1

Consider the following differential equation:

$$(\star) \quad \frac{\partial u}{\partial t} + 2t\,\frac{\partial u}{\partial x} = 0$$

a) Check that any function of the form

$u(x,t) = f(x - t^2)$,

where f is an arbitrary differentiable function of a single variable, is a solution to the equation $(\star)$.

$$u(x,t) = f(x-t^2)$$

$$\frac{\partial u}{\partial t} = f'(x-t^2) \cdot \frac{\partial (x-t^2)}{\partial t} = f'(x-t^2)\cdot(-2t)$$

$$\frac{\partial u}{\partial x} = f'(x-t^2) \cdot \frac{\partial (x-t^2)}{\partial x} = f'(x-t^2)\cdot(1)$$

$$\frac{\partial u}{\partial t} + 2t\,\frac{\partial u}{\partial x} = f'(x-t^2)\,(-2t) + 2t\,f'(x-t^2)$$

$$= f'(x-t^2)\,(-2t+2t) = 0$$

b) Find a particular solution to the differential equation ($\star$) corresponding to the following initial condition:

$u(x, 0) = x^2$

$$u(x,t) = f(x - t^2)$$

$$\Downarrow$$

$$x^2 = u(x,0) = f(x - 0^2) = f(x)$$

$$\Downarrow$$

$$f(x) = x^2$$

$$\Downarrow$$

$$f(x - t^2) = (x - t^2)^2$$

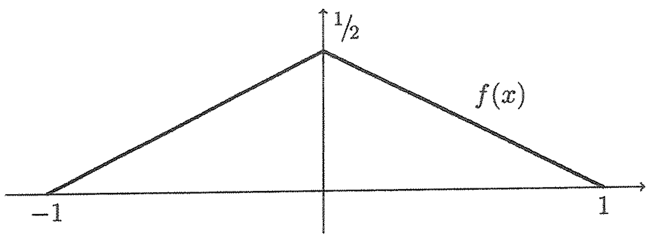
$$\Downarrow$$

$$u(x,t) = (x - t^2)^2 = x^2 - 2xt^2 + t^4$$

$$\boxed{u(x,t) = (x - t^2)^2}$$

2

Let $f(x)$ be the periodic function with period 2 whose graph in the interval $(-1,1)$ is the following:



Find the Fourier series of $f(x)$.

$$f(x) = \begin{cases} \frac{1}{2}(1-x) & \text{if} \quad 0 \leq x \leq 1 \\ \frac{1}{2}(1+x) & \text{if} \quad -1 \leq x \leq 0 \end{cases}$$

PERIOD $= 2 \Rightarrow \boxed{P = 1}$ | $f(x)$ EVEN $\Rightarrow b_n = 0$

$$a_0 = \frac{1}{P}\int_0^P f(x)\,dx = \int_0^1 \frac{1}{2}(1-x)\,dx =$$

$= ($AREA OF TRIANGLE OVER $(0,1)) =$

$$= \frac{1}{2} \cdot \frac{1}{2} \cdot 1 = \frac{1}{4}$$

$$a_n = \frac{2}{P}\int_0^P f(x)\left(\cos\frac{n\pi}{P}x\right)dx = 2\int_0^1 \frac{1}{2}(1-x)(\cos n\pi x)\,dx$$

$$= \int_0^1 (\cos n\pi x)\,dx - \int_0^1 x(\cos n\pi x)\,dx$$

$$= \left[\frac{1}{n\pi}\sin n\pi x\right]_0^1 - \left[\frac{1}{(n\pi)^2}(\cos n\pi x) + \frac{x}{n\pi}(\sin n\pi x)\right]_0^1$$

$$= \left[\frac{1}{n\pi}(\sin n\pi) - \frac{1}{n\pi}(\sin 0)\right] - \left[\frac{1}{n^2\pi^2}(\cos n\pi) + \frac{1}{n\pi}(\sin n\pi) -\right.$$

$$\left. - \frac{1}{n^2\pi^2}(\cos 0) + \frac{0}{n\pi}(\sin 0)\right] = \begin{cases} \sin n\pi = 0 \\ \sin 0 = 0 \\ \cos 0 = 1 \end{cases}$$

$$= -\frac{1}{n^2\pi^2}(\cos n\pi) + \frac{1}{n^2\pi^2} = \frac{1 - \cos n\pi}{n^2\pi^2} = \quad \Big| \cos n\pi = (-1)^n$$

$$= \frac{1-(-1)^n}{n^2\pi^2} = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{2}{n^2\pi^2} & \text{if } n \text{ is odd} \end{cases}$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n(\cos n\pi x) =$$

$$= \frac{1}{4} + \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^2\pi^2}(\cos n\pi x)$$

$$= \frac{1}{4} + \frac{2}{\pi^2}\sum_{n \text{ odd}} \frac{1}{n^2}(\cos n\pi x)$$

$$= \frac{1}{4} + \frac{2}{\pi^2}\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}\left(\cos(2k-1)\pi x\right).$$

3

Find the sine series expansion of $f(x) = e^{-x}$ in the interval $0 \leq x \leq \pi$.

$$p = \pi$$

$$b_n = \frac{2}{p} \int_0^p f(x) \left(\sin \frac{n\pi}{p} x \right) dx =$$

$$= \frac{2}{\pi} \int_0^\pi e^{-x} (\sin nx)\, dx =$$

$$= \frac{2}{\pi} \left[\frac{e^{-x}}{(-1)^2 + n^2} \left(-(\sin nx) - n(\cos nx) \right) \right]_0^\pi =$$

$$= \frac{2}{\pi} \left[\frac{e^{-\pi}}{1+n^2} \left(-(\sin n\pi) - n(\cos n\pi) \right) - \right.$$

$$\left. - \frac{e^{-0}}{1+n^2} \left(-\sin 0 - n \cos 0 \right) \right] =$$

$$\begin{cases} \sin n\pi = 0 \\ \sin 0 = 0 \\ \cos 0 = 1 \\ e^{-0} = e^0 = 1 \end{cases}$$

$$= \frac{2}{\pi} \left[- \frac{e^{-\pi} n \cos n\pi}{1+n^2} + \frac{n}{1+n^2} \right]$$

$$= \frac{2}{\pi} \frac{n}{1+n^2} \left(1 - e^{-\pi} \cos n\pi \right) \underset{\cos n\pi = (-1)^n}{=} \frac{2}{\pi} \frac{n}{1+n^2} \left(1 - e^{-\pi} (-1)^n \right)$$

$$f(x) \sim \sum_{n=1}^{\infty} b_n (\sin nx) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{n}{1+n^2} \left[1 - e^{-\pi} (-1)^n \right] (\sin nx)$$

4

Find a solution to the one-dimensional wave equation with $c = 1$ and $L = \pi$,

$$\frac{\partial^2 u}{\partial^2 t} = \frac{\partial^2 u}{\partial x^2} \qquad \text{for} \quad 0 \le x \le \pi, \quad t \ge 0,$$

satisfying the usual boundary conditions,

$$u(0,t) = 0, \qquad u(\pi,t) = 0, \qquad \text{for } t \ge 0,$$

and satisfying the following initial conditions:

$$u(x,0) = 2(\sin 3x) - 3(\sin 5x), \qquad \frac{\partial u}{\partial t}(x,0) = 0, \qquad \text{for } 0 \le x \le \pi.$$

GENERAL SOLUTION:

$$u(x,t) = \sum_{n=1}^{\infty} b_n \left(\sin \frac{n\pi}{L} x\right)\left(\cos \frac{n\pi c}{L} t\right) = \left\{\begin{matrix} L = \pi \\ c = 1 \end{matrix}\right.$$

$$= \sum_{n=1}^{\infty} b_n (\sin nx)(\cos nt)$$

IN OUR CASE:

$$2(\sin 3x) - 3(\sin 5x) = u(x,0) = \sum_{n=1}^{\infty} b_n (\sin nx)$$

$$\Rightarrow b_3 = 2, \quad b_5 = -3, \quad \text{all other } b_n\text{'s are zero.}$$

HENCE:

$$\boxed{u(x,t) = 2(\sin 3x)(\cos 3t) - 3(\sin 5x)(\cos 5t)}$$