Math 3220 — Study guide for midterm 1

1. Is the set $\{ (x,0) \in \mathbb{R}^2 \mid -1 < x < 1 \}$ open, closed, both, or neither in $\mathbb{R}^2$?

2. Is the following set connected in $\mathbb{R}^2$: $\{ (x,y) \mid x^2 + y^2 < 1 \} \cup \{ (x,1) \mid x \in \mathbb{R} \}$.

3. Define *convergence sequence*. Does the sequence $\{x_n\}_{n\in\mathbb{N}}$,

$$x_n = \left(1 + \frac{(-1)^n}{n}, \sin\frac{1}{n} \right) \in \mathbb{R}^2,$$

converge? If so, to what?

4. Choose one of the following statements to prove. Indicate which one you choose.

   (a) Let A be a set in a topological space X. A is closed if and only if every sequence in A that converges in X converges to a point in A.

   (b) Let A be a set in a metric space X. A point $a \in A$ is an interior point of A if and only if for any sequence $\{x_n\}$ in X converging to a there is some N such that for all $n > N$, $x_n \in A$.

   (c) Let A be a set in a topological space X. A is disconnected if and only if there is a continuous map $f\colon A \to \mathbb{R}$ whose image is the set $\{0, 1\}$.

5. Define *compactness*. Let E be an infinite set in $\mathbb{R}^d$, and suppose that every point of E is isolated. Show that E is not compact.

6. Define *continuity* of a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ at a point $(x_0, y_0) \in \mathbb{R}^2$. Using just the definition, show that $f(x,y) = xy^2$ is continuous at any point $(x_0, y_0) \in \mathbb{R}^2$.

7. Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.

   (a) Suppose $f\colon \mathbb{R}^d \to \mathbb{R}$ is continuous. Then $E = \{ x \in \mathbb{R}^d \mid f(x) \le 0 \}$ is a closed set.

   (b) Suppose $f\colon \mathbb{R}^d \to \mathbb{R}$ is continuous. Then $E = \{ x \in \mathbb{R}^d \mid \|f(x)\| \le 1 \}$ is a connected set.

   (c) Let $E \subset \mathbb{R}^d$, and $\{x_n\}$ a sequence in the boundary ∂E. If the sequence converges to a point $x_n \to x$ in $\mathbb{R}^d$, then $x \in \partial E$.

8. Does the sequence of functions $\{F_n\}$ converge on $B_1((0,0,0)) \subset \mathbb{R}^3$, where $F_n\colon \mathbb{R}^3 \to \mathbb{R}^3$ is defined by

$$F(x,y,z) = \left(\sin\frac{xy}{n}, \frac{1}{n}, x^n + y^n + z^n \right).$$

If so, is the convergence uniform?

Note: The actual midterm will have fewer problems (probably 5 or 6).