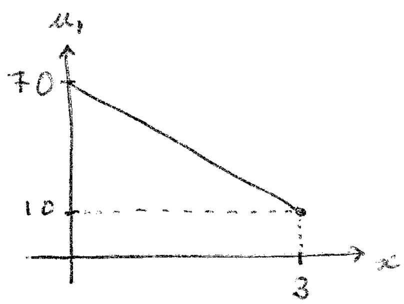
1 A bar of length 3 with insulated lateral ends and no internal sources of heat has its $x = 0$ end held at a constant 70 degrees and its $x = 3$ end held at a constant 10 degrees. The initial temperature of the bar is given by the function

$$u(x, 0) = 20\,(1 - x) \qquad 0 \le x \le 3$$

The constant c in the heat equation is 2 for this bar.

a) Find the steady-state temperature distribution.

$u_1(x)$ = linear function with $\begin{cases} u_1(0) = 70 \\ u_1(3) = 10 \end{cases}$



$$u_1(x) = 70 + \frac{10 - 70}{3}\, x$$

$$\boxed{u_1(x) = 70 - 20\,x}$$

b) Find the function $u(x,t)$ that describes the temperature of point x of the bar at time t.

$$u_2(x,t) = u(x,t) - u_1(x)$$

$$b_n = \frac{2}{L}\int_0^L \left(f(x) - u_1(x)\right)\left(\sin \frac{n\pi}{L} x\right) dx$$

$$= \frac{2}{3}\int_0^3 \left(20(1-x) - 70 + 20x\right)\left(\sin \frac{n\pi}{3} x\right) dx$$

$$= \frac{2}{3}\int_0^3 (-50)\left(\sin \frac{n\pi}{3} x\right) dx = -\frac{100}{3}\left[-\frac{3}{n\pi}\cos \frac{n\pi}{3} x\right]_0^3$$

$$= \frac{100}{n\pi}\left((\cos n\pi) - 1\right) = \frac{100}{n\pi}\left((-1)^n - 1\right) = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{-200}{n\pi} & \text{if } n \text{ is odd} \end{cases}$$

$$\lambda_n = c\frac{n\pi}{L} = \frac{2n\pi}{3}$$

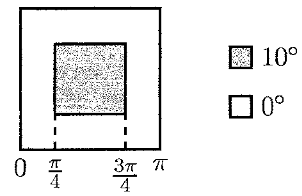
$$u_2(x,t) = -\frac{200}{\pi}\sum_{n \text{ odd}} \frac{1}{n} e^{-\left(\frac{2n\pi}{3}\right)^2 t}\left(\sin \frac{n\pi}{3} x\right)$$

$$\boxed{u(x,t) = 70 - 20x - \frac{200}{\pi}\sum_{n \text{ odd}} \frac{1}{n} e^{-\left(\frac{2n\pi}{3}\right)^2 t}\left(\sin \frac{n\pi}{3} x\right)}$$

2 Solve the heat equation in a $\pi \times \pi$ square plate with zero boundary conditions and the following initial condition:

$$u(x,y,0) = \begin{cases} 10 & \text{if } \frac{\pi}{4} < x \le \frac{3\pi}{4} \text{ and } \frac{\pi}{4} < y \le \frac{3\pi}{4} \\ 0 & \text{otherwise} \end{cases}$$

Assume that the constant c in the heat equation is 1.



$a = b = \pi \qquad c = 1 \qquad \lambda_{mn} = c\pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} = \sqrt{m^2+n^2}$

$$A_{mn} = \frac{4}{ab}\int_0^a\int_0^b f(x,y)\left(\sin\frac{m\pi}{a}x\right)\left(\sin\frac{n\pi}{b}y\right)dy\,dx$$

$$= \frac{40}{\pi^2}\left(\int_{\pi/4}^{3\pi/4}\sin mx\,dx\right)\cdot\left(\int_{\pi/4}^{3\pi/4}\sin ny\,dy\right)$$

$$= \frac{40}{\pi^2}\left[-\frac{1}{m}\cos mx\right]_{\pi/4}^{3\pi/4}\cdot\left[-\frac{1}{n}\cos ny\right]_{\pi/4}^{3\pi/4}$$

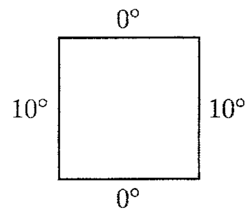
$$= \frac{40}{mn\pi^2}\left(\cos\frac{m\pi}{4} - \cos\frac{3\pi m}{4}\right)\left(\cos\frac{n\pi}{4} - \cos\frac{3n\pi}{4}\right)$$

$$\boxed{u(x,y,t) = \frac{40}{\pi^2}\sum_{n,m}\frac{1}{nm}\left(\cos\frac{m\pi}{4} - \cos\frac{3m\pi}{4}\right)\left(\cos\frac{n\pi}{4} - \cos\frac{3n\pi}{4}\right)(\sin mx)(\sin ny)e^{-t(m^2+n^2)}}$$

3 Determine the steady-state temperature distribution in a 5×5 square plate with the following boundary conditions:

$$f_1(x) = f_2(x) = 0$$

$$g_1(y) = g_2(y) = 10$$



$a = b = 5$

$f_1(x) = f_2(x) = 0 \Rightarrow A_n = B_n = 0 \quad \Big\| \quad g_1(y) = g_2(y) \Rightarrow D_n = C_n$

$$D_n = C_n = \frac{2}{b \sinh \frac{n\pi a}{b}} \int_0^b g_1(y) \sin \frac{n\pi}{b} y \, dy =$$

$$= \frac{2}{5 \sinh n\pi} \int_0^5 10 \sin\left(\frac{n\pi}{5} y\right) dy = \frac{4}{\sinh n\pi} \left[-\frac{5}{n\pi} \cos\left(\frac{n\pi}{5} y\right) \right]_0^5 =$$

$$= -\frac{20}{n\pi \sin n\pi} \left((\cos n\pi) - 1 \right) = \frac{20\left(1 - (-1)^n\right)}{n\pi \sin n\pi} = \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{40}{n\pi \sinh n\pi} & \text{if } n \text{ is odd} \end{cases}$$

$$\boxed{\begin{aligned} u(x,y) = &\sum_{n \text{ odd}} \frac{40}{n\pi \sinh n\pi} \left(\sinh \frac{n\pi}{5}(5-x) \right) \left(\sin \frac{n\pi}{5} y \right) \\ &+ \sum_{n \text{ odd}} \frac{40}{n\pi \sinh n\pi} \left(\sinh \frac{n\pi}{5} x \right) \left(\sin \frac{n\pi}{5} y \right) \end{aligned}}$$

4 Find a solution to the homogeneous one-dimensional wave equation with $c = 1$ and $L = \pi$,

$$\frac{\partial^2 u}{\partial^2 t} = \frac{\partial^2 u}{\partial x^2} \quad \text{for} \quad 0 \le x \le \pi, \quad t \ge 0, \qquad u(0,t) = 0, \qquad u(\pi,t) = 0, \qquad \text{for} \quad t \ge 0,$$

satisfying the following initial conditions:

$$u(x,0) = x, \qquad \frac{\partial u}{\partial t}(x,0) = -x, \qquad \text{for } 0 \le x \le \pi.$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x \sin nx \, dx = \frac{2}{\pi}\left[\frac{1}{n^2}\sin nx - \frac{x}{n}\cos nx\right]_0^{\pi} =$$

$$= \frac{2}{\pi}\left(\cancel{\frac{1}{n^2}\sin n\pi} - \frac{\pi}{n}\cos n\pi - \cancel{\frac{1}{n}\sin 0} + \cancel{\frac{0}{n}\cos 0}\right) = -\frac{2}{n}(-1)^n = \frac{2}{n}(-1)^{n+1}$$

$$b_n^* = \frac{2}{n\pi}\int_0^{\pi}(-x)\sin nx \, dx = -\frac{1}{n} b_n$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} (\sin nx) \, \frac{2(-1)^{n+1}}{n}\left(\cos nt - \frac{1}{n}\sin nt\right)}$$

5 Use the method of separation of variables to transform the following PDE into a system of ODEs.

$$\frac{\partial^2 u}{\partial t^2} = a\frac{\partial u}{\partial x} + b\frac{\partial^3 u}{\partial y^3}$$

$$u(x, y, t) = X(x) \cdot Y(y) \cdot T(t)$$

$$\frac{\partial^2 u}{\partial t^2} = X\,Y\,T'' \qquad \frac{\partial u}{\partial x} = X'YT \qquad \frac{\partial^3 u}{\partial y^3} = X\,Y'''T$$

$$XYT'' = a\,X'YT + b\,XY'''T$$

$$\Big\downarrow \; : \frac{1}{XYT}$$

$$\frac{T''}{T} = a\frac{X'}{X} + b\frac{Y'''}{Y} \longrightarrow \frac{T''}{T} = K \longrightarrow \underline{T'' - KT = 0}$$

$$\downarrow$$

$$a\frac{X'}{X} + b\frac{Y'''}{Y} = K$$

$$a\frac{X'}{X} = \mu \longrightarrow \underline{aX' - \mu X = 0}$$

$$b\frac{Y'''}{Y} = \lambda \longrightarrow \underline{bY''' - \lambda Y = 0}$$

$$\underline{K = \mu + \lambda}$$

$$\boxed{\begin{array}{ll} T'' - KT = 0 & \\ aX' - \mu X = 0 & K = \mu + \lambda \\ bY''' - \lambda Y = 0 & \end{array}}$$