MATH 1030: Midterm 2 Practice Exam

The following are practice problems for the second exam.

1. Linus has a savings account that compounds monthly at an APR of 5.32%.

   (a) Find the APY of Linus' account.

   By definition, the APY is the annual percent yeild, or the amount earned each year as a percent. If Linus starts out with \$1000 for example, then in one year, he will have

   $$A = 1000 \cdot \left(1 + \tfrac{.0532}{12}\right)^{12\cdot 1} = \$1053.26$$

   Thus, in one year, the Linus earns \$53.26. As a percent of the original amount invested, he has earned $\frac{\$53.26}{\$1000} = .0532 = 5.32\%$. The APY of the account is then 5.32%.

   (b) If Linus deposits \$350 per month for 18 years, how much will he have after the 18 years? How much of that will have been earned in interest?

   Linus is making multiple deposits (one each month), so we need to use the savings plan formula. The formula says the amount in the account after 18 years is

   $$A = PMT\frac{\left(1+\frac{APR}{n}\right)^{nY} - 1}{\frac{APR}{n}} = \$350\frac{\left(1+\frac{.0532}{12}\right)^{12\cdot 18} - 1}{\frac{.0532}{12}} = \$126,310$$

   To calculate the amount that was earned in interest, we first calculate how much Linus deposited. He deposited \$350 each month for 18 years. In total he deposited $\$350\cdot 12\cdot 18 = \$75,600$. The rest of the money in the account must have been earned in interest, so he earned $\$126,310 - \$75,600 = \$50,710$.

   (c) If Linus wants to have \$125,000 10 years from now, how much should he deposit each month for the 10 years?

   Now we are asked how much Linus should put in his account every month in order to have \$125,000 ten years from now. In part (b), Linus saved \$350 per month for 18 years and ended up with just a little more than \$125,000. So we should expect that if he were to save \$350 per month for just 10 years, he would not have enough money. Then whatever our answer is, it should be larger than \$350. We will use this as a check for our calculations. Now, he is still saving money each month, so we need to use the savings plan formula. This time, we have a final account balance in mind (\$125,000) and we need to find what monthly payments Linus must make in order to meet that goal. So we have to solve

   $$\$125,000 = PMT\frac{\left(1+\frac{.0532}{12}\right)^{12\cdot 10} - 1}{\frac{.0532}{12}}$$

   for PMT. Doing this, we get $PMT = \$791.29$ per month. This is larger than \$350 per month, just as expected.

2. Pauline's bank offers her a loan that compounds monthly at an APR of 7.87%.

   (a) If Pauline takes out a 9-year loan for \$48,000, how much will her monthly payment be? How much will she pay over the course of the 9 years?

We are supposed to calculate Pauline's monthly payments towards a loan, so we should use the loan payment formula. We are given the APR, the frequency of compounding, the term of the loan, and the initial principal. Thus, we have

$$PMT = P\frac{\frac{APR}{n}}{1-\left(1+\frac{APR}{n}\right)^{-n\cdot Y}} = \$48,000\frac{\frac{.0787}{12}}{1-\left(1+\frac{.0787}{12}\right)^{-12\cdot 9}} = \$621.67$$

So Pauline must make payments of \$621.67 each month.

(b) If Pauline can afford to pay \$800 per month for 9 years towards a loan, how much can she afford to borrow?

Now Pauline says she can afford monthly payments of \$800 towards her loan (which has the same APR and a term of 9 years). We want to find how much money she can borrow so that her payments will be \$800 each month. We can use our answer from part (a) as a check on our answer. In part (a), she borrowed \$48,000 and her loan payments were less than \$800 per month, so she can definitely afford to borrow more than \$48,000. Now let's calculate. The loan payment formula gives

$$PMT = P\frac{\frac{APR}{n}}{1-\left(1+\frac{APR}{n}\right)^{-n\cdot Y}} \qquad \Longrightarrow \qquad \$800 = P\frac{\frac{.0787}{12}}{1-\left(1+\frac{.0787}{12}\right)^{-12\cdot 9}}$$

We can compute the number on the right, and solve for P. We get

$$\$800 = P\cdot 0.012951 \qquad \Longrightarrow \qquad \$61,769$$

So Pauline can afford to borrow \$61,769.

3. Willie's new bank account compounds continuously with an APR of 6.2%.

(a) If Willie wants to have \$150,000 in 10 years so he can put a down payment on a house, how much should he put into the account now?

Here, we have to extract the relevant information from the problem. What Willie intends to do with the money is not important. What is important is that he is making a *one-time* deposit right now. He wants to have \$150,000 in ten years. Since the account compounds continuously, we must use the compound interest formula. We must find out how much to deposit now (P) in order to have the required amount in 10 years. Therefore, the equation that describes the situation is

$$A = Pe^{APR\cdot Y} \qquad \Longrightarrow \qquad \$150,000 = Pe^{.062\cdot 10}$$

Solving this for P, we get $P = \$80,691$.

(b) Willie decides instead that he wants to retire with this account. He would like to have \$60,000 a year to live off of. How much does he need in his account to live off of the interest alone?

Willie intends to live off the interest alone, and he needs \$60,000 per year for expenses. In other words, he needs enough money in the account to generate \$60,000 in interest every year. Then we should be using the compounding interest fomula (continuously compounding because the account is continuously compounding), with $Y = 1$ and the given APR. We need to find a value of P so that one year later the amount in the account (A) is \$60,000 more than he started with. This means $A = P + \$60,000$. Plugging this into the continuously compounding interest formula, we get

$$\begin{aligned} & P + \$60,000 = Pe^{.062 \cdot 1} = 1.0639P \\ \Longrightarrow \quad & \$60,000 = 1.0639P - P = P(1.0639 - 1) = .0639P \\ \Longrightarrow \quad & P = \$938,052 \end{aligned}$$

Evidently, if Willie has \$938,052 in his account, then he will earn \$60,000 in interest the first year.

4. The number of ants on Harold's ant farm is doubling every 7 months.

(a) Find the time it takes for the number of ants to triple.

Before we do any calculations, let's get an estimate that we will later use to check our answer. We want to know how long it will take for the population to triple. In 7 months the population will have doubled, so it certainly takes more than 7 months for the population to triple. In 7 more months (14 months total) the population will have doubled a second time, so the original population has been multiplied by 4. Then the amount of time for the population to triple should be more than 7 but less than 14 months. We can model the number of ants on Harolds ant farm by $Q = Q_0 \cdot 2^{t/7}$ where t is measured in months. For the purposes of this specific question, we can assume that Harold starts with 100 ants. We then need to find the amount of time it takes until he has 300 ants. (This assumption is justified because the ant population is growing at a constant *relative* rate. So the amount of time it takes for the population to double (or triple) doesn't depend on the number of ants we start with) In terms of the given equation, this boils down to solving

$$300 = 100 \cdot 2^{t/7} \qquad \Longrightarrow \qquad \frac{300}{100} = 3 = 2^{t/7}$$

Taking logarithms of both sides, then solving for t, we get

$$\log 3 = \log 2^{t/7} = \frac{t}{7} \log 2 \qquad \Longrightarrow \qquad t = \frac{7 \log 3}{\log 2} = 11.09$$

So it takes 11.09 months for the population to triple, which agrees with our prediction.

(b) If there were 1,250 ants two months ago, how many ants are on Harold's farm now?

Let's say there are Q_0 ants on Harold's farm now. Then 2 months ago corresponds to $t = -2$, so the statement says that

$$Q_0 \cdot 2^{-2/7} = 1,250$$

We can solve this for the number of ants right now to get $Q_0 = \frac{1,250}{2^{-2/7}} = 1,524$ ants.

(c) If there were 1,250 ants two months ago, how many months ago were there 200 ants?

Using our answer from part (b), we know that if there were 1,250 ants 2 months ago then the number of ants at any time, t, is given by $Q = 1524 \cdot 2^{t/7}$. We want to find a time so that the number of ants is $Q = 200$, so we need to solve the equation

$$200 = 1,524 \cdot 2^{t/7} \qquad \Longrightarrow \qquad \frac{200}{1,524} = .1312 = 2^{t/7}$$

Taking logs of both sides, we get

$$\log .1312 = \log 2^{t/7} = \frac{t \log 2}{7} \qquad \Longrightarrow \qquad t = \frac{7 \log .1312}{\log 2} = -20.51$$

Negative times correspond to the past, so 20.51 months ago, there were only 200 ants.

5. The frequency of ghost sightings is decreasing at a rate of 8.2% per year.

   (a) Find the number of years it takes for the frequency of ghost sightings to be cut in half.

   As before, we can assume there were 100 ghost sightings this year. The number of ghost sightings each year is then given by $Q = 100(1 - .082)^t$, and we need to find how many years until there are only 50 sightings. We get the equation

   $$50 = 100(1 - .082)^t \qquad \Longrightarrow \qquad \frac{50}{100} = 0.5 = .918^t$$

   Taking logarithms and solving for t as in previous problems we get $t = 8.10$ years.

   (b) If there are 88 ghost sightings per month this year, how many ghost sightings per month will there be in 11 years? How many ghost sightings per month were there 5 years ago?

   The frequency of ghost sightings is decreasing at the previously stated rate. This question is tricky because we have to be careful about the units that time is measured in. If there are 88 ghost sightings per month this year, the frequency of ghost sightings (per month) at a time, t, is given by $Q = 88(1 - .082)^t = 88 \cdot .918^t$. Is t measured in months or years? Recall that the units of t are always the same as the units of time in the relative growth rate! Therefore t is measured in years. In 11 years, we have

   $$Q = 88 \cdot .918^1 1 = 34$$

   So eleven years from now, there will be 34 ghost sightings per month. Five years ago, there were $Q = 88 \cdot .918^{-5} = 135$ ghost sightings per month.

   (c) If there are 88 ghost sightings per month this year, when will there only be 2 ghost sightings per month? When were there 500 ghost sightings per month?

   We have to find the time when the frequency of ghost sightings is 2 per month. Using our exponential model from (b), we have to solve for t in the equation

   $$2 = 88 \cdot .918^t \qquad \Longrightarrow \qquad t = \frac{\log(2/88)}{\log 0,918} = 44 \,\text{years}$$

   To find when there were 500 ghost sightings per month, we solve a similar equation with $Q = 500$. We get

   $$500 = 88 \cdot .918^t \qquad \Longrightarrow \qquad t = \frac{\log(500/88)}{\log 0.918} = -20.3 \,\text{years}$$

   So 20.3 years ago, there were 500 ghost sightings per month.

6. The number of fruit flies at Eleanor's compost is increasing at a rate of 2.1% per day.

(a) How often does the number of fruit flies double? How often does the number triple?

The number of fruit flies at time t is modeled by the exponential equation $Q = Q_0(1 + .021)^t$, where t is measured in days. As in problem 4(a), to find the amount of time it takes for the number of fruit flies to triple, we can assume there are 100 fruit flies to start with. Then once they have tripled, there will be 300 fruit flies. So we need to solve

$$300 = 100 \cdot 1.021^t \qquad \Longrightarrow \qquad t = \frac{\log 3}{\log 1.021} = 52.86\,\text{days}$$

I skipped a bunch of steps here because I have shown the same calculation (with different numbers) elsewhere.

(b) If there are 465 fruit flies today, how many fruit flies were there 4 weeks ago? How many fruit flies will there be 1 year from now?

Now we are given $Q_0 = 465$, so the number of fruit flies at any time, t, is given by $Q = 465 \cdot 1.021^t$. Then four weeks ago, there were $Q = 465 \cdot 1.021^{-28} = 260$ fruit flies. One year (365 days) from now, there will be $Q = 465 \cdot 1.021^{365} = 915,902$ fruit flies.

(c) If there are 465 fruit flies today, when will the number of fruit flies reach 2,000?

We want to solve

$$2,000 = 465 \cdot 1.021^t \qquad \Longrightarrow \qquad t = \frac{\log(2,000/465)}{\log 1.021} = 70.19\,\text{days}$$

So in about 70 days, there will be 2,000 fruit flies.

7. Argon-41 has a half-life of 1.827 hours.

(a) How long does it take for a quantity of Argon to decrease to 10% of its original amount?

As before, we can suppose that we start with any initial amount of Argon we like, say 10 pounds. Then we need to find tha amount of time it takes until we have only 1 pound of Argon. The amount of Argon at time t is modeled by $Q = 10 \cdot \left(\frac{1}{2}\right)^{t/1.827}$. So we must solve

$$1 = 10 \cdot \left(\tfrac{1}{2}\right)^{t/1.827} \qquad \Longrightarrow \qquad 0.1 = \left(\tfrac{1}{2}\right)^{t/1.827}$$

Now we can take the logarithm of both sides, then solve for t. This gives

$$\log 0.1 = \log 0.5^{t/1.827} \qquad \Longrightarrow \qquad t = \frac{1.827 \log 0.1}{\log 0.5} = 6.069\,\text{hours}$$

(b) Suppose that 1 hour ago Jasper had 50mg of Argon-41. How much Argon-41 is left right now? When will Jasper only have 10mg of Argon-41?

If Jasper has 50 mg of Argon-41 one hour ago, the amount he has at time t is modeled by $Q = 50 \cdot 0.5^{t/1.827}$ where t is measured in hours with $t = 0$ corresponding to 1 hour ago. Then right now should correspond to $t = 1$, so right now Jasper has

$$Q = 50 \cdot 0.5^{1/1.827} = 34.21\,\text{mg}$$

To find when Jasper will only have 10 mg or Argon-41 left, we need to solve

$$10 = 50 \cdot 0.5^{t/1.827} \qquad \Longrightarrow \qquad t = \frac{1.827 \log(10/50)}{\log(0.5)} = 4.42\,\text{hours}$$

But remember, this is 4.42 hours from 1 hour ago, so it is 3.32 hours from right now.

(c) If Jasmine has 100mg of Argon-41 right now, how long ago did she have 150mg of Argon-41. How much will she have 1 day from now?

The amount of Argon-41 Jasmine has at time t is modeled by the exponential equation $Q = 100 \cdot 0.5^{t/1.827}$, where t can be either positive or negative. To find out when Jasmine had 150 mg of Argon-41, we must solve

$$150 = 100 \cdot 0.5^{t/1.827} \qquad \Longrightarrow \qquad t = \frac{1.827 \log(150/100)}{\log 0.5} = -1.07$$

So Jasmine had 150 mg of Argon 1.07 hours ago. One day (or 24 hours) from now, she will have

$$Q = 100 \cdot 0.5^{t/1.827} = 0.0111\,\mathrm{mg of Argon} - 41$$

(d) What is the domain and range of this problem? What are the dependent and independent variables?

In this problem, the amount of Argon-41 depends on the time. Therefore, time is the independent variable and the quantity of Argon-41 is the dependent variable. The domain of the problem (or the set of values which make sense for t) is all real numbers, which we can write as $(-\infty, \infty)$ or $\mathbb{R}$. You do not need to worry about domain.

8. Suppose that the number of wormless apples on Exeter's apple tree is decreasing at a linear rate. Further suppose that he had 85 wormless apples 4 days after he began harvesting and 60 wormless apples 8 days after he began harvesting.

(a) Find a linear equation that describes the number of wormless apples in Exeter's tree as a function of the number of days since he began harvesting.

We need a linear equation to describe the number of wormless apples as a funtion of time. Let's first graph the information we are given. The independent variable (time) should go on the x-axis and the dependent variable (# of wormless apples) should go on the y-axis. We are given two points on the graph of the line. First, the line should go through $(4, 85)$, corresponding to the fact that there were 85 wormless apples four days after Exeter started harvesting. The second point we are given is $(8, 60)$. Now we can find the slope of our linear equation by calculating

$$m = \frac{\text{rise}}{\text{run}} = \frac{60 - 85}{8 - 4} = \frac{-25}{4} = -6.25$$

Now we can use point-slope formula to find the linear equation. We just calculated the slope and we can use either of the two points we already have. I will use the point $(8, 60)$. The formula is $y - y_0 = m(x - x_0)$. Plugging in our data gives

$$y - 60 = -6.25(x - 8) \qquad \Longrightarrow \qquad y = -6.25x + 50 + 60 = -6.25x + 110$$

(b) How many wormless apples did he have the day he began harvesting? How many wormless apples did he have 15 days after he began harvesting?

Now, the hard work is done. The equation from (a) tells us how many wormless apples he had on any given day. For example, the day he began harvesting ($x = 0$), he had

$$y = -6.25 \times 0 + 110 = 110\,\text{wormlessapples}$$

Fifteen days after he began harvesting, he had

$$y = -6.25 \times 15 + 110 = 16.25 \approx 16\,\text{wormlessapples}$$

(c) How many days after he began harvesting apples were there 30 wormless apples?

Now we need to find the time (x) when the number of wormless apples is 30 (i.e., when $y = 30$). So we solve

$$30 = -6.25x + 110 \qquad \Longrightarrow \qquad x = 12.8$$

or 12.8 days from when Exeter began harvesting.

(d) What is the domain and range of this problem? What are the dependent and independent variables?

The independent variable in this problem is time, and the dependent variable is the quantity of wormless apples. The domain of the *function* we wrote down is all real numbers. However, that is not the domain of the problem. This is because Exeter cannot have a negative number of wormless apples. So we need to find when Exeter has zero wormless apples

$$0 = -6.25x + 110 \qquad \Longrightarrow \qquad x = \frac{-110}{-6.25} = 17.6$$

So 17.6 days after he began harvesting, Exeter would have zero wormless apples. Presumably, Exeter was not measuring the number of wormless apples before he started harvesting. The domain of this problem is then $(0, 17.6)$ which we could also write as $\{x \in \mathbb{R} \mid 0 \leq x \leq 17.6\}$. I would also accept $(-\infty, 17.6)$ because Exeter may have been measuring the number of wormless apples before he began harvesting.

9. The price of robot vacuums is increasing by \$4.10 per month and on average they already cost \$280!

(a) Find a linear equation that describes the price or robot vacuums as a function of the number of months from right now.

The rate of change in price of robot vacuums is the slope of our line. We are given an initial value of \$280. We can then use slope-intercept formula to find the linear equation that models the situation. We have

$$y = mx + b \qquad \Longrightarrow \qquad y = 4.10x + 280$$

where y is the cost of robot vacuums and x is the times in months from right now.

(b) How much will robot vacuums cost one year from now?

We can use our linear model from (a). One year (12 months) from now, the price will be

$$y = 4.10 \cdot 12 + 280 = \$329.2$$

(c) When will robot vacuums cost \$1,000?

Again, we can use our linear model from (a). We want to know when (i.e., which x value) the price will be $\$1,000$. So we need to solve

$$1000 = 4.10x + 280 \qquad \Longrightarrow \qquad x = \frac{1000 - 280}{4.10} = \frac{720}{4.10} = 175.61$$

So in 175.61 months, the price of a robot vacuum will be $\$1,000$.