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Transition waves in bistable structures. I. Delocalization of damage

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Abstract

We consider chains of dimensionless masses connected by breakable bistable links. A non-monotonic piecewise linear constitutive relation for each link consists of two stable branches separated by a gap of zero resistance. Mechanically, this model can be envisioned as a ''twin-element'' structure which consists of two links (rods or strands) of different lengths joined by the ends. The longer link does not resist to the loading until the shorter link breaks. We call this construction the *waiting link structure*. We show that the chain of such strongly non-linear elements has an increased in-the-large stability under extension in comparison with a conventional chain, and can absorb a large amount of energy. This is achieved by two reasons. One is an increase of dissipation in the form of high-frequency waves transferring the mechanical energy to heat; this is a manifestation of the inner instabilities of the bonds. The other is delocalization of the damage of the chain. The increased stability is a consequence of the distribution of a partial damage over a large volume of the body instead of its localization, as in the case of a single neck formation in a conventional chain. We optimize parameters of the structure in order to improve its resistance to a slow loading and show that it can be increased significantly by delocalizing a damage process. In particular, we show that the dissipation is a function of the gap between the stable branches and find an optimal gap corresponding to maximum energy consumption under quasi-static extension. The results of numerical simulations of the dynamic behavior of bistable chains show that these chains can withstand without breaking the force which is several times larger than the force sustained by a

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conventional chain. The formulation and results are also related to the modelling of compressive destruction of a porous material or a frame construction which can be described by a two-branched diagram with a large gap between the branches. We also consider an extension of the model to multi-link chain that could imitate plastic behavior of material.

Keywords: Dynamics; Phase transition; Bistable-bond chain; Asymptotic analysis

1. Introduction

We consider a chain as an assembly of concentrated masses connected by massless links. Each link satisfies a piecewise linear constitutive relation: the force–elongation dependence for the link is characterized by two stable branches, a basic low-strain branch and a high-strain branch as in Fig. 1. Such a dependence can be achieved, in particular, in the *waiting link structure* considered below. While theoretically a material can absorb energy until it melts, the strain localization dramatically decreases the limit in conventional materials. To the contrary, designed here constitutive relations are of hardening type which leads to the delocalization of strain. In addition, such a non-monotonic dependence leads to a pronounced energy

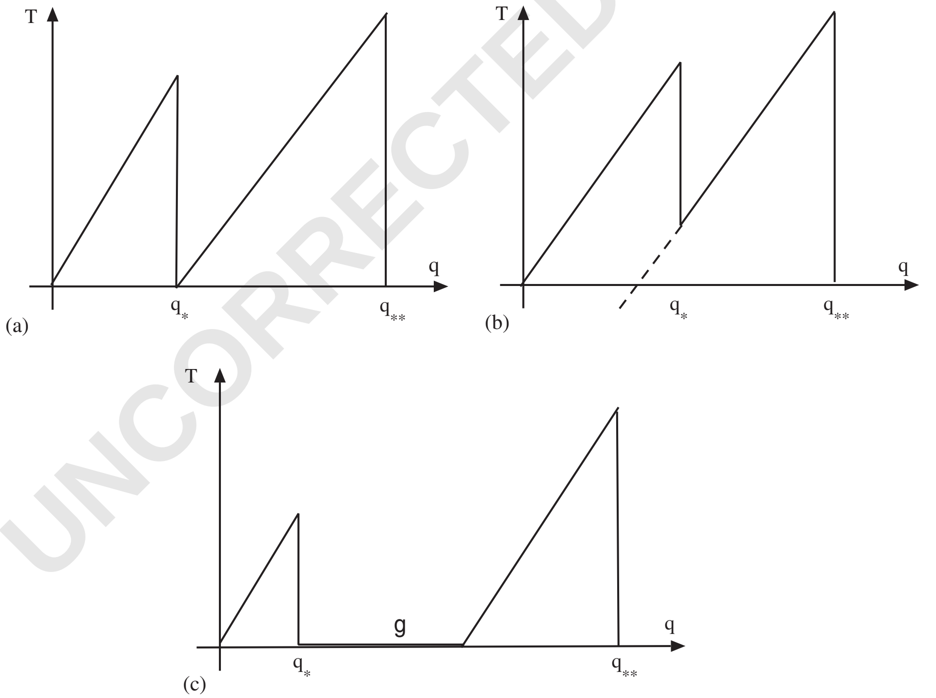

Fig. 1. (a–c) Two-branch piece-wise linear functions show bistable force–elongation diagrams.

dissipation. Both these factors considerably rise the limit of the energy consumption. A stable state corresponds to one of the stable branches of the constitutive relation. Under a dynamic action, each element of the chain transits from the basic lower-strain state to the higher-strain state. This generates a transition wave that propagates along the chain. This transition wave is accompanied by high-frequency dissipative waves which intensity depends on the slopes of the stable branches of the constitutive relation and on the gap between them. Hence, the maximal energy consumption and the ability of the structure to withstand impacts depend on the design of structural elements.

A discrete chain represents a simplest adequate model suitable for both analytical studies and numerical simulations of transition waves which are also present in more complicated models of periodic bistable structures. This model possesses some advantages in comparison with a continuous model of a bistable material. First of all, in contrast to a continuous model, it leads to a unique solution. It needs less assumptions, allows one to see the details of the damage, and to account for the high-frequency models that are "invisible" in the continuum limit. Some bistable-link chain models were considered in a number of works (see Slepyan and Troyankina, 1984, 1988; Puglisi and Truskinovsky, 2000; Slepyan, 2000, 2001, 2002; Balk et al., 2001a,b; Charlotte and Truskinovsky, 2002; Ngan and Truskinovsky, 1999; Cherkaev and Zhornitzkaya, 2003). In the present paper we consider a chain with waiting links, formulate the dynamic equations, and estimate the gap role in the resistance of the chain to quasi-static and dynamic extension. We also present results of numerical simulations of the dynamic behavior of bistable chains demonstrating their high resistance to extension. Due to the damage delocalization and energy dissipation, these chains can withstand an impact several times larger than the impact sustained by a conventional chain.

Motivation: The considered models and results relate to different areas where the transition path contains an unstable region. In particular, the following related problems can be mentioned:

(a) A structure designed to withstand impacts or explosions. For such dynamic actions the major quality of a material or a construction is the limiting energy consumption which could be increased by the design.

(b) Fracture of a porous material or a frame construction under compression can be described by a similar two-branch diagram. In the case of a porous material the dissipation is pronounced and temperature effects can be considerable; the dissipation rises the temperature so significantly that the *heat pressure* becomes enormous (see Zel'dovich and Raizer, 1966, 1967).

(c) The constitutive model with a large gap between the branches corresponds to a framed construction of a multi-story building.

(d) Stick-slip sliding in friction and earthquake behavior is a similar type phenomenon: in this case a large amount of energy is released due to the transition from one stable state to another one. Lastly,

(e) The phase transition process in a material may also be considered using the bistable chain models.

2. Model: bistable-link chain

2.1. Equilibria of bistable chain

Chain with bistable links: Consider a periodic chain of equal masses, M, connected by equal bistable links of length a (see Fig. 2). The tensile force $T(q)$ acting in each link is a non-monotonic function of the elongation q containing two stable (increasing) branches separated by an unstable region. Under a monotonic elongation $T(q)$ is characterized by

$$\frac{\mathrm{d}T}{\mathrm{d}q}\begin{cases} >0 & \text{when } q<q_* \text{ the first, original stable branch,} \\ \leqslant 0 & \text{when } q_*<q<q_*+\mathcal{G} \text{ unstable region,} \\ >0 & \text{when } q_*+\mathcal{G}<q<q_{**} \text{ the second stable branch} \end{cases} \tag{1}$$

and $T=0$ when $q>q_{**}$. Here $\mathcal{G}$ is the gap between the stable branches. This dependence is shown in Fig. 1(c). Note that this non-monotonic force-elongation dependence corresponds to a non-convex strain energy of the link.

Damage parameter: In the initial state the elongation, q, is smaller than the critical value q_*, $q<q_*$, which marks the first stable branch of the diagram. When the link is monotonically elongated and the elongation reaches the critical value q_*, the link transits to the second stable branch passing through the instable region. Due to instability, the transition is characterized by an abrupt jerk-type motion that creates a significant kinetic energy of the masses connected by the link. It is remarkable that the related dynamic effects are significant independently of the rate of loading. This is a characteristic feature of a bistable structure.

The state of the link depends on the elongation history: once damaged, the link stays damaged. Therefore, if the elongation reaches the second stable branch, the unloading follows a different path. We assume that for unloading the tensile force $T_{\mathrm{uld}}(q)$ is a monotonic function with

$$T_{\mathrm{uld}}(q)\leqslant T(q).$$

The difference in the loading and unloading reflects irreversibility. This dependence is shown by a dashed line in Fig. 1(b). In general, the function $T_{\mathrm{uld}}(q)$ also depends on the maximal elongation reached during the loading; we, however, assume that

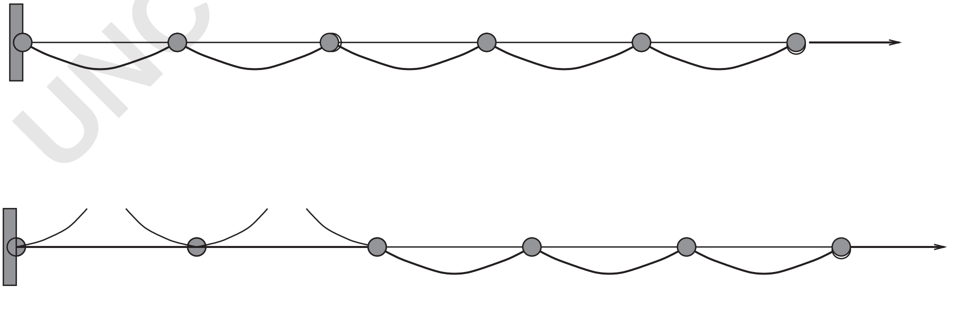

Fig. 2. Waiting link structure.

after the link reaches the second stable state the force is completely defined by the strain independently of the maximal elongation (if the link is not broken, i.e. if the maximal elongation is below q_{**}). Hence, the second-branch state is assumed brittle-elastic as well as the first one.

In the present paper, we use the simplest model of damage—the breakage. We assume that the point $q = q_*$ is a point of discontinuity where the tensile force suddenly drops to zero. Then it remains to be at zero in the gap region, $q_* < q < q_* + \mathcal{G}$, see Fig. 1(c). To account for an irreversible damage we introduce a time-dependent damage indicator $\mathcal{D}(t, q)$ which depends on the elongation history. Damage indicator is equal to zero in the beginning of the elongation and until the elongation reaches the critical value q_* first time. Then it becomes equal to one:

$$\mathcal{D}(t) = \begin{cases} 0 & \text{when } \max_{\tau \in [0,t)} q(\tau) < q_*, \\ 1 & \text{otherwise.} \end{cases} \tag{2}$$

Alternatively, one may account for a non-instant damage that corresponds to the differential equation for the accumulated damage (see Cherkaev and Zhornitzkaya, 2003):

$$\dot{\mathcal{D}}(t) = Y(q, \mathcal{D}), \quad \mathcal{D}(0) = 0, \tag{3}$$

where

$$Y(q, \mathcal{D}) = \begin{cases} W & \text{if } q > q_* \text{ and } \mathcal{D} < 1, \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

and $W \geqslant 0$ is the rate of damaging. It states that the rate of accumulation of the damage increases when the elongation $q = q(t)$ is above the threshold, q_*, and the link is not fully damaged, that is $\mathcal{D}(t) < 1$. The breakage corresponds to the limiting case, $W = \delta(t - t_*)$, where δ is the delta function and t_* is the moment when the elongation reaches the value q_*. This model with a continuous damage parameter is computationally more stable than that in Eq. (2).

The state of the link under any time-dependent loading is described as

$$T(q, \mathcal{D}) = [1 - \mathcal{D}_1(t)]T_1(q) + [1 - \mathcal{D}_2(t)]T_2(q), \tag{5}$$

where $T_1(q)$ is the dependence for the first stable region (the first branch) and $T_2(q)$ is that for the second branch with $T_2 = 0$ for $q < q_* + \mathcal{G}$. The first damage parameter $\mathcal{D} = \mathcal{D}_1$ is defined in Eq. (2), and the damage parameter $\mathcal{D} = \mathcal{D}_2$ is defined by the same relation, but with q_{**} instead of q_*. The unloading follows the first branch if $\mathcal{D}_1 = 0$ and the second one if $\mathcal{D}_1 = 1, \mathcal{D}_2 = 0$. The link is completely broken if $\mathcal{D}_2 = 1$. Recall that $T_2(q)$ is monotonic, and the unloading path smoothly continues the second branch of the loading path. This formulation in terms of the damage parameters is exploited in the computational scheme.

2.2. The waiting-link structure

A bistable bond can be realized as a "waiting link" or "waiting element" structure (see Cherkaev and Slepyan, 1995). An element of the structure, the link or the bond, consists of two generally parallel links, one straight and the other slightly curved, joined by their ends, see Fig. 2(a). The straight (basic) link resists the elongation as an elastic-brittle one. It is assumed that the longer (waiting) link does not resist until it is straightened (the bending stiffness is neglected). Then it starts to resist similarly to the basic link. This curved link is called the waiting one because it 'waits' for the right time to start to resist.

Elongation of the basic link: Assume that the material is elastic-brittle: the tensile force depends on the elongation q as:

$$T_b(t) = \begin{cases} \mu_b q & \text{if } 0<q<q_* \text{ and } q(\tau)<q_* \text{ for } \tau<t, \\ 0 & \text{otherwise,} \end{cases} \tag{6}$$

where μ_b is the modulus of the basic link. Recall that this link resists only in a limited range of elongations. It does not resist when contracted due to the loss of Euler stability under compression, and does not resist when extension is too large because it breaks.

Using the introduced damage parameter $\mathscr{D}(t) = \mathscr{D}_b$ we express the stress–strain relation in the basic link as:

$$T_b(q, \mathscr{D}_b) = [1 - \mathscr{D}_b(t)]\, \mu_b\, q\, H(q), \tag{7}$$

where the subscript b denotes a value belonging to the basic link. The energy E_b stored in this link is equal to

$$U_b(q, \mathscr{D}_b) = \left[(1 - \mathscr{D}_b)\left(\tfrac{1}{2}\mu_b q^2\right) + \mathscr{D}_b\left(\tfrac{1}{2}\mu_b q_*^2\right)\right] H(q). \tag{8}$$

Waiting link: The force–elongation dependence for the waiting link is similar to that for the basic link, but is shifted by the length $\mathscr{G} + q_*$. The force–elongation dependence $T_w(q, \mathscr{D}_w)$ corresponding to this link is

$$T_w(q, \mathscr{D}_w) = [1 - \mathscr{D}_w(t)]\mu_w(q - q_* - \mathscr{G})H(q - q_* - \mathscr{G}), \tag{9}$$

where μ_w and $\mathscr{D}_w$ are the modulus and the damage parameter of the waiting link. The damage parameter $\mathscr{D}_w$ is defined as

$$\mathscr{D}_w(t) = \begin{cases} 0 & \text{when } \max_{\tau\in[0,t)} q(\tau)<q_{**}, \\ 1 & \text{otherwise.} \end{cases} \tag{10}$$

Recall that the waiting link starts to resist when the basic link is broken, i.e. $\mathscr{G} \geqslant 0$. The whole assembly is *bistable* because the tensile force T is a sum of T_b and T_w:

$$T(q, \mathscr{D}_b, \mathscr{D}_w) = T_b(q, \mathscr{D}_b) + T_w(q, \mathscr{D}_w). \tag{11}$$

Note that to achieve the bistability, an additional (hidden) length of the waiting link has been utilized. However, this does not mean that this additional length should be large if the gap is large, because the large gap means a relatively large elongation, but not the length.

The design parameters of the waiting link chain include: the elongation parameters, $q_*, \mathscr{G}$ and q_{**}, and the modules, μ_b and μ_w. For the case when the basic and waiting links are made of the same material, we use the representation

$$\mu_b = \alpha\mu, \quad \mu_w = (1 - \alpha)\mu, \tag{12}$$

where μ is the modulus of the element of the combined thickness of the links, and α is a fraction of material that is put in the basic link. In this case α is a design parameter in optimization process. We assume that the waiting link is stronger than the basic one, namely, $\alpha < 1/2$,

$$(1-\alpha)\mu q_* = \max\ T_w > \max\ T_b = \alpha\mu q_*. \tag{13}$$

3. Quasi-static behavior

3.1. Multiple equilibria and delocalization

Equilibria: A bistable-link chain is characterized by multiple equilibria. Two locally stable states in each link correspond to the basic branch in the interval $0 \leqslant q < q_*$ and to the waiting branch of the force–elongation function in the interval $q_* + \mathcal{G} < q < q_{**}$. If the chain consists of N links, it has 2^N states of equilibrium. The total elongation of the chain q_{total} can take $N+1$ different values depending on how many links are in the first (or in the second) state:

$$q_{\text{total}}(T) = \frac{T}{\mu_b}k + \left[\frac{T}{\mu_w} + q_* + \mathcal{G}\right](N-k), \tag{14}$$

where $k = 0, \ldots, N$ is the number of undamaged links. A real state can be determined considering the loading history. This implies consideration of dynamics. Below, we address this issue accounting for the inertia of the masses in the chain. The delocalization phenomenon is analyzed using a quasi-static approximation.

Multiple reloading under monotonic elongation (Delocalization): Consider the chain with the fixed left mass, and let the right mass start to move slowly to the right. Until the tensile force in the chain reaches the critical value $T(q_* - 0)$ only the basic links resist. At some point, one of the basic links breaks and the corresponding waiting link is activated. The damaged link elongates, while other links contract to keep the total elongation reached by the chain. The tensile force in each link of the chain and hence the strain energy is decreased due to this break, however the work of the external force stays the same. The difference between the work and the strain energy is the dissipated energy. Though the connection between the thermal energy and lattice dynamics is more complex, we can view the dissipated energy as energy of oscillations which finally transfers to heat. The elongation increases further, and when the tensile force again reaches the threshold, the basic link in some other bond breaks and is replaced by the waiting link, and so on. Each break results in loss of a part of the strain energy of the chain. The process continues until the whole structure transits to the second stable branch of the force–elongation relation. Then the elongation of the waiting links reaches the critical value, q_{**}, and the chain breaks completely.

In this process, multiple periodic breaks and multiple reloading in the chain before its final rupture reflects the delocalization of large strains. If the number of links is large, this delocalization results in a dramatic increase of the total consumed energy in comparison with a chain of regular links of the same material and weight. The constraint is that the strength of the second branch, $T(q_{**} - 0)$, must be large enough to withstand the dynamic overshoot in the waiting links caused by the repeated sudden breaks of the basic links. Note, however, that the overshoot can be suppressed by internal inelastic resistances which speed up the energy transfer from the mechanical oscillations to heat (see Slepyan, 2000, 2002). A paper by Friesecke and Matthies (2002) has interesting insights as to why neglecting sinusoidal waves is justified.

3.2. Quasi-static estimation of the optimal gap

Here we estimate the work of a slowly growing external force stretching the chain. Consider the initial length of the basic link a as the length unit. In these terms, the elongation of a link q plays the role of strain; this also concerns q_*, q_{**} and $\mathcal{G}$. The non-dimensional length of the waiting link is thus $(1 + q_* + \mathcal{G}) > 1$. Consider the chain just before the break of the nth basic link when the tensile force reaches its critical value $\mu_b q_*$. At this moment the length of the chain consisting of N links is

$$L_n = (1 + q_*)(N - n + 1) + (1 + q_* + \mathcal{G} + q_*\mu_b/\mu_w)(n - 1). \tag{15}$$

Just after the nth basic link breaks the length remains the same, but the tensile force decreases. The unknown force T_n is thus defined by the equation

$$(1 + T_n/\mu_b)(N - n) + (1 + q_* + \mathcal{G} + T_n/\mu_w)n = L_n. \tag{16}$$

From these two relations we obtain

$$T_n = \left\{\left[N - n + \frac{\mu_b}{\mu_w}(n - 1)\right]q_* - \mathcal{G}\right\}\left(\frac{N - n}{\mu_b} + \frac{n}{\mu_w}\right)^{-1}. \tag{17}$$

The number of links is assumed large enough to obtain a non-negative tensile force from this expression. This condition is satisfied if for any n

$$N \geqslant n - \frac{\mu_b}{\mu_w}(n - 1) + \frac{\mathcal{G}}{q_*}. \tag{18}$$

The last inequality is valid if

$$N \geqslant \begin{cases} 1 + \mathcal{G}/q_* & \text{for} \quad \mu_b \geqslant \mu_w, \\ (1 + \mathcal{G}/q_*)\mu_w/\mu_b & \text{for} \quad \mu_w \geqslant \mu_b. \end{cases} \tag{19}$$

If this condition is satisfied the total work, A, of the external force is a sum of the work produced by extension of the chain before the break of a basic link plus the work of the final extension of the chain of waiting links:

$$A = \frac{1}{2}\sum_{n=0}^{N-1}(T_n + q_*\mu_b)(L_{n+1} - L_n) + \frac{1}{2}(T_N + T_{\max})(L_{\max} - L_N) \tag{20}$$

with

$$T_0 = 0, \quad L_0 = N, \quad T_{\max} = \mu_w(q_{**} - q_* - \mathcal{G}), \quad L_{\max} = (1 + q_{**})N. \tag{21}$$

At the same time, if the basic and waiting links are not connected with each other, the total strain energy is

$$U = \tfrac{1}{2}N[\mu_b q_*^2 + \mu_w(q_{**} - q_* - \mathcal{G})^2(1 + q_* + \mathcal{G})]. \tag{22}$$

We assume for simplicity that the basic and waiting links made of elastic material with the same modulus, $\mu_b = \mu_w = \mu$, but the waiting link is of a higher strength, $q_{**} > q_*$. In this case,

$$\begin{aligned} L_n &= (1 + q_*)(N - n + 1) + (1 + 2q_* + \mathcal{G})(n - 1), \\ T_n &= \frac{\mu}{N}[(N - 1)q_* - \mathcal{G}], \\ A &= \frac{\mu}{2}\{(1 - 1/N)[(2N - 1)q_* - \mathcal{G}](q_* + \mathcal{G}) + Nq_*^2 \\ &\quad + [q_{**} - \mathcal{G} - (q_* + \mathcal{G})/N][N(q_{**} - 2q_* - \mathcal{G}) + q_* + \mathcal{G}]\}, \\ U &= \frac{\mu}{2}N[q_*^2 + (q_{**} - q_* - \mathcal{G})^2(1 + q_* + \mathcal{G})]. \end{aligned} \tag{23}$$

In addition, the material weight, Q, is assumed to be proportional to the total length of the links, that is

$$Q = C_0\,(2 + q_* + \mathcal{G})N, \tag{24}$$

where C_0 is a constant.

The optimal value of the gap $\mathcal{G} = \mathcal{G}_{\text{opt}}$ between the branches can now be found maximizing the ratio A/Q. Recall that we consider a chain with a large number of links. Both the basic and waiting link materials are rigid; they are of the same modulus, but the critical elongation of the basic link $q_{**} - q_* - \mathcal{G}$, is larger then the critical elongation of the waiting link. So, it is assumed that

$$N \gg 1, \quad q_* \ll 1, \quad q_{**} - q_* - \mathcal{G} = C_1 q_*, \tag{25}$$

where $C_1 > 1$ is a moderate constant, say, $C_1 = 1.5$ (see, for example, Fig. 1c). In this case, $\mathcal{G}_{\text{opt}} \gg q_*$, and the considered values can be expressed as

$$\begin{aligned} A &\sim \frac{\mu}{2}(2Nq_* - \mathcal{G})\mathcal{G}, \\ Q &\sim C_0(2 + \mathcal{G})N, \\ U &\sim \frac{\mu}{2}Nq_*^2[1 + C_1^2(1 + \mathcal{G})] < \frac{\mu}{2}\,Nq_*^2 C_1^2(2 + \mathcal{G})]. \end{aligned} \tag{26}$$

Now, up to the $\mathcal{G}$-independent multiplier

$$\frac{A}{Q} \sim \frac{(2Nq_* - \mathcal{G})\mathcal{G}}{2 + \mathcal{G}}, \tag{27}$$

and the optimal value of the gap is

$$\mathcal{G}_{\text{opt}} \sim 2\left(\sqrt{1 + q_* N} - 1\right). \tag{28}$$

For a large N, this value of the gap satisfies the condition in (19). In terms of dimensional distances, i.e. for an arbitrary value of a, the optimal gap is

$$\mathscr{G}_{\text{opt}} \sim 2a(\sqrt{1+\varepsilon_* N} - 1), \quad \varepsilon_* = q_*/a, \tag{29}$$

where ε_* is the critical strain of the basic link. For the work A and the strain energy U in the chain with the optimal gap and $N \gg 1$ we obtain

$$\begin{aligned} A &\sim 2\mu\sqrt{1+q_* N}(\sqrt{1+q_* N} - 1)^2, \\ U &< \mu C_1^2 N q_*^2 \sqrt{1+q_* N}. \end{aligned} \tag{30}$$

Consider for example the optimal-gap chain with $q_* = 0.01, N = 300, \mathscr{G} = \mathscr{G}_{\text{opt}} = 2$. Under the quasi-static elongation of the chain, q_{chain}, the elongation–tensile force diagram for the chain is presented in Fig. 3. The dependence is valid until all 300 basic links are broken; then the tensile force monotonically increases up to the waiting link critical value and becomes zero when it breaks.

We compare the energy consumption in the optimal bistable-link chain and in the conventional chain consisting of regular links. The conventional chain breaks when one of the links is broken, hence the energy consumption is limited by the strain energy corresponding to the breakage of a regular link. To obtain a more transparent result we use a larger estimate of the strain energy U as the last expression in Eq. (26), thus decreasing the optimal efficiency $\mathscr{E}$ of the bistable structure. If the gap is as in Eq. (28), for $N \gg 1$ the efficiency $\mathscr{E}$ is an increasing function of N,

$$\mathscr{E} = \frac{A}{U} \approx \frac{(\sqrt{1+q_* N} - 1)^2}{q_*^2 N} \sim \begin{cases} N/4 & (q_* N \ll 1), \\ 1/q_* & (q_* N \to \infty). \end{cases} \tag{31}$$

This ratio can be made very large. For example, if the basic link limiting strain is $q_* = 5 \times 10^{-3}$ and $N = 32$, the efficiency $\mathscr{E} \approx 8$, while it approaches a much higher value, $\mathscr{E} \approx 200$, in the case of a long chain, $N \gg 200$.

Thus, if the number of links N is large and the chain is properly structured, the energy, A, equal to the work of the cyclic reloading of the bistable-link chain under a monotonic extension, can be much greater than the limiting strain energy of the basic and waiting links, $A \gg U$. This is the advantage of the considered structure. The difference $A - U$ is the energy corresponding to the dynamic effects; finally this

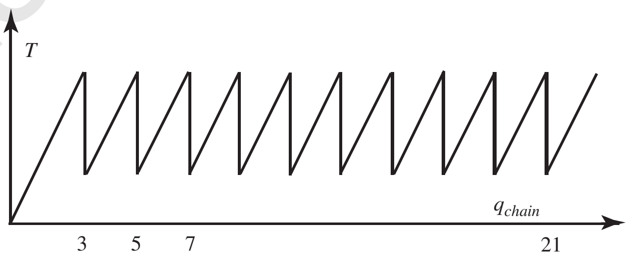

Fig. 3. The tensile force under elongation of the chain: $q_* = 0.01, N = 300, \mathscr{G} = \mathscr{G}_{\text{opt}} = 2$.

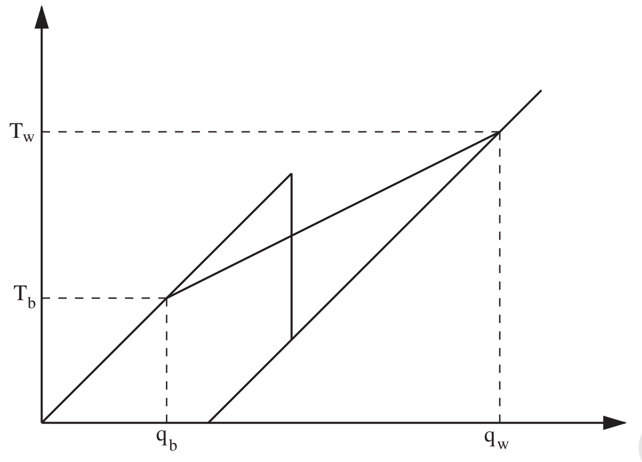

Fig. 4. Transition from one stable branch to the other occurs at the critical elongation q_b. The corresponding tensile force behind the transition front of the transition is T_b, the force ahead of the transition is T_w.

energy is transferred to heat. The dynamic effects are described in more details based on the dynamic formulation in Section 4. The quasi-static approach allows us to determine only the total 'dynamic' energy under the condition of a low extension rate. Recall that we consider here a brittle linearly elastic material. The fracture energy itself was not taken into account as well as the energy of possible plastic deformations. Both these factors will further increase the efficiency because plastic necks and ruptures are distributed (repeated N times) in the bistable-link chain, contrary to one neck and one rupture in a non-structured chain.

Finally, it is remarkable that the quasi-static behavior of the elastic-brittle bistable chain resembles that of an ideally plastic rod (see Fig. 4). In both cases, there is almost constant resistance to a continued elongation and the irreversibility of strain. The main difference between natural plasticity and the artificial ideal plasticity of a waiting-link chain is the discussed above delocalization. If the bistable-link chain is made of a plastic material, large plastic strains (necks) in the basic link are developed in each link, while a single plastic neck is developed in a conventional plastic rod.

4. Dynamic behavior

We now use the dynamic formulation to determine the process of the transition from the low-elongation basic state to the high-elongation state. In dynamics, this process is characterized by a system of waves which includes long step waves and oscillating dissipative waves. Recall that, due to instabilities caused by non-monotonic character of the force–elongation relation, the dynamic effects are considerable even in the case of an arbitrary low, quasi-static loading rate. The above quasi-static considerations allowed us to estimate the total dissipated energy, but not the wave structure.

The dynamics of the chain is described by a system of difference–differential equations with respect to the displacements, u_m, of the masses and elongations, $q_m = u_m - u_{m-1}$. It has the form

$$M\ddot{u}_m + \gamma\dot{u}_m = T(q_{m+1}, \mathscr{D}^b_{m+1}, \mathscr{D}^w_{m+1}) - T(q_m, \mathscr{D}^b_m, \mathscr{D}^w_m). \tag{32}$$

Here m is the number of the mass, γ is the coefficient of viscosity introduced to stabilize numerical simulations, and $\mathscr{D}^b_m$ and $\mathscr{D}^w_m$ are the damage parameters of the basic and waiting links in mth link (these parameters are defined in Section 3.1). In the below numerical simulations, the left mass $m = 0$ is assumed unmoving, while the right one, $m = 32$, is subjected to an impact by a rigid mass.

Generally, a subcritical step wave is found propagating ahead of the transition wave. In a finite chain, when the step wave reaches the opposite (fixed) end of the chain, it reflects and its magnitude increases. This increase can initiate a contra-directional transition wave moving towards the initial impact point. Transition wave can be also initiated when two reflected elastic waves meet. Not that such a reflection phenomenon can be observed, for example, in a cylindrical shell which can lose the stability under an axial impact; it is revealed also in numerical simulations of the chain dynamics shown below. Before the description of the numerical results we estimate the role of the gap between the branches of the force–elongation relation in dynamics.

4.1. Dynamic estimation of the gap role

We estimate here how the dynamic dissipation depends on the gap between the stable branches of the constitutive relation. Consider an elastic bistable-link chain under a time-independent external action given as a longitudinal force applied to the end chain mass or as a given speed of this mass. Using particle velocities $\dot{u}_m(t)$ and elongations $q_m(t)$, we represent the solution as a sum of a long-wave approximation (as the step wave) and oscillating structure-associated waves. The first term corresponds to an 'equivalent' continuous material rod (as the 'macrolevel' description of the chain), while the second one can be referred to the microlevel. Propagation of the step wave in a bistable (with a non-convex energy) waveguide is accompanied by an energy release. In the case of a continuous material, this energy is lost. Considering the discrete elastic chain where there is no energy loss one can see that the energy released from the propagation of the step wave goes to the excitation of the microlevel oscillations. This transformation of the energy is here called the dissipation.

In principle, the dissipation can be determined by examining the step wave in a continuous material rod. The difficulty is that the transition point (or the critical elongation) cannot be exactly determined in the framework of a continuous model because the microlevel oscillations which facilitate overcoming the barrier are not taken into account in this model. This drawback, however, can be neglected in a rough estimation of the gap role.

Let the external force and the tensile force, $T_w = \mu_w(q_w - q_* - \theta)$, in the link behind the transition front be given as a constant. The step wave ahead of the front is characterized by the tensile force, $T_b = \mu_b q_b$, where q_b is the unknown transition point: $0 < q_b \leqslant q_*$ (see Fig. 4). We now use the mass and momentum conservation laws. Note that these laws are valid for the long-wave approximation, as well as for the complete solution for the chain, since the oscillating waves give no contribution for a 'long-term' (macrolevel) consideration. Specifically, we assume that the external force is applied at the right end of the chain, and the front moves to the left. Then,

$$v_w - v_b = \frac{q_w - q_*}{a} V,$$

$$T_w - T_b = \frac{M}{a}(v_w - v_b)V, \tag{33}$$

where v_w and v_b are the particle velocities behind and ahead of the transition front, respectively, V is the front speed, and M/a is the mass per unit length (in this section we do not assume $a = 1$). It follows that

$$V = \sqrt{\frac{a^2(T_w - T_b)}{M(q_w - q_b)}}. \tag{34}$$

Further, the particle velocity in the wave ahead of the front

$$v_b = \frac{aT_b}{Mc}, \quad c = \sqrt{\frac{a^2\mu_b}{M}}, \tag{35}$$

where c is the sound speed of the long-wave in the initial-phase chain. We obtain

$$v_w = \frac{aT_b}{Mc} + \frac{a(T_w - T_b)}{MV}. \tag{36}$$

For a large gap $\mathscr{G}$, the first term in Eq. (36) becomes negligible in comparison with the second one, and the transition front speed and the particle velocity behind the front can be approximated as

$$V \sim \sqrt{\frac{a^2(T_w - T_b)}{M\mathscr{G}}},$$

$$v_w \sim \sqrt{\frac{(T_w - T_b)\mathscr{G}}{M}}. \tag{37}$$

At the same time, the energy release per unit length G, is asymptotically equal to the area under the transition line between the points (q_b, T_b) and (q_w, T_w) in Fig. 4

$$G \sim \tfrac{1}{2}(T_w + T_b)\mathscr{G}, \tag{38}$$

and the dissipation D per unit time is

$$D = GV \sim \tfrac{1}{2}(T_w + T_b)\sqrt{\frac{a^2(T_w - T_b)\mathscr{G}}{M}}. \tag{39}$$

We thus come to the following results. With an increase of the gap, $\mathscr{G}$, between the stable branches of the constitutive relation (under the same remaining conditions), the transition front speed decreases as $1/\sqrt{\mathscr{G}}$, while the particle velocity behind the front increases as $\sqrt{\mathscr{G}}$. This shows that the larger is the gap, the higher is the speed of an impact that can be sustained by the link of a given critical tensile force. The total dissipation is asymptotically proportional to gap (the dissipation per unit time is proportional to $\sqrt{\mathscr{G}}$). Recall that the gap is large if it is much greater than the critical elongation, q_*; at the same time it can be much smaller the inter-particle distance, a.

Finally, the dissipation per unit length of the chain is asymptotically proportional to $G/(2+\mathscr{G})$ (recall that the waiting link is slightly longer than the basic one). In these calculations, we did not take into account an increase of the maximal tensile force in the second phase caused by the structure-associated oscillations. These oscillations can be suppressed by internal inelastic resistance which speeds up the transfer of the mechanical energy into heat. With an inelastic material in mind, the energy of the oscillations existing in the elastic model can be viewed as the energy transferred to heat that does not influence the material strength so much as the additional macrolevel stresses. Note that the similar phenomenon can arise due to internal friction when the structure is comprised of a number of such chains.

5. Results of numerical simulations

We now discuss numerical simulations of the transition waves in the chain. A rested chain of N bistable links of the diagram shown in Fig. 1(c) is impacted by a large mass M_0 moving with initial velocity v_0. This means that the end mass, $m = N$, of the chain is equal to $M + M_0$ and has initial velocity

$$\dot{u}_N(0) = \frac{v_0 M_0}{M_0 + M} \tag{40}$$

directed to the right. We start the simulations using an impact mass $M_0 = 120$; this mass is sustained by the conventional chain without waiting elements. The mass $M_0 = 125$ breaks the conventional chain. Then we model a chain with the waiting links impacted with the same mass and demonstrate that it sustains the impact. Increasing the loading mass and adjusting parameters of the links we find the configurations of the waiting link structures which support an impact of a large mass without breaking. With various constraints for the parameters of the structure, the waiting link chain sustains an impact of the mass several times larger than the conventional chain with non-structured links does.

The equations: The dynamics of the chain is described by Eq. (32), condition for the mass in the root $u_0 = 0$, the equation for the loaded mass

$$(M_0 + M)\ddot{u}_N = -T(q_N, \mathscr{D}_N^b, \mathscr{D}_N^w), \tag{41}$$

and Eq. (4) for the damage indicators $\mathscr{D}_m^b$ and $\mathscr{D}_m^w$ where $m = 1, \ldots, N$. Recall that the elongation q_m is expressed through the displacements u_m of the masses as $q_m = u_m - u_{m-1}$ and the tensile force is defined in Eqs. (7) and (9).

The initial conditions are

$$u_m = 0, \quad m = 0, \ldots, N \tag{42}$$

$$\dot{u}_m = 0, \quad m = 1, \ldots, N-1, \quad \dot{u}_N(0) = \frac{v_0 M_0}{M_0 + M}. \tag{43}$$

In numerical simulation, we assume the following:

1. The number N of elements is $N = 32$.
2. The impacting mass M_0 is much larger than the mass M of a chain particle.
3. The speed v_0 of the impacting mass is much below the sound speed (the long wave speed) in the undamaged chain: $v_0 \ll c = a\sqrt{\mu/M}$.
4. There exists a gap $\mathcal{G}$ in the force-elongation relation, so that the basic link is completely broken before the waiting link is activated.

For numerical simulations we use non-dimensional values

$$\hat{u}_m = \frac{u_m}{a}, \quad \hat{q}_m = \frac{q_m}{a}, \quad \hat{q}_* = \frac{q_*}{a}, \quad \hat{q}_{**} = \frac{q_{**}}{a}, \quad \hat{\mathcal{G}} = \frac{\mathcal{G}}{a}. \tag{44}$$

In these terms, the initial distance between neighboring masses of the chain is equal to one (as $a = 1$).

Results: The computer experiments of the dynamics of the chain simulated using MATLAB, show that for different constraints for the structural parameters, the structures with waiting links sustain the loading masses several times greater than the conventional structures. We stress that this result is achieved using the same amount of the same material, the only difference is the morphology of the structure.

The first series of experiments investigates the influence of the relative thickness α (see (Fig. 12)) of the basic link on the strength of the chain. It is assumed that the element is characterized by an elongation–force function shown in Fig. 8 with $\mu = 5, q_* = 0.2$, the total weight of the elements is constant; the material is only redistributed between the basic and the waiting links. The parameter α describes a fraction of material in the bond which is put in the basic link. The value $\alpha = 1$ corresponds to the conventional elastic-brittle element (absence of the waiting link), since the thickness of the waiting link is zero. The value $\alpha < 1/2$ describes the bond with waiting link stronger than the basic link, and $\alpha > 1/2$ characterizes the chain in which the waiting links are weaker than the basic ones; this type of structure does not delocalize the damage. Different values of the loading mass equal to 120 and 125 correspond to Figs. 5 and 6, respectively. One can see that the chain sustains the impact in the first case and is broken in the second case. Trajectories of the knots in the chain with waiting elements, $\alpha = 0.25$, $\mathcal{G} = 0.3$, impacted by the mass $M_0 = 350$ are shown in Fig. 7. The chain sustains the impact. The elastic wave generates two waves of a partial damage at both ends of the chain: at the impact point and near the wall where the magnitude of the elastic wave is maximal. Intensive oscillations of the masses after all links are partially broken lead to dissipation of the energy of the impact. Next, we optimize the value of α maximizing the loading mass M_0 that the chain can sustain. We find, that the choice $\alpha = 0.17$ allows us to increase the limits of applied mass more than three times (see Fig. 9). We stress that this result is achieved

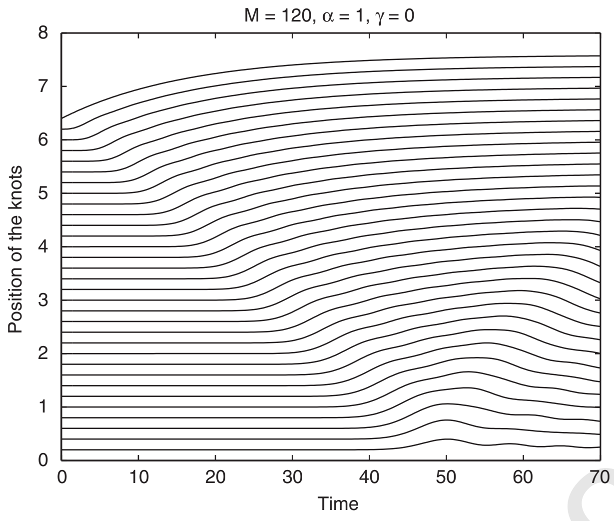

Fig. 5. The motion of the knots of the chain without waiting elements impacted by the mass $M_0 = 120$. The chain sustains the impact.

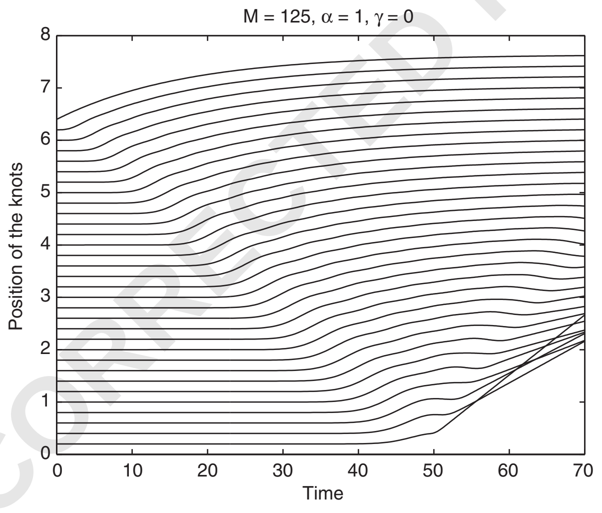

Fig. 6. The motion of the knots in the chain without waiting elements impacted by the mass $M_0 = 125$. The chain is broken. After the break, the bottom masses move up without resistance. However, possible collisions with neighboring masses are not taken into account in the calculations. As a result, the trajectories crossing occurs.

using the same amount of the same material, the only difference is the morphology of the structure.

Modification of the model (Dissipation): The dynamics of the chain with partially broken links indicates that the masses are involved in a chaotic motion after the basic

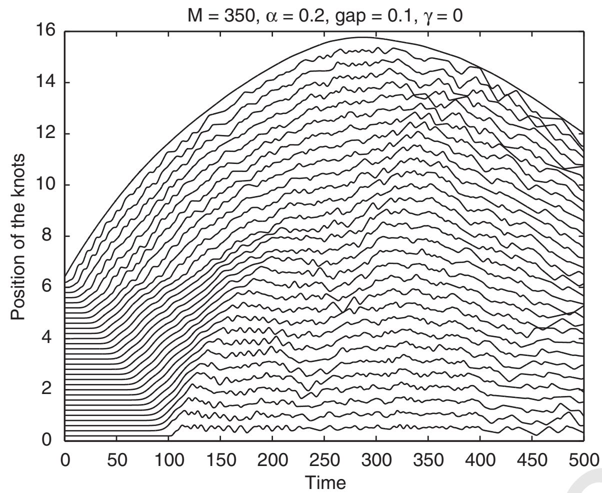

Fig. 7. Trajectories of the knots in the chain with waiting elements, $\alpha = 0.2$, $\mathcal{G} = 0.1$, impacted by the mass $M_0 = 350$. The chain sustains the impact. Observe the elastic wave and two waves of a partial damage originated at both ends of the chain where the magnitude of the elastic wave is maximal. Notice intensive oscillations of the masses after all links are partially broken.

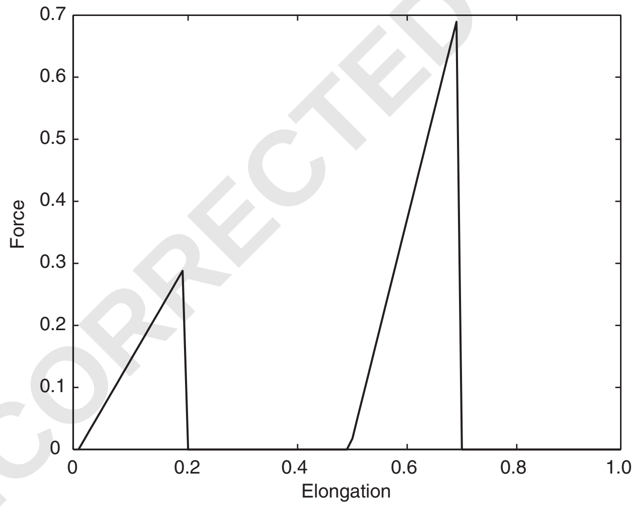

Fig. 8. Example of the constitutive relation corresponding to the links in the chain with waiting elements used in simulations. Shown constitutive relation is described by the following parameters: $\mu = 5$, $\alpha = 0.3$, $\mathcal{G} = 0.25$.

links are broken. It is exactly the chaotic motion phase that leads to the final breakage of the chain due to excessive elongation of a link. Observing this phenomenon, we may question the adequacy of the model that does not take into account a small dissipation that is always present in mechanical systems; this

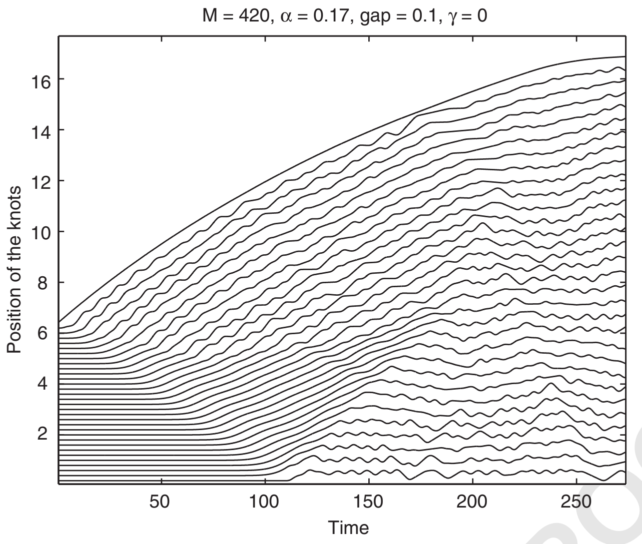

Fig. 9. Trajectories of the knots in the chain with waiting elements: $\alpha = 0.17$, $\mathscr{G} = 0.1$, $M_0 = 420$. The chain sustains the impact.

dissipation would reduce the chaotic motion, especially in a long time range. Accounting for this effect, we introduce a small dissipation into numerical scheme. "Small" means that it practically does not influence the initial wave of the damage but reduces the chaotic afterward motion. The small dissipation would have practically no influence on the behavior of chains made of a stable material, if the excitation is slow. However, the chain of unstable materials excites intensive fast-propagating waves no matter how low is the impact speed (see Balk et al., 2001). To avoid chaotic motion after phase transition, we introduce a small dissipation $\gamma = 0.03$. As shown in Figs. 10 and 11 this dissipation suppresses oscillations but does not prevent the chain from breaking.

With the assumed gap $\mathscr{G} = 0.55$, we investigated dependence of the strength of the chain on the parameter α describing the fraction of material put in the basic damageable links. Numerically estimated velocity of the impacting mass $M_0 = 900$ at the moment when the chain breaks is shown in Fig. 12. The value $\alpha = 0.245$ corresponding to the minimum of velocity actually allows the chain to sustain the damage. The behavior of the chain with waiting links, $\alpha = 0.245$, under impact of the mass equal to 900 units is shown in Fig. 13. The chain sustains the extension demonstrating 4.5 times greater efficiency comparing with the behavior of the conventional chain, see Figs. 10 and 11.

Large dissipation: Large dissipation significantly increases the ability of the chain to resist the impact. In the model with large dissipation, $\gamma = 1$, the conventional chain was impacted by a slow moving mass with the initial velocity 0.01. The behavior of the chain impacted by the masses $M_0 = 70,000$ and 80,000 is shown in

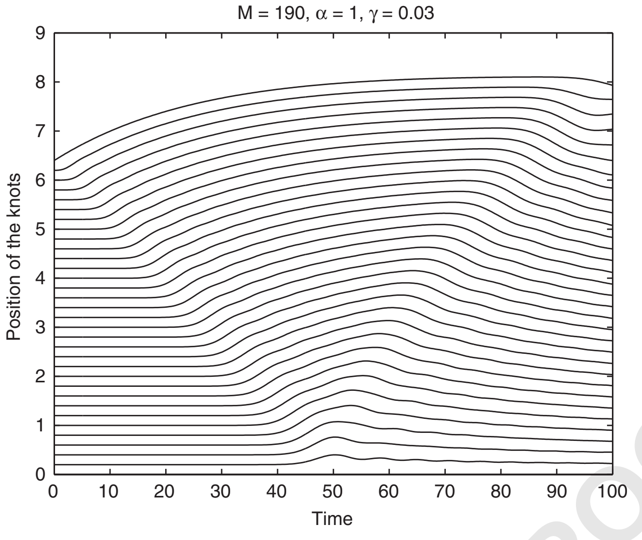

Fig. 10. The conventional chain without waiting links with small dissipation, $\gamma = 0.03$, sustains the impact of the mass of 190 units. The dissipation suppresses oscillations and allows the chain to support higher load.

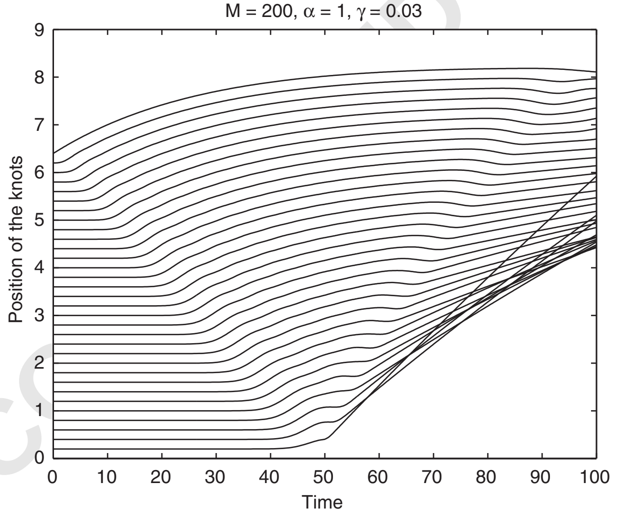

Fig. 11. With dissipation $\gamma = 0.03$, the chain without waiting links breaks being impacted by the mass of 200 units. The dissipation suppresses oscillations but does not prevent the chain from breaking.

Figs. 14 and 15. The dissipation suppresses the oscillations, and the chain resists to the impact in the first case; however, it is broken in the second case.

The wave of phase transition in the model with waiting links is presented in Figs. 16 and 17. The increase of the dissipation coefficient allows for more orderly

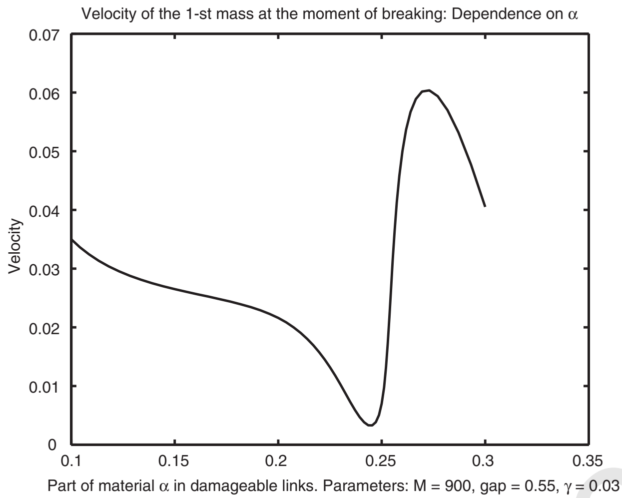

Fig. 12. Velocity of the large mass, $M_0 = 900$, at the moment of break of the chain for different values of the parameter α modelling the fraction of the material in the basic damageable links. The model includes a small dissipation, $\gamma = 0.03$, the parameters of the constitutive function are $\mu = 5$, $\mathscr{G} = 0.55$.

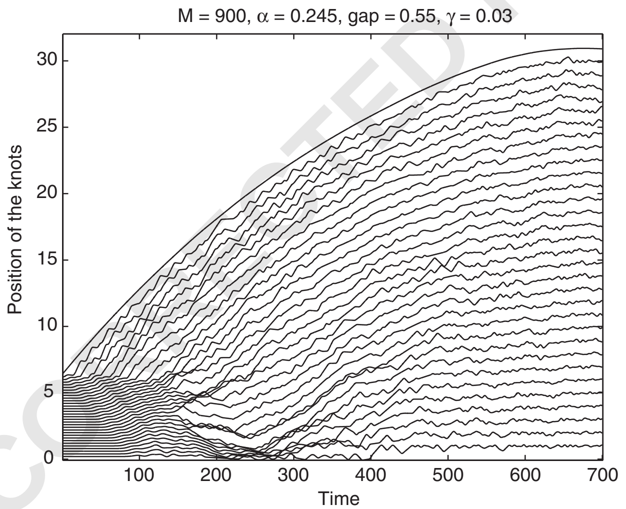

Fig. 13. Trajectories of the knots of the chain with waiting elements, impacted by the mass $M_0 = 900$. The chain sustains the extension. The model includes a small dissipation, $\gamma = 0.03$, the parameters of the constitutive function are $\mu = 5, \alpha = 0.245, \mathscr{G} = 0.55$.

transition since the propagating and reflected waves are suppressed. The viscous-elastic wave originated at the point of impact of the chain with the mass $M_0 = 250,000$ propagates to the root of the chain, reaches the far end and then reflects; this results in a partial damage of the second to the root link. This partial damage dissipates some part of the energy. Eventually partial damage of every link in the

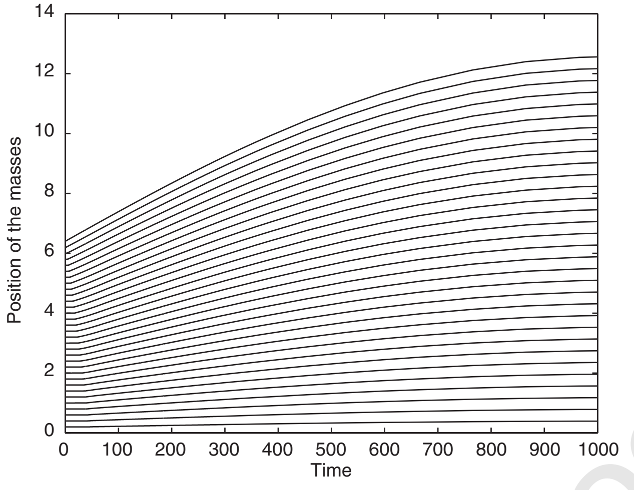

Fig. 14. With large dissipation, $\gamma = 1$, the chain impacted by a slow moving mass $M = 70,000$ with the initial velocity 0.01. The chain sustains the impact; the dissipation suppresses the oscillations.

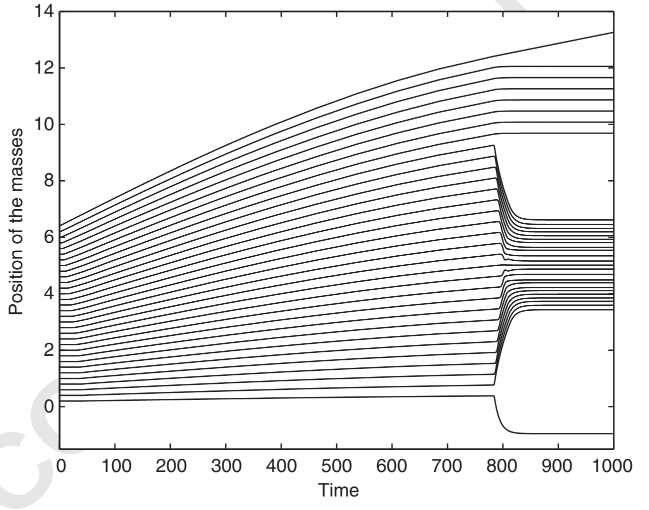

Fig. 15. The same chain without links breaks if it is impacted by a mass $M = 80,000$.

chain leads to dissipation of all energy of the impact, but the chain keeps its integrity. This can be compared with the chain impacted by the mass $M_0 = 80,000$, shown in Fig. 15, when in a similar situation, the damage of the second to the root link leads to complete damage of the chain.

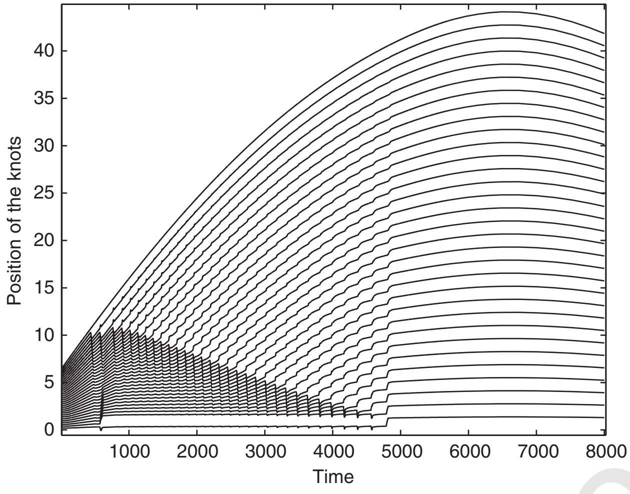

Fig. 16. Trajectories of the knots in the chain with waiting elements, $\alpha = 0.3$, and with a large dissipation, $\gamma = 1$, impacted by a slow moving mass $M = 250,000$ with the initial velocity 0.01. The chain sustains the impact; the dissipation suppresses the oscillations. Observe the wave of partial breakage that propagates starting from the impact point. The closest to the root link experiences a partial damage because of the viscous-elastic wave.

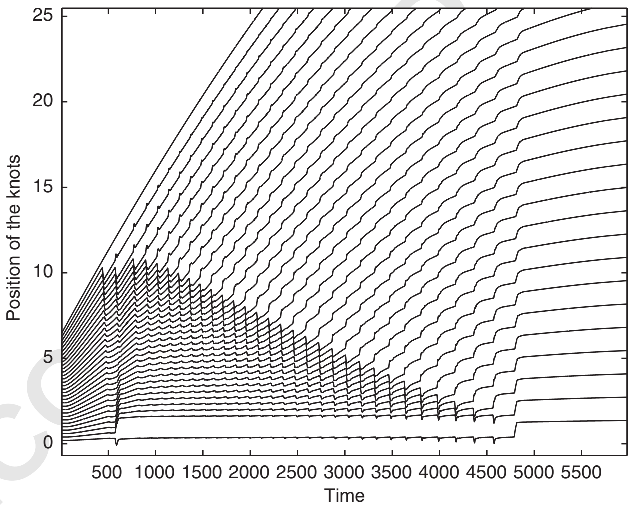

Fig. 17. Propagation of the wave of partial damage. Magnification of the detail of the previous picture.

6. Discussion

The results show a dramatic increase of the strength of the chain with waiting elements compared to the conventional design. The dissipation further increases the effect. By delocalization of partial damage these specially structured chains are able to absorb and dissipate great amounts of energy and sustain impacts of masses several times larger than regular chains of non-structured material.

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