the boundary will generate a noticeable difference in voltages for the media with conductivity $\gamma + \sigma_1$ and $\gamma + \sigma_2$.

Let us consider the functional (6). When the system of functions $\{f_\alpha\}$ forms a basis in $L_2\,(\partial\Omega)$ the functional (6), being an euclidean distance between the operators R_{γ^*} and R_γ, is equal to the norm $\|\cdot\|_2$:

$$\| V_{\gamma^*-\gamma} \|_2^2 = \Sigma_{f_i} \| V_{\gamma^*-\gamma}(f_i) \|^2 \tag{18}$$

for the system of functions $\{f_i\}$ which are the eigenfunctions of the operator $V_{\gamma^*-\gamma}$. But the solutions of the inverse problem for the functional (18) and for a similar sum with only the first q terms are solutions which are equivalent up to the noise level. It means that the euclidean norm can be considered for only functions f_i which do not belong to the ϵ-null space $\mathcal{N}_\epsilon$ of the operator $V_{\gamma^*-\gamma}$:

$$\Sigma_{f_i \notin \mathcal{N}_\epsilon} \| V_{\gamma^*-\gamma}(f_i) \|^2. \tag{19}$$

Therefore given a particular conductivity distribution γ the functional (19) can be considered instead of the euclidean norm $\| V_{\gamma^*-\gamma} \|_2$, and this functional is equal to the sum of the first q dominant eigenvalues:

$$\Sigma_{f_i \notin \mathcal{N}_\epsilon} \| V_{\gamma^*-\gamma}(f_i) \|^2 = \sum_{i=1}^{q} M_i^{\,2} = \sum_{i=1}^{q} \| w_i - u_i \|^2_{L_2(\partial\Omega)}, \tag{20}$$

where w_i and u_i are solutions of the equations (11)-(12) for the eigenfunctions f_i, $f_i = \frac{1}{\mathcal{M}_i}(w_i - u_i)$.

For the norm $\| \cdot \|_1$ we formulated an inverse problem in a previous paragraph. Similarly, using the norm $\| \cdot \|_2$ an inverse solution can be constructed as a solution of the problem:

$$J_2 = \| V_{\gamma^*-\gamma} \|_2^2 \to \min_\gamma \tag{21}$$

and constructing a numerical solution we can restrict ourselves to functions $f_i \notin \mathcal{N}_\epsilon$.

The problems with least squares functionals are usually solved using some gradient method. In any of these methods we use a Frechet derivative of the functional and we need to calculate a linear increment of the solution when the function of conductivity γ in the equation (11) is changed to $\gamma+\delta\gamma$, where $\delta\gamma$ is a small perturbation of the conductivity function.

5 Linearization of the problem and subspace containing the solution

Let us consider a linearized problem assuming that the conductivity perturbation is small, as was done in [3].

Let v be the additional scattering potential, $v = w-u$, and the potentials w and u be the solutions of (11) corresponding to conductivity distributions $\gamma + \delta\gamma$ and γ, respectively.

We assume that $supp\,(\delta\gamma) \in \tilde{\Omega}$ for some subdomain $\tilde{\Omega} \subset \Omega$, and $\delta\gamma(x) = 0$, for $x \in \Omega \setminus \tilde{\Omega}$. We suppose that $\delta\gamma \in L_2(\tilde{\Omega})$. Then

$$\begin{aligned} \nabla \cdot \gamma \nabla u &= 0 \text{ in } \Omega, \quad \gamma \frac{\partial u}{\partial n} = f \text{ on } \partial\Omega, \\ \nabla \cdot \gamma \nabla v &= -\nabla \cdot \delta\gamma \, \nabla u \text{ in } \Omega, \quad \gamma \frac{\partial v}{\partial n} = 0 \text{ on } \partial\Omega \,. \end{aligned} \tag{22}$$

The inverse linearized problem is to find a conductivity distribution $\delta\gamma$ which fits the measured potential difference v on the surface for different applied currents f, which are related by the equations (22).

Using Green's second formula the solution of the system (22) can be represented in integral form,

$$v(x) \;=\; \int_\Omega \nabla \cdot \delta\gamma \, \nabla u(y) \; G(x,y) \; dy$$

$$= -\int_\Omega \delta\gamma(y) \int_{\partial\Omega} f(z) \; \nabla_y G(z,y) \cdot \nabla_y G(x,y) \; dz \; dy, \tag{23}$$

where $G(x,y)$ is the Green's function for the model problem with conductivity γ.

We can see that the linear increment v of the solution when the conductivity function γ is changed to the function $\gamma + \delta\gamma$ is given by the integral linear operator (23), denoted as

$$v = L \, \delta\gamma. \tag{24}$$

Let us consider now the voltage difference operator $V_{\gamma^*-\gamma}$. Linearized in γ , it has values v satisfying the linearized equation (22): $V_{\gamma^*-\gamma} f = v \,|_{\,\partial\Omega}$.

For a current which is an eigenfunction f_i of the operator $V_{\gamma^*-\gamma}$ we have $f_i(x) \;=\; \frac{1}{\mathcal{M}_i} \, v_i(x)\,,\; x \in \partial\Omega$. Here $\mathcal{M}_i$ is the corresponding eigenvalue of the operator $V_{\gamma^*-\gamma}$.

Hence in the linearized problem the eigenfunctions satisfy the following system of equations:

$$\nabla \cdot \gamma \, \nabla u_i \;=\; 0 \;\text{ in }\; \Omega, \;\; \gamma \, \frac{\partial u_i}{\partial n} \;= f_i =\; \frac{1}{\mathcal{M}_i} \, v_i \;\text{ on }\; \partial\Omega, \tag{25}$$

$$\nabla \cdot \gamma \, \nabla v_i \;=\; -\nabla \cdot \delta\gamma \, \nabla u_i \;\text{ in }\; \Omega, \;\; \gamma \, \frac{\partial v_i}{\partial n} \;=\; 0 \;\text{ on }\; \partial\Omega. \tag{26}$$

Integrating (26) by parts we obtain the first eigenvalue and the $\| \cdot \|_1$ norm of the operator $V_{\gamma^*-\gamma}$:

$$\| \, V_{\gamma^*-\gamma} \, \|_{\,1} = M_1 = |\mathcal{M}_1| \;=\; | \int_\Omega \delta\gamma \; \nabla u_1 \cdot \nabla u_1 \; dy \, |, \tag{27}$$

and similarly for any eigenfunction f_i

$$\mathcal{M}_i \;=\; -\int_\Omega \delta\gamma \; \nabla u_i \cdot \nabla u_i \; dy. \tag{28}$$

Thus the eigenvalues of the voltage difference operator $V_{\gamma^*-\gamma}$ are scalar products (in $L_2\,(\tilde{\Omega})$) of the function $\delta\gamma \;=\; \gamma^* - \gamma$ with functions $\phi_i \;=\; \nabla u_i \cdot \nabla u_i$.

$$\mathcal{M}_i \;=\; \langle \delta\gamma, \, \phi_i \rangle. \tag{29}$$

Functions ϕ_i are the squared electric fields generated by the eigencurrents. From this point of view the previous result on equivalency of solutions means that if for some q the spectral values $M_i, \; i > q$, are less than the noise level

ϵ the corresponding projections of $\delta\gamma$ cannot be restored from the data, because the corresponding scalar products are not distinguishable from zero.

In this case we can put $\langle \delta\gamma, \phi_i \rangle = 0$ for $i > q$ within the equivalency of the solutions.

For $i \neq j$ and functions $\psi_k = \nabla u_i \cdot \nabla u_j$ it follows from (23) and orthogonality of the functions $\{f_i\}$ in L_2 that

$$\langle \delta\gamma, \psi_k \rangle_{L_2(\tilde{\Omega})} = \mathcal{M}_i \langle f_i, f_j \rangle_{L_2(\partial\Omega)} = 0. \tag{30}$$

Hence, it turns out that the function $\delta\gamma$ which can be restored from noisy data is orthogonal to a subspace T_0 in $L_2(\tilde{\Omega})$ spanned by the functions $\phi_i,\ i > q$, and the functions $\psi_k : T_0 = \mathrm{Sp}[\phi_i,\ \psi_k],\ i = q+1, ...,\ k = 1, 2, ...$

We will see in the next section that the functions $\phi_i,\ i > q$, are nothing but the results of the adjoint L^* to the Frechet derivative operator L (24) applied to the sets of data corresponding to eigencurrents with negligible eigenvalues $\phi_n = L^*\, v_n$ for $n = q+1, ...$

6 Construction of a numerical solution

Based on previous consideration, we assume a cut system of restrictions in constructing the numerical solution of the inverse problem. Thus in constructing an iterative scheme we use only data of the eigencurrents corresponding to the eigenvalues of the voltage difference operator which are greater than the noise level.

We consider two functionals F_1 and F_2, with F_1 being based on the norm $\|\cdot\|_1$ of the voltage difference operator which is the maximal eigenvalue, and F_2 being based on the $\|\cdot\|_2$ norm. The functional F_1 is a least squares functional based on data of the first eigencurrent, while the functional F_2 is a least squares functional based on the system of q dominant eigenfunctions. We can compare the results of reconstruction using F_1 and F_2 with reconstruction using a functional F_{f_α} based on data due to an arbitrary function f_α .

The solutions are developed with a gradient method. Gradient or quasi-Newton methods are usually exploited when dealing with minimization of a nonlinear functional of least squares type [8, 18]. In [5] the convergence rate for a quasi-Newton method of solution of the least squares problem for electrical tomography is estimated independently of the applied currents.

Applying some gradient method for minimization of the functional F_1 we obtain a 'backprojected' or 'backpropagated' solution ([18]). The functional F_1 is

$$F_1 = \frac{1}{2} \| V_{\gamma^* - \gamma} \|_1^2 = \frac{1}{2} || w_1 - u_1 ||^2, \tag{31}$$

where w_1 is the measured potential or the potential in the real medium of unknown conductivity γ^* corresponding to the optimal current, while u_1 is the background potential or calculated voltage corresponding to the optimal current and the medium of known conductivity γ. This means that the functions w_1 and u_1 are solutions of the problem (11)-(12) for the first eigenfunction f_1 of the voltage difference operator $V_{\gamma^*-\gamma}$.

Using the 'chain rule' for Frechet differentiation or varying the functional F_1 with respect to γ and using the linearized equation to calculate the gradient we obtain the following:

$$\delta_\gamma F_1 = \langle\, (w_1 - u_1),\ \delta_\gamma (w_1 - u_1)\,\rangle, \quad \text{and} \quad \delta_\gamma (w_1 - u_1) = L_1\, \delta\gamma, \tag{32}$$

where L_1 is linearized operator (23) when the current f in (23) equal to the first eigencurrent f_1. Now substituting into δF_1 we obtain the gradient of the functional with respect to $\delta\gamma$:

$$\delta F_1 = \langle\, (w_1 - u_1),\ L_1\, \delta\gamma\,\rangle = \langle\, L_1^*\,(w_1 - u_1),\ \delta\gamma\,\rangle = \langle\, \Gamma,\ \delta\gamma\,\rangle, \tag{33}$$

where Γ is the gradient $\Gamma = L_1^*\,(w_1 - u_1)\,.$

Steepest descent search gives $\delta\gamma = -\tau\Gamma$ and standard methods ([8, 10]) can be used to determine the parameter of relaxation τ.

Using the explicit expression (23) with the first eigencurrent f_1 and the property of the eigenfunctions (7) we have:

$$\begin{aligned} L_1^*\,(w_1 - u_1) &= \mathcal{M}_1 \int_{\partial\Omega} f_1(z)\, \nabla_y G(z,y)\cdot \nabla_y u_1(y)\, dz \\ &= \mathcal{M}_1\, \nabla u_1 \cdot \nabla u_1. \end{aligned} \tag{34}$$

Let us consider the functional (6) for one function f_α. We will show that all directions in $L_2\,(\Omega)$ which are different from those given by expressions similar to (34) for the first q eigencurrents are orthogonal to the difference $\gamma^* - \gamma$.

Any function f_α and the corresponding voltage difference response $V_{\gamma^*-\gamma}(f_\alpha)$ can be presented as

$$f_\alpha = \Sigma_i\, \alpha_i f_i\ , \quad \text{with} \quad \Sigma_i\, \alpha_i^2 = 1\ , \quad \text{and} \quad V_{\gamma^*-\gamma}(f_\alpha) = \Sigma_i\, \alpha_i\, \mathcal{M}_i\, f_i. \tag{35}$$

For a functional $F_{f_\alpha} = \|\, V_{\gamma^*-\gamma}(f_\alpha)\,\|^{\,2}$ we have

$$\delta_\gamma F_{f_\alpha} = \langle (w_{f_\alpha} - u_{f_\alpha}),\, \delta_\gamma (w_{f_\alpha} - u_{f_\alpha})\rangle = \langle\, \Sigma_i\, \alpha_i \mathcal{M}_i f_i\ ,\ \Sigma_i\, \alpha_i L_i \delta\gamma\,\rangle, \tag{36}$$

because from (22) and (35) it follows for the linearized difference $v_{f_\alpha} = w_{f_\alpha} - u_{f_\alpha}$ that:

$$\begin{aligned} v_{f_\alpha}(x) \;&=\; \int_\Omega \delta\gamma\, \nabla u_{f_\alpha}(y)\cdot \nabla G(x,y)\, dy \\ &=\; \Sigma_i\, \alpha_i \int_\Omega \delta\gamma\, \nabla u_i \cdot \nabla G(x,y)\, dy \end{aligned} \tag{37}$$

and the last integral in (37) is the operator L_i:

$$L_i \delta\gamma = \int_\Omega \delta\gamma\, \nabla u_i \cdot \nabla G(x,y)\, dy \tag{38}$$

corresponding to the applied eigencurrent f_i with generated potential u_i.

Now

$$\delta_\gamma F_{f_\alpha} = \langle\, \Sigma_{i,j}\, \alpha_i\, \alpha_j\, \mathcal{M}_i\, L_j^*\, f_i\,,\ \delta\gamma\,\rangle, \tag{39}$$

and

$$L_j^*\, f_i = \int_{\partial\Omega} f_i(x)\, \nabla u_j(y) \cdot \nabla G(x,y)\, dy. \tag{40}$$

We recall now that the eigenfunctions $\{f_i\}$ form an orthogonal system in $L_2\,(\partial\Omega)$. From this fact it follows that the true function $\delta\gamma$ should be orthogonal to $L_j^*\, f_i$ for $i \neq j$.

$$\begin{aligned} \langle\, L_j^*\, f_i\,,\, \delta\gamma\,\rangle = \int_\Omega \delta\gamma(y) \int_{\partial\Omega} f_i(x)\, \nabla u_j(y) \cdot \nabla G(x,y)\, dx\, dy = \\ \int_{\partial\Omega} f_i(x)\, \mathcal{M}_j\, f_j(x)\, dx = 0. \end{aligned} \tag{41}$$

Therefore the solution $\delta\gamma$ is orthogonal to the functions $\phi_i = L_i^*\, f_i$ for f_i from the ϵ-null subspace $\mathcal{N}_\epsilon$ and to the functions $\psi_k = L_j^*\, f_i$, for arbitrary $i,\ j$ such that $i \neq j$.

For the second functional F_2 we deal only with nonzero projections of the solution:

$$F_2 \,=\, \frac{1}{2} \sum_{i=1}^{q} ||\, w_i - u_i\, ||^2, \quad \delta_\gamma F_2 \,=\, \sum_{i=1}^{q} \langle\, (w_i - u_i),\ \delta_\gamma (w_i - u_i)\,\rangle. \tag{42}$$

We use linearized equations (24) to calculate the gradient:

$$\delta_\gamma F_2 \,=\, \sum_{i=1}^{q} \langle\, (w_i - u_i),\ L_i \delta\gamma\,\rangle, \tag{43}$$

where L_i is given by (38).

Similarly to the previous derivation for F_1 we have:

$$\sum_{i=1}^{q} \langle\, L_i^*\, (w_i - u_i),\ \delta\gamma\,\rangle \,=\, \langle \sum_{i=1}^{q} L_i^*\, (w_i - u_i),\ \delta\gamma\,\rangle \,=\, \langle\, \Gamma,\ \delta\gamma\,\rangle, \tag{44}$$

where Γ is the gradient $\Gamma = \sum_{i=1}^{q} L_i^*\, (w_i - u_i)$. Therefore in this case the gradient does not include directions which are orthogonal to the true solution.

7 Results of computer simulations

We apply an algorithm based on F_1 minimization to the following problem of monitoring a contaminated area. Let us consider a region of the earth containing a waste deposit which can probably leak. We assume that the waste container is resistive, and that the probable leakage is also

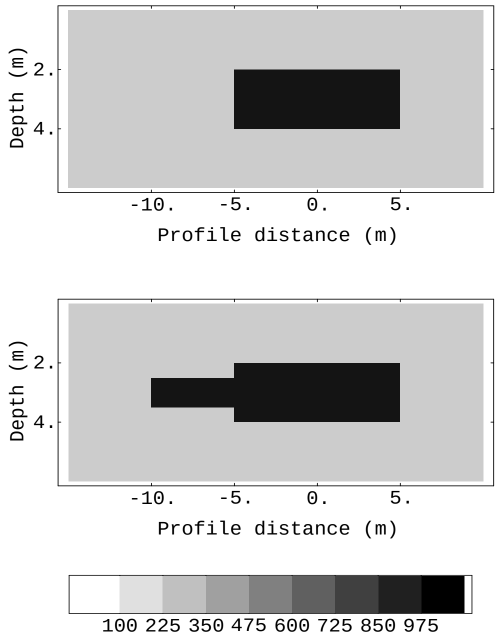

FIG. 1. *Cross section of the starting model (top) and the real medium with leak (bottom). The scale on the bottom shows the values of resistivity in ohm.m.*

resistive. Thus the model background resistivity distribution consists of a homogeneous earth material of resistivity 100 *ohm.m* with an embedded container of resistivity 1000 *ohm.m*. The size of the container is 5 m by 2 m by 1 m. We assume that the container leaks, and we take as the real medium the previous model with the leakage modeled as an additional inclusion of resistivity 1000 *ohm.m* near the center of the right vertical side of the waste container. The size of the leakage is 2.5 m by 1 m by 1 m.

Figure 1 shows a cross-section of the models - a starting initial model and a real model with an inclusion which models leakage.

The electric current is assumed to be injected on the surface of the Earth through a set of point electrodes located on a profile going along the long horizontal axis of the container. We use an integral equations forward modeling algorithm [12] in order to calculate the voltage response of the medium and of the 'real' medium.

We find the optimal current distribution on the surface using singular value decomposition of the impedance matrix as described in [2]. Figure 2 shows eigenvalues of the voltage difference operator for this particular conductivity distribution. We can see that only a few eigenvalues are significantly different from zero, which means that only data of the corresponding eigencurrents provide some information about the inclusion.

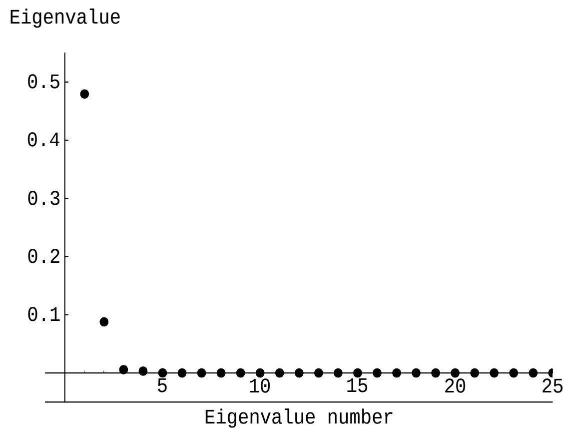

FIG. 2. *Distribution of eigenvalues M_i before starting the inversion.*

Knowing the optimal intensities of the current injected at different electrodes we calculate the optimal background electric field inside the earth as a weighted combination of the electric fields due to the unit currents. We use a weight function in the vicinity of the electrodes to ensure $supp\,(\delta\gamma) \subset \tilde{\Omega}$. From calculation of the gradient the perturbation of the conductivity function $\delta\gamma$ is proportional to the optimal background electric field. The coefficient of proportionality is calculated from the first eigenvalue and the squared norm of the background field. This is the first step of the iteration.

On the next step we take the 'improved' conductivity $\gamma + \delta\gamma$ as a known medium γ_1 and repeat the process starting from the calculation of the current distribution which is optimal for distinguishing this new background from the real medium.

Figure 3 shows the cross-section of the medium as a result of calculations after 1, 3, 9, and 24 iterations.

Figure 4 gives the values of the first eigenvalues for the different iterations. This plot shows the rms norm of the difference between the voltages generated by the real medium and by the medium with conductivity calculated during the described iterative process.

Figure 5 shows the actual difference in voltage data on the surface before starting the inversion and after 3, 9, 18, and 27 iterations.

8 Conclusions

In the present paper a solution of the inverse electric tomography problem is suggested for measurements containing noise. In this case only a part of the conductivity function can be restored from the data, and only a part of

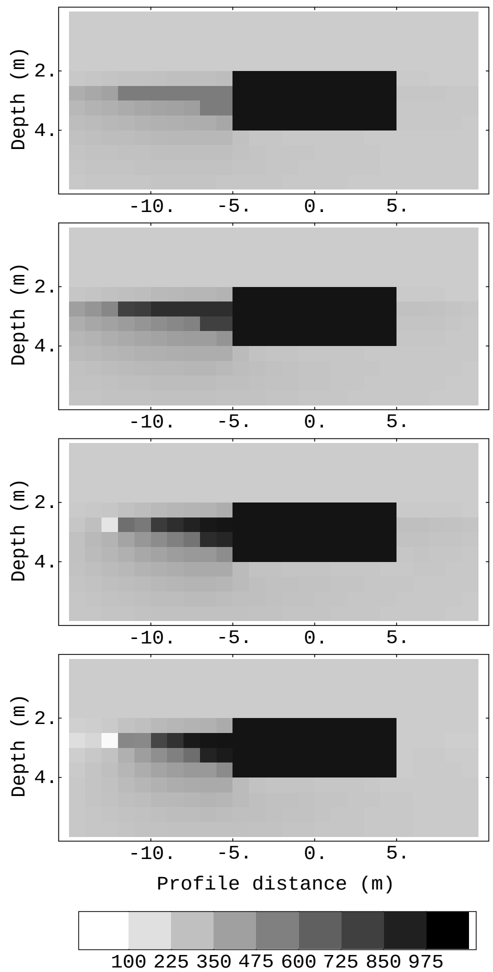

Fig. 3. *Cross sections given by the solution of the inverse problem after 1, 3, 9, and 24 iterations. The scale on the bottom shows the values of resistivity in ohm.m.*

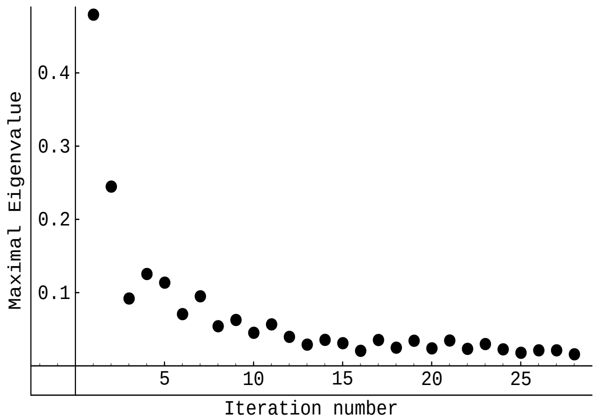

FIG. 4. *Size of the maximal eigenvalue with respect to the number of iterations.*

the data contain information about the unknown function of conductivity distribution.

The eigenvalues M_i of the current to voltage mapping show what data components are informative and what projections of the conductivity distribution γ can be determined from the data at a given noise level.

The algorithm for constructing an inverse solution for noisy data provides a conductivity function from a set of solutions which are equivalent up to the noise level. It is based only on those data which contain valuable information, hence reducing the dimension of the problem and avoiding unnecessary computations.

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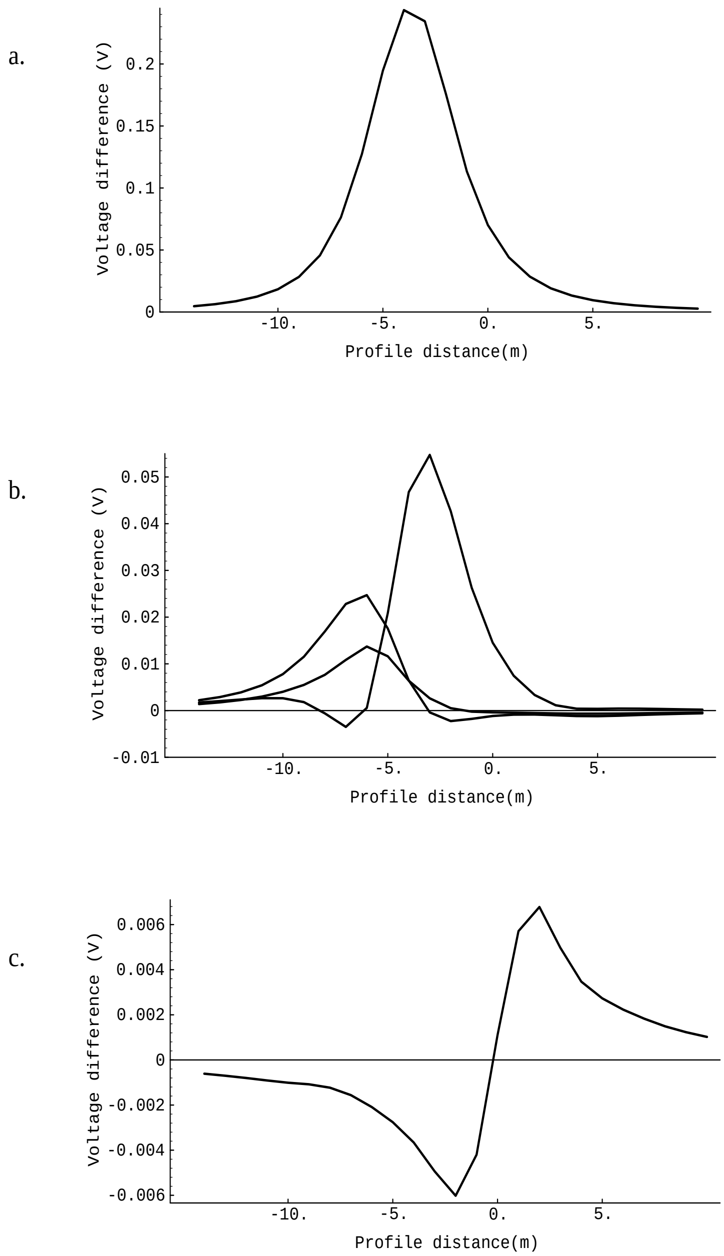

FIG. 5. *Difference in voltage data on the surface: before starting the inversion (a), after 3, 9, 18 iterations (b), after 27 iterations(c). Note the different scales on the graphs.*

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