Math 3220 — Study guide for midterm 2

1 Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Show that both $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist everywhere, but f is not differentiable at $(0,0)$.

2 Find the differential of the function $G\colon \mathbb{R}_{>0} \times \mathbb{R} \to \mathbb{R}^3$ given by

$$G(x,y) = (\ y \ln x,\ xe^{xy},\ \sin(xy)\ ).$$

Find the best affine approximation to G at the point $(1, \frac{\pi}{2})$.

3 Assume the function $F\colon \mathbb{R}^p \to \mathbb{R}$ differentiable everywhere, and that its gradient dF is a constant vector. Show that that F is an affine function.

4 Find the differential of the real valued function $f(x,y,z) = xy^2 \cos(xz)$. Then find the best affine approximation to f at the point $(1, 1, \frac{\pi}{2})$.

5 From the definitions, prove that if a function $F\colon U \to \mathbb{R}^q$, $U \subset \mathbb{R}^p$ open, is differentiable at $a \in \mathbb{R}^p$, then F is continuous at a.

6 If $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable, and $g(x,y) = f(xy)$, show that $x\frac{\partial g}{\partial x} - y\frac{\partial g}{\partial y} = 0$.

7 Let (x,y) be the Cartesian coordinates in $\mathbb{R}^2$, and (r,θ) be the polar coordinates.

(a) Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function. Find formulas for the partial derivatives of f with respect to r and θ in terms of the partial derivatives with respect to x and y.

(b) The Laplacian differential operator is given in Cartesian coordinates by $\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$. Prove that in polar coordinates the Laplacian is given by $\Delta f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}$.

8 For the curve

$$\gamma(t) = (\cos t, \sin 2t),$$

find a parametric equation of the tangent line at $(0,0)$ if the domain of $\gamma(t)$ is $\{\, t \in \mathbb{R} \mid \pi < t < 2\pi \,\}$.

9 Find an equation for the tangent plane to the surface $x^2 + y^2 - z^2 = 1$ at each point (a,b,c) on the surface.

10 Find the degree $n = 2$ Taylor's formula for $f(x,y) = \ln(x + y + 1)$ at the point $(0,0)$.

11 Consider the function $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $F(x,y) = (e^x \cos y, e^x \sin y)$.

(a) Does F have an inverse?

(b) Does F have a smooth local inverse function near every point (x,y)?

12 Consider $F\colon \mathbb{R}^2 \to \mathbb{R}^2$, $F(r,\theta) = (r\cos\theta, r\sin\theta)$. Does a smooth local inverse function exist near $(1, 2\pi)$? If it does, find one explicitly.

13 Consider the function $F\colon U \to \mathbb{R}^2$, where

$$F(x,y) = \left(x^2, \frac{y}{x}\right), \qquad U = \{\, (x,y) \in \mathbb{R}^2 \mid x \neq 0 \,\}.$$

(a) Find all the points near which F has a smooth local inverse.

(b) If F is restricted to $V = \{\, (x,y) \mid x > 0, y > 0 \,\}$, find a smooth inverse for F. What is the domain of F^{-1}?

(c) As in (b), find the differential of F^{-1} directly, and also by applying the Inverse Function Theorem. Compare the results.

14 Problems 4 and 5 in 9.7.