Sample Midterm 3 — Solutions

1

Solve the wave equation on a disk of radius 2 with $c = 1$ and initial conditions:

$$u(r,\theta,0) = 3\ J_0\left(\tfrac{\alpha_{0,2}}{2} r\right) - 2\ J_0\left(\tfrac{\alpha_{0,1}}{2} r\right)$$

$$\frac{\partial u}{\partial t}(r,\theta,0) = J_0\left(\tfrac{\alpha_{0,3}}{2} r\right) - 4\ J_0\left(\tfrac{\alpha_{0,1}}{2} r\right)$$

Solution: Notice that the initial conditions do not depend on θ, so we are in the symmetric case (section 4.2). From the formula in page 202, using that $a = 2$ and $c = 1$, we know that the general solution is of the form:

$$u(r,\theta,t) = \sum_{n=1}^{\infty} \left(A_n \cos\left(\tfrac{\alpha_{0,n}}{2} t\right) + B_n \sin\left(\tfrac{\alpha_{0,n}}{2} t\right)\right)\ J_0\left(\tfrac{\alpha_{0,n}}{2} r\right).$$

To solve the problem we need to find A_n and B_n. When we set $t = 0$ we get:

$$u(r,\theta,0) = \sum_{n=1}^{\infty} A_n\, J_0\left(\tfrac{\alpha_{0,n}}{2} r\right).$$

and:

$$\frac{\partial u}{\partial t}(r,\theta,0) = \sum_{n=1}^{\infty} B_n \tfrac{\alpha_{0,n}}{2}\ J_0\left(\tfrac{\alpha_{0,n}}{2} r\right).$$

Direct inspection of the initial conditions, gives that:

$$A_1 = -2 \qquad A_2 = 3 \qquad B_1 \tfrac{\alpha_{0,1}}{2} = -4 \qquad B_3 \tfrac{\alpha_{0,3}}{2} = 1 \qquad B_1 = -\tfrac{8}{\alpha_{0,1}} \qquad B_3 = \tfrac{2}{\alpha_{0,3}}$$

and all other A_n's and B_n's are 0. Therefore the solution is:

$$\begin{aligned} u(r,\theta,t) = {} & 3\ \cos\left(\tfrac{\alpha_{0,2}}{2} t\right)\ J_0\left(\tfrac{\alpha_{0,2}}{2} r\right) - 2\ \cos\left(\tfrac{\alpha_{0,1}}{2} t\right)\ J_0\left(\tfrac{\alpha_{0,1}}{2} r\right) \\ & + \tfrac{2}{\alpha_{0,3}}\ \sin\left(\tfrac{\alpha_{0,3}}{2} t\right)\ J_0\left(\tfrac{\alpha_{0,3}}{2} r\right) - \tfrac{8}{\alpha_{0,1}}\ \sin\left(\tfrac{\alpha_{0,1}}{2} t\right)\ J_0\left(\tfrac{\alpha_{0,1}}{2} r\right) \end{aligned}$$

2

Solve the wave equation on a disk of radius 1 with $c = 1$ and initial conditions:

$$u(r,\theta,0) = J_1(\alpha_{1,2}\, r)\ \sin\theta - 2\, J_0(\alpha_{0,1}\, r) + 3\, J_2(\alpha_{2,2}\, r)\ \cos 2\theta$$

$$\frac{\partial u}{\partial t}(r,\theta,0) = 0$$

Solution: The strategy for this problem is the same as for problem 1, except that one needs to use equation (16) on page 213 as the general solution (the initial condition depends on θ, so we are not in the symmetric case). As in problem 1, direct inspection of the initial conditions gives that:

$$b_{1,2} = 1 \qquad a_{0,1} = -2 \qquad a_{2,2} = 3$$

and all other coefficients are zero. The final solution is:

$$u(r,\theta,t) = J_1(\alpha_{1,2}\, r)\ \sin\theta\ \cos(\alpha_{1,2}\, t) - 2\, J_0(\alpha_{0,1}\, r)\ \cos(\alpha_{0,1}\, t) + 3\, J_2(\alpha_{2,2}\, r)\ \cos 2\theta\ \cos(\alpha_{2,2}\, t)$$

3

Solve the wave equation on a disk of radius 1 with $c = 1$ and initial conditions:

$$u(r, \theta, 0) = 0$$

$$\frac{\partial u}{\partial t}(r, \theta, 0) = r \sin\theta - r^2 \cos 2\theta$$

Solution: As I explained in class, there are two approaches one can take to solve this problem. The first is to use the formulas of section 4.3 (this involves computing 6 integrals). The second, which is the one I will use here, is to express the initial conditions as Bessel-Fourier series and then proceed in the spirit of problems 1 and 2.

We start by computing the Bessel series of order 1 for the function r. For this we use equation (17) on page 253. We have that

$$r = \sum_{j=1}^{\infty} A_j \, J_1(\alpha_{1,j} r)$$

where

$$\begin{aligned} A_j &= \frac{2}{J_2^2(\alpha_{1,j})} \int_0^1 r \, J_1(\alpha_{1,j} r) \, r \, dr = \frac{2}{\alpha_{1,j}^3 \, J_2^2(\alpha_{1,j})} \int_0^{\alpha_{1,j}} s^2 \, J_1(s) \, ds \\ &= \frac{2}{\alpha_{1,j}^3 \, J_2^2(\alpha_{1,j})} \left[s^2 \, J_2(s) \right]_0^{\alpha_{1,j}} = \frac{2}{\alpha_{1,j} \, J_2(\alpha_{1,j})} \end{aligned}$$

Hence:

$$r = \sum_{j=1}^{\infty} \frac{2}{\alpha_{1,j} \, J_2(\alpha_{1,j})} \, J_1(\alpha_{1,j} r)$$

Analogously, if we write

$$r^2 = \sum_{j=1}^{\infty} A_j \, J_2(\alpha_{2,j} r)$$

then

$$\begin{aligned} A_j &= \frac{2}{J_3^2(\alpha_{2,j})} \int_0^1 r^2 \, J_2(\alpha_{2,j} r) \, r \, dr = \frac{2}{\alpha_{2,j}^4 \, J_3^2(\alpha_{2,j})} \int_0^{\alpha_{2,j}} s^3 \, J_2(s) \, ds \\ &= \frac{2}{\alpha_{2,j}^4 \, J_3^2(\alpha_{2,j})} \left[s^3 \, J_3(s) \right]_0^{\alpha_{2,j}} = \frac{2}{\alpha_{2,j} \, J_3(\alpha_{2,j})} \end{aligned}$$

and we get

$$r^2 = \sum_{j=1}^{\infty} \frac{2}{\alpha_{2,j} \, J_3(\alpha_{2,j})} \, J_2(\alpha_{2,j} r)$$

Therefore, we can rewrite the initial condition as:

$$\frac{\partial u}{\partial t}(r, \theta, 0) = \sum_{n=1}^{\infty} \frac{2}{\alpha_{1,n} \, J_2(\alpha_{1,n})} \, J_1(\alpha_{1,n} r) \, \sin\theta + \sum_{n=1}^{\infty} \frac{2}{\alpha_{2,n} \, J_3(\alpha_{2,n})} \, J_2(\alpha_{2,n} r) \, \cos 2\theta$$

Now, comparing this expression with equation (16) on page 213, we get that:

$$b^*_{1,n}\,\alpha_{1,n} = \frac{2}{\alpha_{1,n}\,J_2(\alpha_{1,n})} \qquad b^*_{1,n} = \frac{2}{\alpha^2_{1,n}\,J_2(\alpha_{1,n})}$$

$$a^*_{2,n}\,\alpha_{2,n} = \frac{2}{\alpha_{2,n}\,J_3(\alpha_{2,n})} \qquad a^*_{2,n} = \frac{2}{\alpha^2_{2,n}\,J_3(\alpha_{2,n})}$$

All other coefficients are 0. Finally, putting these coefficients back in equation (16) of page 213, we get the final answer:

$$u(r,\theta,t) = \sum_{n=1}^{\infty} \frac{2}{\alpha^2_{1,n}\,J_2(\alpha_{1,n})}\,J_1(\alpha_{1,n}r)\,\sin\theta\,\sin(\alpha_{1,n}t) + \sum_{n=1}^{\infty} \frac{2}{\alpha^2_{2,n}\,J_3(\alpha_{2,n})}\,J_2(\alpha_{2,n}r)\,\cos 2\theta\,\sin(\alpha_{2,n}t)$$

4

Find a solution to Dirichlet's problem on a disk of radius 2 and boundary condition:

$$u(2,\theta) = \begin{cases} \pi - \theta & \text{if } 0 < \theta < \pi \\ 0 & \text{if } \pi < \theta < 2\pi \end{cases}$$

Solution: We use the formula on page 218. For this we need to compute the Fourier series for $f(\theta) = u(2,\theta)$. This Fourier series is computed in example 4 on page 33:

$$u(2,\theta) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \left\{ \frac{1}{\pi}\left(\frac{1}{n^2} - \frac{(-1)^n}{n^2}\right)\cos n\theta + \frac{1}{n}\sin n\theta \right\}$$

By (4) on page 218 we get that the final answer is:

$$u(r,\theta) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \left(\frac{r}{2}\right)^n \left\{ \frac{1}{\pi}\left(\frac{1}{n^2} - \frac{(-1)^n}{n^2}\right)\cos n\theta + \frac{1}{n}\sin n\theta \right\}$$

5

Find a solution to Dirichlet's problem on a disk of radius 1 and boundary condition:

$$u(1,\theta) = \sin 2\theta + \cos 2\theta$$

Determine which points in the disk have temperature 0 (i.e., determine which points belong to the isotherm given by $T = 0$).

Solution: Notice that $f(\theta) = u(1,\theta)$ is already in Fourier series form. Therefore the solution can be computed immediately from equation (4) on page 218:

$$u(r,\theta) = r^2\,(\sin 2\theta + \cos 2\theta)$$

To compute the isotherm given by $T = 0$ we need to solve the equation:

$$r^2\,(\sin 2\theta + \cos 2\theta) = 0$$

This can be done in either polar coordinates or in Cartesian coordinates. In polar coordinates, if $r \neq 0$ (i.e., if the point is not the origin), we get:

$$\sin 2\theta + \cos 2\theta = 0$$

which is equivalent to:

$$\tan 2\theta = -1$$

This means that:

$$2\theta = -\frac{\pi}{4} + n\pi$$

or

$$\theta = -\frac{\pi}{8} + n\frac{\pi}{2}$$

In the range $0 \leq \theta \leq 2\pi$, the only possibilities for θ are:

$$\theta = \frac{3\pi}{8} \qquad \theta = \frac{7\pi}{8} \qquad \theta = \frac{11\pi}{8} \qquad \theta = \frac{15\pi}{8}$$

In other words, the temperate is zero whenever the polar angle θ is any of the previous values.
If we want to use Cartesian coordinates, we can transform our equation using the double angle formulas:

$$\begin{aligned} 0 &= r^2 (\sin 2\theta + \cos 2\theta) = r^2 \left(2 \sin\theta \cos\theta + \cos^2\theta - \sin^2\theta\right) \\ &= 2(r\sin\theta)(r\cos\theta) + (r\cos\theta)^2 - (r\sin\theta)^2 = 2yx + x^2 - y^2 \end{aligned}$$

Hence the points that have temperature 0 are those that verify the equation

$$y^2 - 2xy - x^2 = 0$$

Using the quadratic formula, we get that:

$$y = \frac{2x \pm \sqrt{2x^2 + 2x^2}}{2} = \left(1 \pm \sqrt{2}\right) x$$

In other words, the temperature is zero if and only if the point belongs to one of the following two lines:

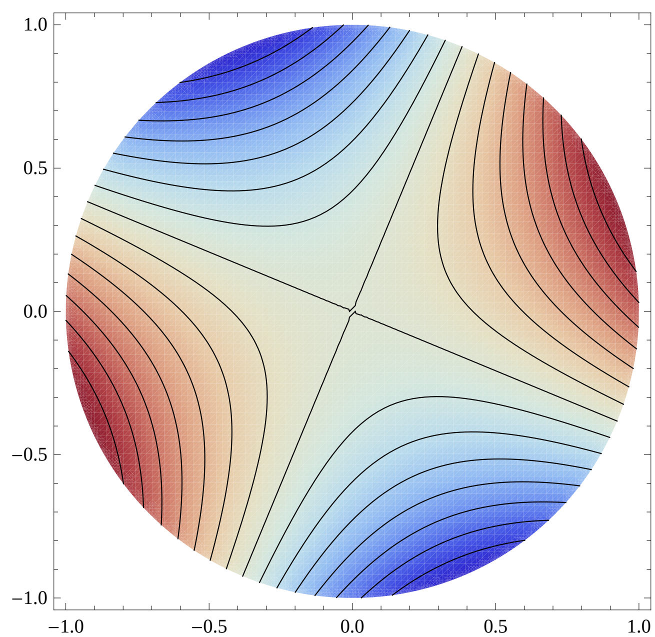
$$y = \left(1 + \sqrt{2}\right) x \qquad y = \left(1 - \sqrt{2}\right) x$$

Notice that

$$\tan \tfrac{3\pi}{8} = \tan \tfrac{11\pi}{8} = 1 + \sqrt{2}$$

$$\tan \tfrac{7\pi}{8} = \tan \tfrac{15\pi}{8} = 1 - \sqrt{2}$$

so we get the same solution as when using polar coordinates.
In the picture, positive temperatures are represented in red, while negative temperatures are blue. The two straight lines going through the origin are the isotherms $T = 0$.



6

Using the definition for the Bessel function of order n,

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!} \left(\frac{x}{2}\right)^{2k+n},$$

show that:

$$\frac{d}{dx}\left[J_0(x)\right] = -J_1(x)$$

Use this fact to show that the local maxima and minima of $J_0(x)$ occur at the zeros of $J_1(x)$.

Solution: See page 249 in the book.