THE DYNAMICS OF THREE-PHASE TRIPLE JUNCTION AND CONTACT POINTS
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Abstract. We use the method of matched asymptotic expansions to study the sharp interface limit of the three-phase system modelled by the Cahn-Hilliard equations with the relaxation boundary condition. The dynamic laws for the interfaces, the triple junction, and the contact points are derived at different time scales. In particular, we show, at $O(t)$ time scale, the dynamic of the triple junction is determined by the balance of the chemical potential gradient along the three interfaces meeting at the triple junction. At faster time scale $O(\epsilon t)$, the motion of the triple junction is controlled by the contact point motions and geometric constraints.

Key words. Cahn-Hilliard equations, three-phase, triple junction, contact point, sharp interface limit

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1. Introduction. The phase field model has been widely used to describe the multiphase problems. The basic idea is to represent the interface implicitly by an order parameter $\phi$ (e.g. scalar function to model two-phase problems or vector valued functions to model three-phase problems) which varies continuously over thin interfacial layers and is mostly uniform in the bulk phases. The evolution of $\phi$ is driven by the gradient of a total free energy of the system. The total free energy is then a sum of three terms: a bulk free energy which is usually taken as a multi-well potential function, an interface energy term depending on the gradient of $\phi$, and a surface energy on the solid boundary. Consider a three-phase system in a two dimensional domain $\Omega$ with a closed and smooth solid boundary $\partial \Omega$, the total free energy functional can be written as

$$F(\phi) = \int_\Omega \left( \frac{\epsilon}{2} |\nabla \phi|^2 + \frac{1}{\epsilon} F(\phi) \right) d\Omega + \int_{\partial \Omega} l(\phi) d\partial \Omega \quad (1.1)$$

where $\phi = (\phi^1, \phi^2, \phi^3)$ is a vector valued order parameter which typically represents the relative concentration of the three phases with $\phi^1 + \phi^2 + \phi^3 = 1$ and $\phi^i$ represents the mass fraction of the $i^{th}$ phase; $\epsilon$ measures the interface thickness between the two phases; $F(\phi)$ is a triple well potential function with three equal minima at $\vec{a}_i (i = 1, 2, 3)$ that denote the three different phases (e.g. $a_1 = (1, 0, 0), a_2 = (0, 1, 0)$, and $a_3 = (0, 0, 1)$); and $l(\phi)$ gives the surface energy density on the solid surface. The evolution of $\phi$ of the system is then described by the Cahn-Hilliard equation

$$\phi_t = \nabla \cdot (M(\phi) \nabla \mu) \quad \text{in} \quad \Omega, \quad (1.2)$$

where $M(\phi)$ is the diffusion mobility factor and can take various form in different physical situations (for example, in [2], $M(\phi)$ is chosen as $M(\phi) = M\phi$ with $M$ being the constant mobility). $\mu := \frac{\delta F}{\delta \phi} = -\epsilon \Delta \phi + \frac{1}{\epsilon} f(\phi)$ is the chemical potential. There are many studies of (1.2) with general mobility factor, see, for example, [8, 9, 10, 11] and

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the references therein. In this paper, for mathematical simplicity, we only consider the case when $M(\phi) = 1$ and focus on the dynamics of the contact angles, contact points, and the triple junction. The boundary conditions are given by

$$\frac{\partial \mu}{\partial \vec{n}} = 0 \quad \text{on} \quad \partial \Omega,$$

where $\vec{n}$ is the outward normal direction of $\partial \Omega$. The variation of (1.1) leads to the boundary condition for $\phi$ (see [7] and the references therein)

$$\epsilon \partial_n \phi + \nabla \phi l(\phi) = 0 \quad \text{on} \quad \partial \Omega.$$  

This will impose the equilibrium contact angle condition (i.e. Young’s equation) [26, 24, 25] in the sharp interface limit. To allow a dynamic condition for the contact angle, one can use the following relaxation boundary condition:

$$\phi_t = -\delta(\epsilon \partial_n \phi + \nabla \phi l(\phi)) \quad \text{on} \quad \partial \Omega,$$

where $\delta \sim O(1)$ determines the relaxation time of the interface intersecting with the boundary.

Then, the dynamics of a three-phase system on a solid surface can be modelled by the Cahn-Hilliard equation with the relaxation boundary condition as follows:

$$\begin{cases}
\phi_t = \Delta \mu, & \text{in} \quad \Omega \times (0, +\infty) \\
\mu = -\epsilon \Delta \phi + \frac{1}{2} f(\phi) & \text{in} \quad \Omega \times (0, +\infty), \\
\phi_t = -\delta(\epsilon \partial_n \phi + \nabla \phi l(\phi) - \frac{1}{3}(\nabla \phi F(\phi) \cdot \vec{e})\vec{e}) & \text{on} \quad \partial \Omega \times (0, +\infty), \\
\partial_n \mu = 0 & \text{on} \quad \partial \Omega \times (0, +\infty), \\
\phi = \phi_0 & \text{on} \quad \Omega \times \{0\}.
\end{cases} \quad (1.3)$$

Because of the constraint $\phi^1 + \phi^2 + \phi^3 = 1$, we can choose $f(\phi) = \nabla \phi F(\phi) - \frac{1}{3}(\nabla \phi F(\phi) \cdot \vec{e})\vec{e}$ to ensure the consistency condition $\mu^1 + \mu^2 + \mu^3 = 0$. This can be realized by projecting $\nabla \phi F(\phi)$ onto the plane $\Sigma = \{\phi \in \mathbb{R}^3 | \phi \cdot \vec{e} = 0\}$ where $\vec{e} = (1, 1, 1)$ (more details can be found in [27, 3]). In the relaxation boundary condition, $l(\phi)$ is also projected in a similar way.

The above system is a special case of a more general diffusive interface model for the multiphase flow consisting of a coupled Cahn-Hilliard-Navier-Stokes system with the generalized Navier boundary conditions (GNBC) introduced in [20, 21, 23] to model the moving contact line problem. In the slow dynamics, one can neglect the effect of the flow and the system is reduced to (1.3) which enables us to study the evolution of the interface along the solid boundary and the dynamic contact angle[7].

For the two-phase system, this problem will be simplified to the classical Cahn-Hilliard equation for the phase field order parameter [6]. The sharp interface limit is studied by Pego in a classical paper [19] using the method of multiscalar expansions for the Neumann and Dirichlet boundary conditions. In [7], Chen, Xu and Wang analyze the Cahn-Hilliard equation with relaxation boundary condition and they also derived the dynamics of the contact angle when the interface has contact with the solid boundary.

For the three-phase system modelled by vector-valued Ginzburg-Landau Equation, asymptotic behavior of the solution and dynamics of the interface motion was studied by Rubinstein, Sternberg and Keller [22] in the sharp interface limit. Owen,
Rubinstein and Sternberg [18] studied the behaviour of the contact line when the boundary condition is of the Neumann type or Dirichlet type. They showed that the interface meet the boundary of domain with a $\pi/2$ angle when the boundary condition is of the Neumann type, and when the boundary condition is of the Dirichlet type, the angle should depend on the boundary condition and potential. Lia Bronsard and Fernando Reitich [4], presented a formal asymptotical analysis of the Ginzburg-Landau equation and studied the behavior around the triple junction.

For the three-phase system modelled by Allen-Cahn / Cahn-Hilliard system [5], Cohen [17] obtained that the limiting behaviour of the triple junction is governed to the lowest order by Young’s law, a mass flux balance, and a condition on the sum of mean curvatures. Several studies of the Cahn-Hilliard equations have also appeared, see for example Alt and Pawlow [1], Eyre [14], Elliott and Luckhaus [12], Elliott and Garcke [13], Garcke and Novick-Cohen [15], Bronsard [4] et al. 

The main purpose of this paper is to study the sharp interface limit of the three-phase system in a two-dimensional bounded domain, modelled by the Cahn-Hilliard equations with relaxation boundary condition on the solid boundary. The methods in [19, 4] will be generalized to derive the dynamic laws for interfaces, contact points and the triple junction at different time scales. We will focus on an ideal case where the solid boundary is a unit circle. First, at $O(t)$ time scale, by using the multi-scale analysis and the matched asymptotic expansions, we obtain that, in the sharp interface limit, the dynamics of interfaces, triple junction and contact points is described by a Hele-Shaw type equation for the free boundaries, with the angles between three interfaces at the triple junction obeying the Herring condition. We also show that the contact points do not show noticeable motion at $O(t)$ time scale, and the interfaces will approach circular curves as $t \to +\infty$. We then study the dynamics at the fast time scale $O(\epsilon t)$. Under the assumption that all three interfaces are circular curves, we shall derive the dynamics of the contact points, contact angles and the triple junction at the fast time scale.

The outline of the paper is as follows. In Section 2, we firstly use the method of matched asymptotic expansions to derive the sharp interface limit of the problem (1.3) at $O(t)$ time scale and derive the dynamics of the triple junction. In Section 3, combining the property of the system derived before with volume preserving of each phase, we study a fast time scale motion of the system and derive a ODE system to govern the dynamics of the contact points, contact angles and the triple junction.

2. The behaviour of the solution at $O(t)$ time scale. In this section, we will first study the sharp interface limit of the problem (1.3) at the $O(t)$ time scale using the method of matched asymptotic expansions. We then study the asymptotic behaviour of contact points and interfaces as $t \to +\infty$.

2.1. Sharp interface limit. In this subsection, we will use the method of matched asymptotic analysis to study the sharp interface limit of the problem (1.3). We will perform outer and inner expansions in three regions and layers as described in Fig.2.1.

2.1.1. Outer expansions. We seek formal expansions in the following form, away from the interfaces and the triple junction,

$$\phi^\epsilon(x,t) = \phi_0(x,t) + \epsilon\phi_1(x,t) + \epsilon^2\phi_2(x,t) + h.o.t.,$$

$$\mu^\epsilon(x,t) = \mu_0(x,t) + \epsilon\mu_1(x,t) + \epsilon^2\mu_2(x,t) + h.o.t..$$
Substituting these expansions into the equations in (1.3), we have

\[(\phi_0)_t + \text{h.o.t.} = \Delta \mu_0 + \text{h.o.t.},\]

\[
\mu_0 + \text{h.o.t.} = \frac{1}{\epsilon} \left( \nabla_{\phi_0} F(\phi_0) - \frac{1}{3}(\nabla_{\phi_0} \cdot \vec{e})\vec{e} \right) + (\phi_1 \cdot \nabla_{\phi_0}) \left( \nabla_{\phi_0} F(\phi_0) - \frac{1}{3}(\nabla_{\phi_0} \cdot \vec{e})\vec{e} \right) + \text{h.o.t.}
\]

Denote by \(\vec{a}_1 = (1, 0, 0), \vec{a}_2 = (0, 1, 0), \vec{a}_3 = (0, 0, 1)\) the three phases 1, 2, 3 respectively.

Collecting all the terms at the \(O(\frac{1}{\epsilon})\) order, we get,

\[
\nabla_{\phi_0} F(\phi_0) - \frac{1}{3}(\nabla_{\phi_0} \cdot \vec{e})\vec{e} = 0,
\]

which implies that \(\phi_0 = \vec{a}_i, i = 1, 2, 3\), since \(F(\phi)\) is a potential function with three local minima at \(\vec{a}_i, i = 1, 2, 3\). This means that at the leading order, three different phases generate locally.

Without loss of generality, we assume that \(\phi_0\) divides the whole domain into 3 regions and we let \(\Gamma_1\) divides phase 2 and phase 3, \(\Gamma_2\) divides phase 3 and phase 1, \(\Gamma_3\) divides phase 1 and phase 2.

At the \(O(1)\) order, from (2.1) we have \((\phi_0)_t = \Delta \mu_0\), which implies

\[\Delta \mu_0 = 0.\]

To solve the leading order outer solution, we need the boundary conditions at the interfaces, which will be derived from the inner expansions near each interface of two phases.
Remark 2.1. Note that $\mu = -\epsilon \Delta \phi + \frac{1}{\epsilon} f(\phi)$, where $f(\phi) = (f_1(\phi), f_2(\phi), f_3(\phi))^T = \nabla \varphi F(\phi) - \frac{1}{3} (\nabla \varphi F(\phi) \cdot \hat{e}) \hat{e}$. Now, denote by $\mu = (\mu^1, \mu^2, \mu^3)^T$ and $\phi = (\phi^1, \phi^2, \phi^3)^T$, then a simple calculation shows that

$$\mu^1 + \mu^2 + \mu^3 = -\epsilon \Delta (\phi^1 + \phi^2 + \phi^3) + \frac{1}{\epsilon} [f_1(\phi) + f_2(\phi) + f_3(\phi)]$$

$$= 0$$

Hence, for the leading order $\mu_0(x,t)$, we also have

$$\mu^1_0 + \mu^2_0 + \mu^3_0 = 0. \quad (2.2)$$

Also, if $\phi^i \equiv 0$, it is easy to see, from the definition of the chemical potential $\mu$, that $\mu^i$ is a constant.

2.1.2. Inner expansions at each interface. Without loss of the generality, we will consider inner expansion around the interface $\Gamma_1$ but away from the triple junction. The expansions at $\Gamma_2, \Gamma_3$ will lead to similar results. Introduce the re-scaled coordinate:

$$z = \frac{1}{\epsilon} d(x,t)$$

where $d$ is the signed distance function to $\Gamma_1$ and $|\nabla d| = 1$ with $\nabla$ denoting the gradient operator in the spatial variable $x$. Consider the inner expansion as follows with $\phi^i(z,x,t)$ and $\mu^i(z,x,t)$ decaying fast as $z \to \pm \infty$:

$$\phi^i = \tilde{\phi}_0(z,x,t) + \epsilon \tilde{\phi}_1(z,x,t) + \epsilon^2 \tilde{\phi}_2(z,x,t) + h.o.t.$$  
$$\mu^i = \tilde{\mu}_0(z,x,t) + \epsilon \tilde{\mu}_1(z,x,t) + \epsilon^2 \tilde{\mu}_2(z,x,t) + h.o.t.$$  

Substituting the above expansions into the equations in (1.3), we have

$$\frac{d}{dz} \frac{\partial \tilde{\phi}_0}{\partial z} + \text{h.o.t.}$$
$$= \frac{1}{\epsilon^2} \frac{\partial^2 \tilde{\phi}_0}{\partial z^2} + \frac{1}{\epsilon} \left( \Delta d \frac{\partial \tilde{\mu}_0}{\partial z} + \frac{\partial^2 \tilde{\mu}_1}{\partial z^2} + 2 \nabla d \cdot \nabla \left( \frac{\partial \tilde{\mu}_0}{\partial z} \right) \right) + \text{h.o.t.},$$
$$\tilde{\mu}_0 + \text{h.o.t.}$$
$$= \frac{1}{\epsilon} \left( - \frac{\partial^2 \tilde{\phi}_0}{\partial z^2} + \left( \nabla \tilde{\varphi}_0 F(\tilde{\phi}_0) - \frac{1}{3} (\nabla \tilde{\varphi}_0 F(\tilde{\phi}_0) \cdot \hat{e}) \hat{e} \right) \right) \quad (2.3)$$
$$- \left( \frac{\partial^2 \tilde{\phi}_1}{\partial z^2} + \Delta d \frac{\partial \tilde{\phi}_0}{\partial z} + 2 \nabla d \cdot \nabla \left( \frac{\partial \tilde{\phi}_0}{\partial z} \right) \right)$$
$$+ \left( \tilde{\phi}_1 \cdot \nabla \tilde{\varphi}_0 \right) \left( \nabla \tilde{\varphi}_0 F(\tilde{\phi}_0) - \frac{1}{3} (\nabla \tilde{\varphi}_0 F(\tilde{\phi}_0) \cdot \hat{e}) \hat{e} \right)$$
$$+ \text{h.o.t.}.$$  

At the $O(\frac{1}{\epsilon^2})$ order, we have

$$\frac{\partial^2 \tilde{\phi}_0}{\partial z^2} = 0,$$  
$$\quad (2.4)$$
which implies that \( \tilde{\mu}_0(z, x, t) \equiv \mu_0(x, t) \).
At the \( \mathcal{O}(\epsilon) \) order, we have
\[
(\tilde{\phi}_0)_zd_t = (\tilde{\mu}_1)_{zz} + \Delta d(\tilde{\mu}_0)_z + 2\nabla d \cdot \nabla (\frac{\partial \tilde{\mu}_0}{\partial z})
\]
which can be reduced to
\[
(\tilde{\phi}_0)_zd_t = (\tilde{\mu}_1)_{zz}
\]
and
\[
(\tilde{\phi}_0)_{zz} = \nabla_{\tilde{\phi}_0} F(\tilde{\phi}_0) - \frac{1}{3}(\nabla_{\tilde{\phi}_0} F(\tilde{\phi}_0) \cdot \tilde{e})\tilde{e}.
\]
\[
\tilde{\phi}_0|_{z=0} = (\tilde{a}_2 + \tilde{a}_3)/2, \quad \lim_{z \to +\infty} \tilde{\phi}_0 = \tilde{a}_3, \quad \lim_{z \to -\infty} \tilde{\phi}_0 = \tilde{a}_2
\]
which uniquely determines \( \tilde{\phi}_0(z, x, t) = \tilde{\phi}_0(z) \) being independent of \( (x, t) \). Denote
\[
\tilde{\phi}_0 = (\tilde{\phi}_0^1, \tilde{\phi}_0^2, \tilde{\phi}_0^3)^T.
\]
By a direct calculation, we have
\[
\tilde{e} \cdot \partial_z \tilde{\phi}_0 = \partial_z (\tilde{\phi}_0^1 + \tilde{\phi}_0^2 + \tilde{\phi}_0^3) = \partial_z (1) = 0.
\]
Thus, we have
\[
(\nabla_{\tilde{\phi}_0} F(\tilde{\phi}_0) \cdot \tilde{e})\tilde{e} \cdot \partial_z \tilde{\phi}_0 = 0.
\]
Multiplying \( (\tilde{\phi}_0)_z \) to both sides of (2.6), integrating with respect to \( z \), and using (2.7), we have
\[
(\frac{\partial \tilde{\phi}_0}{\partial z})^2 = 2F(\tilde{\phi}_0).
\]
Integrating both sides of (2.5) with respect to \( z \) from \(-\infty\) to \(+\infty\), we have
\[
[\tilde{\phi}_0]d_t = [(\tilde{\mu}_1)_z],
\]
where \([\cdot]\) denotes the jump with respect to \( z \) from \(-\infty\) to \(+\infty\).

Now, fixing \( x \) on \( \Gamma_1 \), when \( \epsilon z \) is between \( O(\epsilon) \) and \( o(1) \), we expand \( (\mu_0 + \epsilon\mu_1 + \epsilon^2\mu_2 + h.o.t.)(x + \epsilon z\bar{n}, t) \) in powers of \( \epsilon \) when \( \epsilon z \to 0^+ \) to obtain
\[
(\mu_0)^+ + \epsilon((\mu_1)^+ + zD\bar{n}(\mu_0)^+) + h.o.t.
\]
where \( \bar{n} \) is the normal direction of \( \Gamma_1 \), and
\[
(\mu_i)^+(x, t) = \lim_{s \to 0^+} (\mu_i)(x + s\bar{n}, t).
\]
A similar expansion is obtained for \( \epsilon z \to 0^- \). To match the solution between inner region at two phase interface and outer region, one requires that:
\[
(\mu_0)^\pm(x, t) = \lim_{z \to \pm\infty} \tilde{\mu}_0(z, x, t)
\]
\[
(\mu_1)^\pm(x, t) = \lim_{z \to \pm\infty} (\tilde{\mu}_1(z, x, t) - zD\bar{n}(\mu_0)^\pm(x, t)).
\]
Since \( \tilde{\phi}_0 \) is then reduced to \( \mu_0 \) should be a constant, which is denoted as \( c \), and \( \mu_0^2 = -c - \mu_0^3 \). In this case, (2.12) is then reduced to
\[
d_t = [D\tilde{\pi}(\mu_0^3)] = -[D\tilde{\pi}(\mu_0^2)].
\] (2.13)

At the O(1) order, we have
\[
\tilde{\phi}_0 = -\left( \begin{array}{c} \partial^2 \phi_1 \\
\Delta d \partial \phi_0 \\
2\n\end{array} \right) + \left( \begin{array}{c} \phi_1 \cdot \nabla \phi_0 \\
\nabla \phi_0 F(\phi_0) - \frac{1}{3} (\nabla \phi_0 F(\phi_0) \cdot \vec{e}) \vec{e} \\
\n\end{array} \right).
\] (2.14)

Multiplying \( \partial_z \tilde{\phi}_0 \) on both sides of (2.14), the solvability condition implies:
\[
\int_{-\infty}^{+\infty} \left( \frac{\tilde{\phi}_0 + \Delta d \partial \phi_0}{\tilde{\phi}_0} \right) \frac{\partial \phi_0}{\partial z} dz
= \int_{-\infty}^{+\infty} \left( -\frac{\partial^2 \phi_1}{\partial z^2} \partial \phi_0 \right) \left( \nabla \phi_0 F(\phi_0) \partial \phi_0 \right) dz
= \int_{-\infty}^{+\infty} \left( \frac{\partial \phi_1}{\partial z} \frac{\partial \phi_0}{\partial z^2} \partial \phi_0 \right) \left( \nabla \phi_0 F(\phi_0) \partial \phi_0 \right) dz
= \int_{-\infty}^{+\infty} \left( \phi_1 \cdot \frac{\partial \phi_0}{\partial z^3} \right) dz + \int_{-\infty}^{+\infty} \left( \phi_1 \cdot \nabla \phi_0 F(\phi_0) \partial \phi_0 \right) dz
= \int_{-\infty}^{+\infty} \left( \phi_1 \cdot \nabla \phi_0 F(\phi_0) \partial \phi_0 \right) dz
= 0.
\] (2.15)

Since \( \tilde{\phi}_0 \) is independent of \( z \), we have from (2.15),
\[
\tilde{\phi}_0 \cdot \tilde{\phi}_0 |_{z = +\infty} = (\Delta d) \int_{-\infty}^{+\infty} |\partial \phi_0|^2 dz = 0.
\] (2.16)

Denote by \( \kappa_1 = \Delta d \) the curvature of \( \Gamma_1 \), and \( \Phi_1 = \int_{-\infty}^{+\infty} |\partial \phi_0^2| \partial \phi_0 \partial \phi_0 | dz \) the interface tension of \( \Gamma_1 \) (these quantities can be similarly defined for \( \Gamma_2 \) and \( \Gamma_3 \)). Combining (2.16) with (2.4) we have:
\[
\mu_0 \cdot [\tilde{\phi}_0] = -\Phi_1 \cdot \kappa_1.
\] (2.17)
2.1.3. The behaviour around the triple junction. Let \( m(t) \) denote the coordinates of the triple junction and the scaled new coordinate \( \eta = \frac{x - m(t)}{\epsilon} \), we then seek the following expansions around the triple junction

\[
\phi^\prime = \hat{\phi}_0(\eta, x, t) + \epsilon \hat{\phi}_1(\eta, x, t) + \epsilon^2 \hat{\phi}_2(\eta, x, t) + \text{h.o.t.},
\]

\[
\mu^\prime = \hat{\mu}_0(\eta, x, t) + \epsilon \hat{\mu}_1(\eta, x, t) + \epsilon^2 \hat{\mu}_2(\eta, x, t) + \text{h.o.t.}.
\]

Substitute these expansions into (1.3), it follows

\[
\left( -\frac{m'(t)}{\epsilon} \right) \nabla_\eta \hat{\phi}_0 + \text{h.o.t.}
= \frac{1}{\epsilon^2} \Delta_\eta \hat{\mu}_0 + \frac{1}{\epsilon} \Delta_\eta \hat{\mu}_1 + \text{h.o.t.},
\]

\[
\hat{\mu}_0 + \text{h.o.t.}.
\]

\[
(\Delta_\eta \hat{\phi}_0 - \left( \nabla_\eta F(\hat{\phi}_0) - \frac{1}{3} (\nabla_\eta F(\hat{\phi}_0) \cdot \hat{e}) \hat{e} \right)
+ \Delta_\eta \hat{\phi}_1 - \left( \hat{\phi}_1 \cdot \nabla_\eta \right) \left( \nabla_\eta F(\hat{\phi}_0) - \frac{1}{3} (\nabla_\eta F(\hat{\phi}_0) \cdot \hat{e}) \hat{e} \right) + \text{h.o.t.}.
\]

At the \( O(\frac{1}{\epsilon^2}) \) order of (2.18), we have:

\[
\Delta_\eta \hat{\mu}_0 = 0,
\]

which implies that \( \hat{\mu}_0 \) is independent of \( \eta \) by using Liouville’s theorem.

The \( O(\frac{1}{\epsilon}) \) order of (2.18) gives rise to

\[
\Delta_\eta \hat{\phi}_0 - \left( \nabla_\eta F(\hat{\phi}_0) - \frac{1}{3} (\nabla_\eta F(\hat{\phi}_0) \cdot \hat{e}) \hat{e} \right) = 0,
\]

\[
-m'(t) \nabla_\eta \hat{\phi}_0 = \Delta_\eta \hat{\mu}_1.
\]

Next, we will use (2.20) and (2.21) to derive the angle condition at the triple junction and dynamic of the triple junction.

2.1.4. The Herring angle condition at the triple junction. Let \( T \) be a triangle with sides perpendicular to \( \Gamma_1, \Gamma_2, \Gamma_3 \) and containing the triple junction \( m(t) \) as the circumcentre of the triangle (here we only consider the case that angles between three interfaces are between \( \frac{\pi}{2} \) and \( \pi \)). Denote by \( h \) the height of the triangle (See Fig.2.2). Now, we use a new coordinate \( \eta_i = (\xi_i, \zeta_i) \), where \( \zeta_i \) is the tangent direction of \( \Gamma_i \) at \( m(t) \), and \( \xi_i \) is perpendicular to \( \zeta_i \). We have the following matching condition for \( \xi_i \) fixed:

\[
\lim_{\zeta_i \to \infty} \hat{\phi}_0(\xi_i, \zeta_i) = \hat{\phi}_0(\xi_i)
\]

We now consider (2.20) in the \( \eta_i \) variables. Multiply (2.20) by \( \partial_{\xi_i} \hat{\phi}_0 \) and integrate over \( T \) to have:

\[
\int \int_T \left( \nabla_\eta F(\hat{\phi}_0) - \frac{1}{3} (\nabla_\eta F(\hat{\phi}_0) \cdot \hat{e}) \hat{e} \right) \cdot \partial_{\xi_i} \hat{\phi}_0 d\eta
= \int \int_T \partial_{\xi_i}^2 \hat{\phi}_0 \cdot \partial_{\xi_i} \hat{\phi}_0 + \partial_{\xi_i}^2 \hat{\phi}_0 \cdot \partial_{\xi_i} \hat{\phi}_0 d\eta.
\]
which implies

\[
\int \int_T \partial_\xi_1 (F(\hat{\phi}_0) + \frac{1}{2} |\partial_{\xi_1} \hat{\phi}_0|^2 - \frac{1}{2} |\partial_{\xi_1} \hat{\phi}_0|) \, d\eta_1 = \int \int_T \partial_{\xi_1} (\hat{\phi}_0 \cdot \partial_{\xi_1} \hat{\phi}_0) \, d\eta_1. 
\]  
(2.23)

By using the divergence theorem in (2.23), we get

\[
\int_{\partial T} [F(\hat{\phi}_0) + \frac{1}{2} |\partial_{\xi_1} \hat{\phi}_0|^2 - \frac{1}{2} |\partial_{\xi_1} \hat{\phi}_0|] \, v_1 \, ds = \int_{\partial T} [\partial_{\xi_1} \hat{\phi}_0 \cdot \partial_{\xi_1} \hat{\phi}_0] \, v_2 \, ds, 
\]  
(2.24)

where \( v = (v_1, v_2) \) is the outward normal vector to \( \partial T \).

Next, we separate the integral of (2.24) into three parts along three sides of the triangle and parametrize these line integrals in \((\xi_i, \zeta_i)\) coordinates on the corresponding sides. Notice that we have the following relationship

\[
\begin{align*}
\partial_{\xi_1} &= -\sin \gamma_i \partial_{\xi_i} + \cos \gamma_i \partial_{\zeta_i} \\
\partial_{\zeta_1} &= -\cos \gamma_i \partial_{\xi_i} - \sin \gamma_i \partial_{\zeta_i},
\end{align*}
\]

where \( 0 \leq \gamma_i \leq 2\pi \) are the angles between \( \Gamma_i \) and the \( \xi_1 \) axis (See Fig. 2.2). We then have

\[
\begin{align*}
|\partial_{\xi_1} \hat{\phi}_0|^2 - |\partial_{\xi_1} \hat{\phi}_0|^2 \\
= -\cos(2\gamma_i)|\partial_{\xi_1} \hat{\phi}_0|^2 + 4 \sin \gamma_i \cos \gamma_i \partial_{\xi_1} \hat{\phi}_0 \cdot \partial_{\xi_1} \hat{\phi}_0 + \cos(2\gamma_i)|\partial_{\xi_1} \hat{\phi}_0|^2 \\
\partial_{\xi_1} \hat{\phi}_0 \cdot \partial_{\zeta_1} \hat{\phi}_0 \end{align*}
\]

\[
\begin{align*}
= -\cos \gamma_i \sin \gamma_i |\partial_{\xi_1} \hat{\phi}_0|^2 - \cos(2\gamma_i)|\partial_{\xi_1} \hat{\phi}_0 \cdot \partial_{\zeta_1} \hat{\phi}_0 + \cos \gamma_i \sin \gamma_i |\partial_{\xi_1} \hat{\phi}_0|^2.
\end{align*}
\]

Using the matching condition (2.22) and the fact that \( \lim_{|\zeta_i| \to \infty} |\partial_{\zeta_1} \hat{\phi}_0| = 0 \), and by
taking the limit as $h \to \infty$ on both sides of (2.24) (see also [4]), we get:

\[
\mathcal{LHS} = \lim_{h \to \infty} \int_{\partial T} [F(\hat{\phi}_0) + \frac{1}{2} |\partial_{x_i} \hat{\phi}_0|^2 - \frac{1}{2} |\partial_{x_i} \hat{\phi}_0|^2] v_1 ds
\]

\[
= \int_{-\infty}^{+\infty} \frac{1}{2} |\partial_{x_i} \hat{\phi}_0|^2 \cos(2\gamma_2) + F(\hat{\phi}_0) \cos(\gamma_2) d\xi_2
\]

\[
+ \int_{-\infty}^{+\infty} \frac{1}{2} |\partial_{x_i} \tilde{\phi}_0|^2 \cos(2\gamma_3) + F(\tilde{\phi}_0) \cos(\gamma_3) d\xi_3
\]

\[
\mathcal{RHS} = \lim_{h \to \infty} \int_{\partial T} \partial_{x_i} \hat{\phi}_0 \cdot \partial_{x_i} \hat{\phi}_0 v_2 ds
\]

\[
= - \int_{-\infty}^{+\infty} |\partial_{x_i} \hat{\phi}_0|^2 \cos(\gamma_2) \sin(\gamma_2) \sin(\gamma_2) d\xi_2
\]

\[
- \int_{-\infty}^{+\infty} |\partial_{x_i} \tilde{\phi}_0|^2 \cos(\gamma_3) \sin(\gamma_3) \sin(\gamma_3) d\xi_3.
\]

Again denote the interface tension $\Phi_i = \int_{-\infty}^{+\infty} |\partial_{x_i} \tilde{\phi}_0|^2 d\xi_i$ and combining (2.8) with $\mathcal{LHS} = \mathcal{RHS}$, we have:

\[
\cos(\gamma_2) \Phi_2 = - \cos(\gamma_3) \Phi_3.
\]

Since $\gamma_2 + \frac{\pi}{2} = \Theta_3$ and $\gamma_3 - \gamma_2 = \Theta_1$, where $\Theta_i$ is described as in Fig.2.2, then we have:

\[
\sin(\Theta_3) \Phi_2 = \sin(\Theta_2) \Phi_3.
\]

If we rotate $T$ so that its base is around $\Gamma_2$, the same argument will lead:

\[
\sin(\Theta_2) \Phi_1 = \sin(\Theta_1) \Phi_2.
\]

Thus, we have the following Herring angle condition at the triple junction:

\[
\frac{\sin(\Theta_1)}{\Phi_1} = \frac{\sin(\Theta_2)}{\Phi_2} = \frac{\sin(\Theta_3)}{\Phi_3},
\]

which gives the force balance at the triple junction.

2.1.5. **Leading order behaviour.** We now summarize the leading order behaviour at the $O(t)$ time scale. In the leading order, the domain $\Omega$ is divided into three disjoint regions $\Omega_i (i = 1, 2, 3)$ such that $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$. The dynamics of the phase variable $\phi_0$, the chemical potential $\mu_0$, interfaces and triple junction is described by the following Hele-Shaw type equations with free boundaries in $\{(x, t) | x \in \Omega, t > 0\}$,

\[
\begin{aligned}
\phi_0 &= \delta_i, \quad \text{in } \Omega_i, \quad i = 1, 2, 3 \\
\Delta \mu_0 &= 0, \quad \text{in } \Omega_i, \quad i = 1, 2, 3 \\
\frac{\partial \mu_0}{\partial t} &= 0, \quad \text{on } \partial \Omega \\
\mu_0 \cdot [\tilde{\phi}_0 |_{\Gamma_i}] &= - \Phi_i \cdot \kappa_i, \quad \text{on } \Gamma_i (i = 1, 2, 3) \\
[\tilde{\phi}_0 |_{\Gamma_i}] = [D_{\delta}(\mu_0)] |_{\Gamma_i}, \quad \text{on } \Gamma_i (i = 1, 2, 3) \\
\frac{\sin(\Theta_1)}{\Phi_1} &= \frac{\sin(\Theta_2)}{\Phi_2} = \frac{\sin(\Theta_3)}{\Phi_3}, \quad \text{at } \Gamma_1 \cap \Gamma_2 \cap \Gamma_3.
\end{aligned}
\]
where \([.]_{Γ_i}\) represents the jump along Γ_i interface (defined in Section 2.1.1), \(κ_i\) the curvature of Γ_i, Φ_i the interface tension of Γ_i, and \(Θ_i\) the angles at the triple junction between interfaces as shown in Fig. 2.2.

Remark 2.2. The dynamic of triple junction can be implicitly determined by solving the above problem (2.26) from the dynamic of interfaces and the Herring angle condition at the triple junction. On the other hand, we can also use the matched asymptotic expansion to derive an explicit system to describe the dynamic of triple junction in the next subsection.

2.2. Motion of the triple junction. To derive the dynamic law of the triple junction point explicitly, we integrate equation (2.21) over \(T\) with three vertices \(P_1, P_2, P_3\) (See Fig. 2.2), we have

\[-m'(t) \cdot \int_T (\nabla_{\eta_i} \hat{ϕ}_0) d\eta_1 = \int_T \Delta \hat{μ}_1 d\eta_1.\]  

(2.27)

Use the Green theorem, the left hand side is converted into a boundary integral on three sides of the triangle. We then parameterize these line integrals into the local coordinates \((ξ_i, ζ_i)\). Calculate the integral of \(\nabla_{\eta_i} \hat{ϕ}_0\) in each component, we have

\[\int_T \frac{∂\hat{ϕ}_0}{∂ξ_1} d\eta_1 = \int_{P_1}^{P_2} \hat{ϕ}_0(ξ_2) \sin(Θ_3) dξ_2 - \int_{P_1}^{P_3} \hat{ϕ}_0(ξ_3) \sin(Θ_2) dξ_3\]  

(2.28)

\[\int_T \frac{∂\hat{ϕ}_0}{∂ξ_2} d\eta_1 = \int_{P_2}^{P_3} \hat{ϕ}_0(ξ_1) dξ_1 + \int_{P_1}^{P_3} \hat{ϕ}_0(ξ_2) \cos(Θ_3) dξ_2 + \int_{P_1}^{P_2} \hat{ϕ}_0(ξ_3) \cos(Θ_2) dξ_3.\]  

(2.29)

For the right hand side of (2.27), we apply the divergence theorem to get

\[\int_T \Delta \hat{μ}_1 dT = \int_{∂T} \frac{∂\hat{μ}_1}{∂\vec{n}} d\vec{s}\]  

\[= \int_{P_2}^{P_3} \frac{∂\hat{μ}_1}{∂ξ_1} dξ_1 + \int_{P_3}^{P_1} \frac{∂\hat{μ}_1}{∂ξ_2} dξ_2 + \int_{P_1}^{P_2} \frac{∂\hat{μ}_1}{∂ξ_3} dξ_3,\]  

(2.30)

where \(\vec{n}\) is the outward normal vector of the boundary of the triangle.

Note that \(\lim_{ξ_i \to -∞} Φ_0\) is a constant vector. The integrals in (2.28) and (2.29) are divergent. To calculate the limit as the length \(h\) of the height of \(T\) tends to \(∞\), we divide \(h\) on both sides of (2.27). Since \(Θ_1, Θ_2, Θ_3\) are determined by (2.25), we have \(|P_1 P_2| = \frac{h}{\sin Θ_2}, |P_2 P_3| = \frac{h}{\sin Θ_3}, |P_3 P_1| = -\frac{h}{\tan Θ_1} - \frac{h}{\tan Θ_3}\) (we always assume \(Θ_1, Θ_2, Θ_3 \in (\frac{π}{2}, π)\)). Using the matching condition (2.22), rescaling (2.28) by dividing the characteristic length \(h\) (i.e. the height of the T as in Fig. 2.2) on both sides, and using first mean value theorem for definite integrals, it is easy to see that, the scaled (2.28) can be calculated as the follows:

\[\lim_{h \to -∞} \left( \frac{1}{h} \int_{P_1}^{P_2} \hat{ϕ}_0(ξ_2) \sin(Θ_3) dξ_2 - \frac{1}{h} \int_{P_1}^{P_3} \hat{ϕ}_0(ξ_3) \sin(Θ_2) dξ_3 \right)\]  

\[= \lim_{h \to -∞} \left( \frac{|P_3 P_1| \sin(Θ_3) \hat{ϕ}_2}{h} - \frac{|P_1 P_2| \sin(Θ_2) \hat{ϕ}_3}{h} \right)\]  

\[= \hat{ϕ}_2 - \hat{ϕ}_3.\]  

(2.31)
where $\bar{\phi}^2$ is a vector representing the mean value of the integral of $\tilde{\phi}_0$ with respect to $\xi_2$ from $P_3$ to $P_1$ ($\bar{\phi}^3$ and $\bar{\phi}^4$ are also similarly defined). Note that these mean values are determined based on the profile of $\tilde{\phi}_0$ through the corresponding interface.

Similarly, for (2.29), we also have

$$
\lim_{h \to \infty} \frac{1}{h} \left( \int_{P_1}^{P_3} \phi_0(\xi_1)d\xi_1 + \int_{P_1}^{P_3} \phi_0(\xi_2)\cos(\Theta_3)d\xi_2 + \int_{P_1}^{P_2} \phi_0(\xi_3)\cos(\Theta_2)d\xi_3 \right) = -\frac{1}{\tan \Theta_2} + \frac{1}{\tan \Theta_3} \bar{\phi}^1 + \frac{1}{\tan \Theta_3} \bar{\phi}^2 + \frac{1}{\tan \Theta_2} \bar{\phi}^3.
$$

(2.32)

To calculate the limiting behaviour of (2.30) as $h \to +\infty$, the matching conditions for the inner and outer expansions of the chemical potential along $\Gamma_i$ must be developed. In the coordinate $\eta_i = (\xi_i, \zeta_i)$, when $\epsilon \zeta_i$ is between $O(\epsilon)$ and $o(1)$, we expand

$$
[\bar{\mu}_0 + \epsilon \bar{\mu}_1 + h.o.t.] (m(t) + \epsilon \zeta_i \tau_i, \xi_i, t)
$$

(2.33)

in powers of $\epsilon$ when $\epsilon \zeta_i \to 0^+$ to obtain

$$
\bar{\mu}_0(m(t), t) + \epsilon [\bar{\mu}_1(m(t), \xi_i, t) + \zeta_i D_{\tau} \bar{\mu}_0(m(t), t)] + h.o.t.
$$

where $\tau_i$ is the tangent direction of the interface $\Gamma_i$ at the triple junction. To match this expansion to the inner expansion at the triple junction, one requires that

$$
\lim_{\zeta_i \to +\infty}(\bar{\mu}_0(m(t), t) + \epsilon [\bar{\mu}_1(m(t), \xi_i, t) - \zeta_i D_{\tau} \bar{\mu}_0(m(t), t)]) = 0.
$$

(2.35)

From these matching conditions, we have $\frac{\partial \bar{\mu}_1}{\partial \zeta_i} = D_{\tau} \bar{\mu}_0(m(t), t)$ which is independent of $\xi_i$ (as $\zeta_i \to +\infty$). (2.30) can then be calculated as the following (for convenience, we use $D_{\tau} \bar{\mu}_0$ to represent $D_{\tau} \bar{\mu}_0(m(t), t)$ in follows):

$$
\lim_{h \to \infty} \frac{1}{h} \left( \int_{P_2}^{P_3} \frac{\partial \bar{\mu}_1}{\partial \xi_1}d\xi_1 + \int_{P_2}^{P_3} \frac{\partial \bar{\mu}_1}{\partial \xi_2}d\xi_2 + \int_{P_2}^{P_3} \frac{\partial \bar{\mu}_1}{\partial \xi_3}d\xi_3 \right) = -\frac{1}{\tan \Theta_2} + \frac{1}{\tan \Theta_3} D_{\tau} \bar{\mu}_0 + \frac{1}{\sin \Theta_3} D_{\tau_3} \bar{\mu}_0.
$$

(2.36)

Combining (2.27) (2.31) (2.32) and (2.36), we have follows:

$$
-\bar{m}'(t) \cdot \left( \frac{\bar{\phi}^2(\eta)}{\bar{m}^2} - \phi^1 \right) = -\frac{1}{\tan \Theta_2} + \frac{1}{\tan \Theta_3} D_{\tau} \bar{\mu}_0 + \frac{1}{\sin \Theta_3} D_{\tau_3} \bar{\mu}_0.
$$

(2.37)

It is easy to see that $\bar{\phi}^1 = (0, \frac{1}{2}, \frac{1}{2}), \bar{\phi}^2 = (\frac{3}{2}, 0, \frac{1}{2}), \bar{\phi}^3 = (\frac{1}{2}, \frac{1}{2}, 0)$. Denote by $\Upsilon = \Upsilon_{i,j} \in \mathbb{R}^{2 \times 3}$ and $\Xi = \Xi_{i,j} \in \mathbb{R}^{1 \times 3}$ with

$$
\Xi = \left( -\frac{1}{\tan \Theta_2} - \frac{1}{\tan \Theta_3} D_{\tau} \bar{\mu}_0 + \frac{1}{\sin \Theta_3} D_{\tau_3} \bar{\mu}_0 \right)
$$

and

$$
\Upsilon = \left( \begin{array}{ccc}
0 & \frac{1}{\sin \Theta_3} & \frac{1}{\tan \Theta_2} \\
\frac{1}{2 \tan \Theta_2} & -\frac{1}{2 \tan \Theta_3} & -\frac{1}{2 \tan \Theta_2} \\
\end{array} \right).
$$
Then the system (2.37) is written as:

\[-m'(t) \cdot \mathbf{T} = \Xi. \quad (2.38)\]

If we write \(\hat{\mu}_0\) in components \((\hat{\mu}_0^1, \hat{\mu}_0^2, \hat{\mu}_0^3)\) and use the fact that

\[\mathbf{T}_{1,1} + \mathbf{T}_{1,2} + \mathbf{T}_{1,3} = \mathbf{T}_{2,1} + \mathbf{T}_{2,2} + \mathbf{T}_{2,3} = 0, \]

\[\Xi_{1,1} + \Xi_{1,2} + \Xi_{1,3} = 0, \quad (2.39)\]

we then have

\[
m'(t) \cdot \left(\begin{array}{cc}
\frac{1}{2\tan \Theta_1} & \frac{-1}{2\tan \Theta_1} \\
\frac{1}{2\tan \Theta_2} & \frac{-1}{2\tan \Theta_2}
\end{array}\right)
\]

\[= - \left(\frac{1}{\tan \Theta_2} + \frac{1}{\tan \Theta_3}\right) (D_{\tau_1} \hat{\mu}_0^2, D_{\tau_1} \hat{\mu}_0^3) + \frac{1}{\sin \Theta_3} (D_{\tau_2} \hat{\mu}_0^2, D_{\tau_2} \hat{\mu}_0^3)
\]

\[+ \frac{1}{\sin \Theta_2} (D_{\tau_3} \hat{\mu}_0^2, D_{\tau_3} \hat{\mu}_0^3). \quad (2.40)\]

Denote \(\Psi = \left(\begin{array}{c}
-\cos \Theta_2 \sin \Theta_3 \\
\cos \Theta_3 \sin \Theta_2
\end{array}\right)\) and \(\hat{\mu}_0' = (\hat{\mu}_0^2, \hat{\mu}_0^3)\), we can solve (2.40) to give

\[m'(t) = \frac{2}{\sin \Theta_1 \sin \Theta_2 \sin \Theta_3} \left(\sin \Theta_1 D_{\tau_1} \hat{\mu}_0' + \sin \Theta_2 D_{\tau_2} \hat{\mu}_0' + \sin \Theta_3 D_{\tau_3} \hat{\mu}_0'\right) \Psi, \quad (2.41)\]

which gives the dynamics of the triple junction explicitly. We note that this is a balance of the chemical potential flux (along three interfaces) at the triple junction.

2.3. Asymptotic behaviour. In this section, we study the asymptotic behaviour of the system at \(O(t)\) time scale. We show that the interfaces approach circular curves as \(t \to \infty\) and the contact points do not have noticeable motion at this time scale.

2.3.1. Energy estimates. To derive the energy estimate, we multiply the equation \(\phi_t - \Delta \mu = 0\) by \(\mu\) on both sides and integrate over the whole domain to have

\[0 = \int_\Omega \{\mu(\phi_t - \Delta \mu)\} \, d\Omega
\]

\[= \int_\Omega \left\{-\mu \Delta \mu + \frac{-\epsilon \Delta \phi + \frac{1}{\epsilon} \left(\nabla \phi F(\phi) - \frac{1}{3} (\nabla \phi F(\phi) \cdot \mathbf{e}) \mathbf{e}\right) \phi_t\right\} \, d\Omega
\]

\[= \int_{\partial \Omega} \left\{-\frac{\partial \mu}{\partial n} \tau - \frac{\partial \phi}{\partial n} \mathbf{n} \cdot \phi_t\right\} d\Gamma + \int_\Omega \left\{\left|\nabla \mu\right|^2 + \epsilon \nabla \phi \cdot \nabla \phi_t + \frac{1}{\epsilon} \nabla \phi F(\phi) \cdot \phi_t\right\} \, d\Omega
\]

\[= \int_\Omega \left\{\left|\nabla \mu\right|^2 + \epsilon \nabla \phi \cdot \nabla \phi_t + \frac{1}{\epsilon} \nabla \phi F(\phi) \cdot \phi_t\right\} \, d\Omega + \int_{\partial \Omega} \left\{\frac{\phi_t^2}{\delta} + \sigma(x) \nabla \phi l(\phi) \phi_t\right\} \, d\Gamma
\]

\[= \frac{\partial}{\partial t} \left\{\int_\Omega \left(\frac{1}{\epsilon} F(\phi) + \frac{1}{2} \left|\nabla \phi\right|^2 \right) \, d\Omega + \int_{\partial \Omega} \{\sigma(x) l(\phi)\} \, d\Gamma\right\}
\]

\[+ \int_\Omega \left\{\left|\nabla \mu\right|^2\right\} \, d\Omega + \int_{\partial \Omega} \left\{\frac{\phi_t^2}{\delta}\right\} \, d\Gamma,
\]
where we have used the boundary conditions given in (1.1).

Let

$$E(\phi) = \int_{\Omega} \left\{ \frac{1}{\epsilon} F(\phi) + \frac{\epsilon}{2} |\nabla \phi|^2 \right\} d\Omega + \int_{\partial\Omega} \{\sigma(x)\dot{l}(\phi)\} d\Gamma,$$

we then have

$$-\frac{\partial E(\phi)}{\partial t} = \int_{\Omega} \{ |\nabla \mu|^2 \} d\Omega + \int_{\partial\Omega} \left\{ \frac{\phi^2}{\delta} \right\} d\Gamma.$$ 

Integrate with respect to $t$, we have

$$\int_0^{+\infty} \left\{ \int_{\Omega} \{ |\nabla \mu|^2 \} d\Omega + \int_{\partial\Omega} \left\{ \frac{\phi^2}{\delta} \right\} d\Gamma \right\} dt \leq -E(\phi)|_{t=+\infty} + E(\phi)|_{t=0} \leq E(\phi_0), \quad (2.42)$$

which implies that $\int_0^{+\infty} \int_{\partial\Omega} \frac{\phi^2}{\delta} d\Gamma dt$ is a bounded function. We also have that $\int_0^\infty |\nabla \mu|^2$ approaches to 0, when $t \to +\infty$. Consequently, we have $|\nabla \mu|^2 \to 0$ as $t \to +\infty$. This means $\mu$ approaches to a constant when $t \to +\infty$. It follows from (2.17) that the interface becomes circular since the curvature of three interfaces become a constant.

### 2.3.2. The behaviour of contact points.

Consider one of the contact points with position in the polar coordinates $(r, \theta) = (1, \theta(t))$ since the solid boundary $\partial\Omega$ is a unit circle we defined before, and from $t = 0$ to $t = T$, the contact point moves from $\theta(0) = 0$ to $\theta(T) = b > 0$. For each $x_1 \in (0, b)$, denote by $t^\pm(x_1)$ the time at which $\phi_\epsilon(1, x_1, t^\pm_\epsilon(x_1)) = (\frac{1}{2}, \frac{2}{3}, 0), \phi_\epsilon(1, x_1, t^-_\epsilon(x_1)) = (\frac{2}{5}, \frac{1}{3}, 0)$. Obviously, we have

$$\left( \frac{-b}{3}, \frac{b}{3}, 0 \right) = \lim_{\epsilon \to 0} \int_0^b \left[ \phi_\epsilon(1, x_1, t^\pm_\epsilon(x_1)) - \phi_\epsilon(1, x_1, t^-_\epsilon(x_1)) \right] dx_1$$

$$= \lim_{\epsilon \to 0} \int_0^b \int_{t^-_\epsilon(x_1)}^{t^\pm_\epsilon(x_1)} \phi_{\epsilon,t}(1, x_1, t) dt dx_1. \quad (2.43)$$

Denote by $\phi_{\epsilon}(1, x, t) = (\phi_{\epsilon,1}(1, x, t), \phi_{\epsilon,2}(1, x, t), \phi_{\epsilon,3}(1, x, t))^T$. From the second component of the relation (2.43), the boundedness of $\int_0^{+\infty} \int_{\partial\Omega} \frac{\phi^2}{\delta} d\Gamma dt$, and the use of Cauchy-Schwarz inequality, we have

$$\frac{b}{3} = \lim_{\epsilon \to 0} \int_{t^-_\epsilon(x_1)}^{t^\pm_\epsilon(x_1)} \int_0^b \frac{\phi_{\epsilon,2}(t)}{\sqrt{\delta}} \sqrt{\delta} dx_1 dt$$

$$\leq \lim_{\epsilon \to 0} \left[ \int_{t^-_\epsilon(x_1)}^{t^\pm_\epsilon(x_1)} \int_{\partial\Omega} \left\{ \frac{\phi_{\epsilon,2}(t)}{\delta} \right\} d\Gamma dt \right]^{\frac{1}{2}} \delta^{\frac{1}{2}} [A(D_{\epsilon})]^{\frac{1}{2}} \quad (2.44)$$

$$\leq \lim_{\epsilon \to 0} \left[ \int_{t^-_\epsilon(x_1)}^{t^\pm_\epsilon(x_1)} \int_{\partial\Omega} \left\{ \frac{\phi_{\epsilon,2}(t)}{\delta} \right\} d\Gamma dt \right]^{\frac{1}{2}} \delta^{\frac{1}{2}} [A(D_{\epsilon})]^{\frac{1}{2}}$$

$$\leq \lim_{\epsilon \to 0} C_0 \left[ A(D_{\epsilon}) \right]^{\frac{1}{2}},$$

where $(\phi_{\epsilon,2})_t$ represents $\frac{\partial \phi_{\epsilon,2}(1, x_1, t)}{\partial t}$, $C_0$ denotes the upper bound of

$$\sqrt{\delta} \left[ \int_0^{+\infty} \int_{\partial\Omega} \left\{ \frac{\phi_{\epsilon,2}(t)}{\delta} \right\} d\Gamma dt \right]^{\frac{1}{2}},$$
and \( A(D_\epsilon) \) is the area of region:

\[
D_\epsilon := \left\{ (x,t) \mid x \in \partial \Omega, 0 \leq t \leq T, \phi_{\epsilon,2}(1, x, t) \in (\frac{1}{3}, \frac{2}{3}) \right\}.
\]

Obviously, \( D_\epsilon \) has thickness \( O(\epsilon) \) so \( \lim_{\epsilon \to 0} |D_\epsilon| = 0 \). Then, from (2.44), we get \( b = 0 \) as \( \epsilon \to 0 \), which implies that in the limit \( \epsilon \to 0 \), the contact points do not show noticeable motion.

### 3. Fast time motion.

In this section, we will study the behaviour at a fast time scale \( s = \epsilon t \). In the previous section, we have known that when \( t \to +\infty \), the interfaces approach circular and the contact points do not move in the leading order. Note that \( s \in [0, 1] \) is equivalent to \( t \in [0, 1/\epsilon] \). Hence we may assume that the interfaces are circular initially at this fast time scale (as shown in Fig.3.1). We will concentrate on the behavior around contact points on the solid boundary to derive the dynamics of the contact angles and contact points. Firstly, we introduce some constraints which should be kept in this fast time scale. Then we take asymptotical expansions around the contact points to study the behavior around the contact points. Finally, we obtain an ordinary differential system to describe the motion of the whole system at this fast time scale.

![Diagram](image)

**Fig. 3.1** Left: The initial profile of the system where the unit circle is the solid boundary with three circular interfaces inside. Right: A chord with \( S \) as chord length and \( \beta \) as the angle of the chord. Notations with different colors denote the corresponding quantities with the same color.

#### 3.1. Volume conservation.

Integrating both sides of (1.3) in \( \Omega \) and using the boundary condition for \( \mu \), we have

\[
\frac{\partial}{\partial t} \int_\Omega \phi d\Omega = \int_\Omega \frac{\partial \phi}{\partial t} d\Omega = \int_\Omega \Delta \mu d\Omega = \int_{\partial \Omega} \frac{\partial \mu}{\partial n} d\Gamma = 0.
\]

This shows that the volume of three phases are conserved in the system. To calculate the volume of three phases in Fig.3.1, we note that the area of the chord with angle \( \beta \) and length \( S \) is given by

\[
A = \left( \frac{S}{2 \sin \beta} \right)^2 \beta - \left( \frac{S}{2} \right)^2 \frac{\beta}{\tan \beta} = \frac{S^2 (\beta - \sin \beta \cos \beta)}{4 \sin^2 \beta}.
\]

Now, we introduce some notations we will use in this section (See Fig.3.1). We denote three contact points position, which depends on time \( s \), as

\[
C_i(s) = (\cos \theta_i(s), \sin \theta_i(s)), \ i = 1, 2, 3
\]
where \( \theta_i(s) \in [0, 2\pi) \) are the polar angles of the point \( C_i(s) \). Denote the triple junction position as \( m(s) = (m_1(s), m_2(s)) \), and we then denote the three vectors between three contact points and the triple junction as \( \vec{l}_i(s) = C_i(s) - m(s) \). and \( \alpha_i(s) \) as the three angles between the inward direction of the tangent line of interface at the contact point and the clockwise direction of the tangent line of the solid boundary at the contact points. Now, we introduce the calculation of the area of the region between the \( i \)th interface and the \((i + 1)\)th interface. For instance, without loss of generality, the area for the region between the 3rd interface and the 1st interface (see black solid curves in Fig. 3.2) can be calculated by

\[
A_3 = A(D_1) + A(D_2) + A(D_3) - A(D_4)
\]

where \( A(\cdot) \) represents the area of each region in the figure, \( D_1 \) denotes the triangle region bounded by the brown dotted lines in the figure, and \( D_i \) (\( i = 2, 3, 4 \)) denote the regions of three chords in the figure. Using the notations of \( \vec{l}_i(s) \), \( \chi_i(s) \), \( \theta_i(s) \) and \( \alpha_i(s) \), the formula for the area of a chord we introduced, and the area formula for a triangle in the cross product between two vectors, we can have

\[
A(D_1) = \frac{1}{2} |\vec{l}_3(s) \times \vec{l}_1(s)|
\]

\[
A(D_2) = \frac{1}{2} (\theta_1(s) - \theta_3(s) - \sin(\theta_1(s) - \theta_3(s))),
\]

\[
A(D_3) = \frac{|\vec{l}_3(s)|^2 (\chi_3(s) - \sin \chi_3(s) \cos \chi_3(s))}{4 \sin^2 \chi_3(s)},
\]

\[
A(D_4) = \frac{|\vec{l}_1(s)|^2 (\chi_1(s) - \sin \chi_1(s) \cos \chi_1(s))}{4 \sin^2 \chi_1(s)}
\]

where

\[
\chi_i(s) = \pi - \alpha_i(s) - \frac{\theta_{i+1}(s) - \theta_i(s)}{2} - \arccos \left( \frac{\vec{l}_i(s) \cdot (\cos \theta_i(s) - \cos \theta_{i+1}(s), \sin \theta_i(s) - \sin \theta_{i+1}(s))}{|\vec{l}_i(s)| \times |2 \sin(\theta_{i+1}(s) - \theta_i(s))|} \right).
\]
**Remark 3.1.**

1. $\chi_i$ can be negative and thus the area of the chord with the angle $\chi_i$ can be negative. Take the region between 3rd interface and 1st interface in Fig. 3.2 for example, $\chi_3$ is negative if $D_3 \subset D_1 \cup D_2$ and thus $A(D_3)$ is negative. That will make the formula $A_3 = A(D_1) + A(D_2) + A(D_3) - A(D_1)$ be general for arbitrary profiles.

2. We take $i + 1 = 1$ when $i = 3$ in the formula.

In general, we write

$$A_i = \frac{1}{2} |\vec{l}_i(s) \times \vec{l}_{i+1}(s)| + \frac{1}{2} (\theta_{i+1}(s) - \theta_i(s) - \sin(\theta_{i+1}(s) - \theta_i(s)))$$

$$+ \frac{|\vec{l}_i(s)|^2 (\chi_i(s) - \sin \chi_i(s) \cos \chi_i(s))}{4 \sin^2 \chi_i(s)}$$

$$- \frac{|\vec{l}_{i+1}(s)|^2 (\chi_{i+1}(s) - \sin \chi_{i+1}(s) \cos \chi_{i+1}(s))}{4 \sin^2 \chi_{i+1}(s)}.$$  \hspace{1cm} (3.1)

From volume conservation, we know that $A_i$ ($i = 1, 2, 3$) are constants independent of $s$.

**3.2. Angle condition.** From (2.25), we rewrite the angle condition as the following:

$$\frac{\sin(\Theta_1)}{\Phi_1} = \frac{\sin(\Theta_2)}{\Phi_2} = \frac{\sin(\Theta_3)}{\Phi_3},$$

where $(\Theta_1, \Theta_2, \Theta_3)$ are uniquely determined by $(\Phi_1, \Phi_2, \Phi_3)$. It is easy to show that

$$\Theta_i = \arccos \left( \frac{\vec{l}_i(s) \cdot \vec{l}_{i+1}(s)}{|\vec{l}_i(s)||\vec{l}_{i+1}(s)|} \right) + \chi_i(s) - \chi_{i+1}(s). \quad i = 1, 2, 3 \hspace{1cm} (3.2)$$

**3.3. The constraint of curvature and interface tension.** In Section 2.3.1, we showed that $\mu = (\mu^1, \mu^2, \mu^3)$ approaches to a constant vector. Then, using (2.17), we have

$$0 = \mu^1 - \mu^2 + \mu^2 - \mu^3 + \mu^3_3 - \mu^1 = -(\Phi_1 \cdot \kappa_1 + \Phi_2 \cdot \kappa_2 + \Phi_3 \cdot \kappa_3). \hspace{1cm} (3.3)$$

This leads to

$$0 = \Phi_1 \cdot \kappa_1 + \Phi_2 \cdot \kappa_2 + \Phi_3 \cdot \kappa_3$$

$$= \frac{\Phi_1 \cdot 2 \sin \chi_1(s)}{|\vec{l}_1(s)|} + \frac{\Phi_2 \cdot 2 \sin \chi_2(s)}{|\vec{l}_2(s)|} + \frac{\Phi_3 \cdot 2 \sin \chi_3(s)}{|\vec{l}_3(s)|}. \hspace{1cm} (3.4)$$

**3.4. Expansion near contact points.** Now, we take expansions around three contact points $C_i$, $i = 1, 2, 3$. Without loss of the generality, we will consider expansion around the contact point $C_1 = (\cos \theta_1^i(s), \sin \theta_1^i(s))$ but away from the triple junction and other contact points. We use the stretched variable defined by

$$z_1 = \frac{1 - r}{\epsilon},$$

$$z_2 = \frac{- (x - \cos \theta_1^i(s)) \cos(\theta_1^i(s) - \alpha_1^i(s)) - (y - \sin \theta_1^i(s)) \sin(\theta_1^i(s) - \alpha_1^i(s))}{\epsilon},$$
where \( r = \sqrt{x^2 + y^2} \), and \( z_2 \) is a stretched variable along the norm direction of the interface (See Fig. 3.3). We then consider the following expansions:

\[
\begin{align*}
\phi' &= \Phi^0(z_1, z_2, s) + \epsilon \Phi^1(z_1, z_2, s) + \epsilon^2 \Phi^2(z_1, z_2, s) + h.o.t., \\
\mu' &= \nu^0(z_1, z_2, s) + \epsilon \nu^1(z_1, z_2, s) + \epsilon^2 \nu^2(z_1, z_2, s) + h.o.t., \\
\theta_1' &= \theta_1(s) + \epsilon \theta_1^1(s) + \epsilon^2 \theta_1^2(s) + h.o.t., \\
\alpha_1' &= \alpha_1(s) + \epsilon \alpha_1^1(s) + \epsilon^2 \alpha_1^2(s) + h.o.t..
\end{align*}
\]

Substituting these expansions into (1.3), from the leading order equation and the boundary condition we get in \( \{ z_1 > 0, z_2 \in \mathbb{R}, s > 0 \} \),

\[
\begin{align*}
\frac{\partial^2 \Phi^0}{\partial z_1^2} + 2 \cos \alpha_1(s) \frac{\partial^2 \Phi^0}{\partial z_1 \partial z_2} + \frac{\partial^2 \Phi^0}{\partial z_2^2} - f(\Phi^0) &= 0, \\
(\sin \alpha_1(s) \theta_1'(s) - \delta \cos \alpha_1(s)) \frac{\partial \Phi^0}{\partial z_2} &= -\delta \frac{\partial \Phi^0}{\partial z_1} - \delta \nabla_{\Phi^0}(\Phi^0) + \delta (\nabla_{\Phi^0}(\Phi^0) \cdot \vec{e}) \vec{e}, & \text{on} \ z_1 = 0.
\end{align*}
\]

Let \( Q(z_2) = \lim_{z_1 \to +\infty} \Phi^0(z_1, z_2, s) \). Obviously, \( Q \) satisfies the following problem:

\[
\frac{\partial^2 Q}{\partial z_2^2} - f(Q) = 0 \quad \text{on} \ R,
\]

\[
Q(+\infty) = (0, 1, 0), \quad Q(-\infty) = (0, 0, 1), \quad Q(0) = (0, \frac{1}{2}, \frac{1}{2}).
\]

This implies that

\[
\frac{1}{2} \left( \frac{\partial Q}{\partial z_2} \right)^2 = F(Q(z_2)).
\]

If we choose \( l \), such that (see also [7])

\[
\nabla_Q l(Q) - (\nabla_Q l(Q) \cdot \vec{e}) \vec{e} = \sigma_1 \frac{\partial Q}{\partial z_2},
\]

then, (3.5) has a special solution:

\[
\Phi^0(z_1, z_2, s) = Q(z_2),
\]

and \( \theta_1(s) \) is given by:

\[
-\sin \alpha_1(s) \theta_1'(s) = \delta \cos \alpha_1(s) - \delta \sigma_1.
\]
Since, we get

\[ -\sin \alpha_2(s) \theta_2'(s) = \delta \cos \alpha_2(s) - \delta \sigma_2, \quad (3.10) \]

\[ -\sin \alpha_3(s) \theta_3'(s) = \delta \cos \alpha_3(s) - \delta \sigma_3. \quad (3.11) \]

It is easy to see that, given \( \alpha_i(s) \ (i = 1, 2, 3) \), one can solve the five unknown functions \( \theta_i(s) (i = 1, 2, 3) \) from the five equations (3.1), (3.2), (3.4), when they are solvable. Denote the solutions as

\[ \theta_i(s) = \bar{\theta}_i(\alpha_1(s), \alpha_2(s), \alpha_3(s)), \quad i = 1, 2, 3, \quad (3.12) \]

\[ m_i(s) = \bar{m}_i(\alpha_1(s), \alpha_2(s), \alpha_3(s)), \quad i = 1, 2. \quad (3.13) \]

Since

\[ \theta'_1(s) = \sum_{j=1}^{3} \frac{\partial \theta_1}{\partial \alpha_j} \alpha'_j(s), \]

we can write the equations (3.9)-(3.11) as

\[
\begin{pmatrix}
-\sin \alpha_1 \frac{\partial \theta_1}{\partial \sigma_1} & -\sin \alpha_1 \frac{\partial \theta_1}{\partial \sigma_2} & -\sin \alpha_1 \frac{\partial \theta_1}{\partial \sigma_3} \\
-\sin \alpha_2 \frac{\partial \theta_1}{\partial \sigma_1} & -\sin \alpha_2 \frac{\partial \theta_1}{\partial \sigma_2} & -\sin \alpha_2 \frac{\partial \theta_1}{\partial \sigma_3} \\
-\sin \alpha_3 \frac{\partial \theta_1}{\partial \sigma_1} & -\sin \alpha_3 \frac{\partial \theta_1}{\partial \sigma_2} & -\sin \alpha_3 \frac{\partial \theta_1}{\partial \sigma_3}
\end{pmatrix}
\begin{pmatrix}
\alpha'_1(s) \\
\alpha'_2(s) \\
\alpha'_3(s)
\end{pmatrix}
\]

\[ = \begin{pmatrix}
\delta (\cos \alpha_1(s) - \sigma_1) \\
\delta (\cos \alpha_2(s) - \sigma_2) \\
\delta (\cos \alpha_3(s) - \sigma_3)
\end{pmatrix}, \quad (3.14)\]

which can be re-written as

\[
\begin{pmatrix}
-\sin \alpha_1 & 0 & 0 \\
0 & -\sin \alpha_2 & 0 \\
0 & 0 & -\sin \alpha_3
\end{pmatrix}
\begin{pmatrix}
\frac{\partial \theta_1}{\partial \sigma_1} & \frac{\partial \theta_1}{\partial \sigma_2} & \frac{\partial \theta_1}{\partial \sigma_3}
\end{pmatrix}
\begin{pmatrix}
\alpha'_1(s) \\
\alpha'_2(s) \\
\alpha'_3(s)
\end{pmatrix}
\]

\[ = \begin{pmatrix}
\delta (\cos \alpha_1(s) - \sigma_1) \\
\delta (\cos \alpha_2(s) - \sigma_2) \\
\delta (\cos \alpha_3(s) - \sigma_3)
\end{pmatrix}. \quad (3.15)\]

This gives the dynamics of the three contact angles \( \alpha_1(s), \alpha_2(s) \) and \( \alpha_3(s) \). Once \( \alpha_i(s) \) are solved, we can then determine the dynamics of the contact points and the triple junction from (3.12) and (3.13).

**Remark 3.2.** Denote \( \theta_{Y_i} \) as the Young’s angle at the \( i \)th contact point \( C_i \). Then, we actually have \( \sigma_i = \cos(\theta_{Y_i}) \) [7]. Then, it is easy to see, only when \( \alpha_i = \theta_{Y_i} \), the system will be at equilibrium which is the static solution of system (1.3).

**3.5. Numerical example.** In this example, we choose domain \( \Omega \) to be a unit disc with the boundary \( \partial \Omega \) and we set \( \sigma_1 = 1/2, \sigma_2 = -1/2, \) and \( \sigma_3 = 0 \) with three static contact angles equal to \( \pi/3, 2\pi/3, \) and \( \pi/2 \) correspondingly. We set three Herring angles as \( \frac{\pi}{2}, \frac{3\pi}{4}, \) and \( \frac{3\pi}{8} \). The volumes of the three phases are \( A_1 = \frac{\pi}{4} \) and \( A_2 = A_3 = \frac{3\pi}{8} \). The initial position of the triple junction is set at \( (m_1, m_2) = (0, 0) \) and the initial contact points are set at \( C_1 : (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}), C_2 : (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}), C_3 : (0, -1) \)
with the initial contact angles $\alpha_1 = \alpha_2 = \alpha_3 = \frac{\pi}{2}$ (See Fig. 3.4). We then set $\delta = 1$ and solve the system (3.15). The dynamics of contact angles, contact points are then plotted in Fig. 3.5. It is shown that when $s \to \infty$, the triple junction moves toward the equilibrium position $(0, -0.2602)$. Three contact points converge to $C_1 : (0.4328, 0.9015)$, $C_2 : (-0.4328, 0.9015)$, $C_3 : (0, -1)$ with the equilibrium contact angles $\alpha_1 = \frac{\pi}{3}$, $\alpha_2 = \frac{2\pi}{3}$, $\alpha_3 = \frac{\pi}{2}$ respectively (See Fig. 3.6).

**Fig. 3.4.** Initial interface profile by keeping three interfaces as circular with initial contact points: $C_1 : \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, $C_2 : \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, and $C_3 : (0, -1)$, contact angles: $\alpha_1 = \frac{\pi}{3}$, $\alpha_2 = \frac{2\pi}{3}$ and $\alpha_3 = \frac{\pi}{2}$, Young’s angle: $\Theta_1 = \frac{\pi}{2}$, $\Theta_2 = \frac{3\pi}{4}$, and $\Theta_3 = \frac{3\pi}{4}$, volume: $A_1 = \frac{\pi}{4}$ and $A_2 = A_3 = \frac{3\pi}{8}$, and position of triple junction $(m_1, m_2) = (0, 0)$.

**Fig. 3.5.** The interface profile’s evolution governed by the system (3.15) at $s = 0.5, 1.0, 1.5, 2.0, 2.5, 3.0, 3.5, 4.0$ respectively.

**Fig. 3.6.** From left to right: The dynamic of $C_1 = (\cos(\theta_1), \sin(\theta_1))$, the dynamic of the contact angle $\alpha_1$, and the dynamic of $m$ where we use $(0, m)$ to represent the triple junction.
4. Conclusions and further discussions. Using matched asymptotic expansion, we derive the sharp interface models at different time scales for the three-phase Cahn-Hilliard equation with constant mobility factor and the relaxation boundary condition on the solid boundary. At $O(t)$ time scale, the dynamics of the triple junction is determined by the balance of the chemical potential gradient along the three interfaces meeting at the triple junction. At faster time scale $O(\epsilon t)$, the motion of the triple junction is controlled by the contact point motions and geometric constraints. These sharp interface models explicitly determine the dynamics of the triple junctions, contact angles and contact points. We remark that although we focus on the constant mobility factor case in this paper only, the method can be applied to the multi-phase Cahn-Hilliard equation with degenerate mobility or highly disparate diffusion mobility. The dynamics will be quite different from what we obtained in this paper. These will be investigated and reported in the future.

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