Partial solutions for Sample Midterm 1

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A tank initially contains 100 gallons of brine with 1% salt. A pipe pumps a brine solution with 0.5% salt at 2 gallons per minute. The solution gets mixed in the tank and pumps out at a rate of 1 gallon per minute. What is the percentage of salt in the tank after 100 minutes?

Let $y(t)$ be the amount of salt in the tank after t minutes, measured in gallons. Notice that $y(t)$ measures the amount of salt, not the percentage of salt. The amount of salt that is added to the tank every minute is given by:

$$[\text{Rate In}] \;=\; 2\,\frac{\text{gal of brine}}{\text{min}} \times \frac{0.5}{100}\,\frac{\text{gal of salt}}{\text{gal of brine}} \;=\; \frac{1}{100}\,\frac{\text{gal of salt}}{\text{min}}$$

Notice that there is more brine pumped into the tank than pumped out of the tank: we gain one gallon of brine per minute. The number of gallons of brine in the tank after t minutes is $100+t$. The amount of salt that leaves the tank per minute is given by:

$$[\text{Rate Out}] \;=\; \frac{x \;\; \text{gal of salt}}{(100+t)\;\;\text{gal of brine}} \times 1\,\frac{\text{gal of brine}}{\text{min}} \;=\; \frac{y}{100+t}\,\frac{\text{gal of salt}}{\text{min}}$$

Hence the function $y(t)$ verifies the following differential equation:

$$y' = [\text{Rate In}] - [\text{Rate Out}] = \frac{1}{100} - \frac{y}{100+t}$$

This is a first order linear differential equation:

$$y' + \frac{1}{100+t}\,y = \frac{1}{100} \qquad P(t) = \frac{1}{100+t} \qquad Q(t) = \frac{1}{100}$$

The integrating factor is:

$$\mu(t) = \exp\left(\int P(t)\,dt\right) = \exp\left(\int \frac{1}{100+t}\,dt\right) = \exp\left(\ln(100+t)\right) = 100+t$$

After multiplying both sides of the differential equation by the integrating factor we obtain:

$$\begin{aligned}
(100+t)\left(y' + \frac{1}{100+t}\,y\right) &= (100+t)\,\frac{1}{100} \\
(100+t)\,y' + y &= \frac{100+t}{100} \\
\frac{d}{dt}\Big((100+t)\,y\Big) &= 1 + \frac{t}{100} \\
(100+t)\,y &= \int \left(1 + \frac{1}{100}t\right)\,dt \\
(100+t)\,y &= t + \frac{1}{200}t^2 + C \\
\\
y &= \frac{t + \frac{1}{200}t^2 + C}{100+t}
\end{aligned}$$

Now we use the initial condition to find C. We know that at $t = 0$ the amount of brine in the tank is 100 gallons, and that 1% of the brine is salt. This means that we have 1 gallon of salt at $t = 0$:

$$1 = y(0) = \frac{0 + \frac{1}{200}0^2 + C}{100 + 0} = \frac{C}{100} \qquad\qquad C = 100$$

Hence the solution to our differential equation is:

$$y \;=\; \frac{t + \frac{1}{200}t^2 + 100}{100 + t}$$

We need to find the percentage of salt after 100 minutes. First we find the amount of salt:

$$y(100) = \frac{100 + \frac{1}{200}100^2 + 100}{100 + 100} = \frac{250}{200} \text{ gal of salt}$$

The amount of brine after 100 minutes is:

$$100 + 100 = 200 \text{ gal of brine}$$

Therefore the percentage of salt in the brine is:

$$[\,\text{percentage of salt}\,] \;=\; \frac{\frac{250}{200} \text{ gal of salt}}{200 \text{ gal of brine}} \times 100 \;=\; \frac{250}{400}\;\% \;=\; 0.625\;\%$$

The solution is:

$$\boxed{[\,\text{percentage of salt after 100 minutes}\,] \;=\; 0.625\;\%}$$

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(a) $y' = \dfrac{3y+1}{1+x^2}, \quad y(0) = 0$

This equation can be solved using the method of separation of variables. First we find the general solution:

$$\frac{dy}{dx} = \frac{3y+1}{1+x^2}$$

$$\frac{1}{3y+1}\,dy = \frac{1}{1+x^2}\,dx$$

$$\int \frac{1}{3y+1}\,dy = \int \frac{1}{1+x^2}\,dx$$

$$\tfrac{1}{3}\ln(3y+1) = \arctan(x) + C_0$$

$$\ln(3y+1) = 3\arctan(x) + C$$

$$3y+1 = \exp\big(3\arctan(x) + C\big)$$

$$y = \tfrac{1}{3}\Big(\exp\big(3\arctan(x) + C\big) - 1\Big)$$

Now we use the initial condition to find C. In this computation we use the fact that $\arctan(0) = 0$.

$$0 = y(0) = \tfrac{1}{3}\Big(\exp\big(3\arctan(0) + C\big) - 1\Big) = \tfrac{1}{3}\Big(\exp\big(C\big) - 1\Big) = \tfrac{1}{3}\left(e^C - 1\right)$$

$$0 = e^C - 1$$

$$e^C = 1$$

$$C = \ln 1 = 0$$

Hence the solution to the differential equation is:

$$\boxed{y = \tfrac{1}{3}\Big(\exp\big(3\arctan(x)\big) - 1\Big)}$$

(b) $y' - \dfrac{2y}{x} - 3x^4 = 0, \quad y(1) = 3$

This is a first order linear differential equation. It can be written as:

$$y' - \frac{2}{x}\,y = 3x^4 \qquad\qquad P(x) = -\frac{2}{x} \qquad\qquad Q(x) = 3x^4$$

The integrating factor is:

$$\mu(x) = \exp\left(\int P(x)\,dx\right) = \exp\left(\int -\frac{2}{x}\,dx\right) = \exp\left(-2\ln x\right) = \exp\left(\ln x^{-2}\right) = x^{-2} = \frac{1}{x^2}$$

After multiplying both sides of the equation by the integrating factor we obtain:

$$\frac{1}{x^2}\left(y' - \frac{2}{x}\,y\right) = \frac{1}{x^2}\left(3x^4\right)$$

$$\frac{1}{x^2}\,y' - \frac{2}{x^3}\,y = 3x^2$$

$$\frac{d}{dx}\left(\frac{1}{x^2}\,y\right) = 3x^2$$

$$\frac{1}{x^2}\,y = \int 3\,x^2\,dx$$

$$\frac{1}{x^2}\,y = x^3 + C$$

$$y = x^5 + C\,x^2$$

Now we use the initial condition to find C:

$$3 = y(1) = 1^5 + C\,1^2 = 1 + C$$

$$C = 2$$

Hence the solution to the differential equation is:

$$\boxed{y = x^5 + 2\,x^2}$$

(c) $y'' - 7y' + 6y = 0, \qquad y(0) = 10, \qquad y'(0) = 20$

This is a second order linear homogeneous differential equation with constant coefficients. The auxiliary equation is

$$r^2 - 7r + 6 = (r-6)(r-1),$$

which has roots $r = 6$ and $r = 1$. Hence the general solution to our differential equation is

$$y = C_1\, e^{6x} + C_2\, e^{x}.$$

We now use the initial conditions to find C_1 and C_2. For this we first need to find the derivate y':

$$y' = 6\, C_1\, e^{6x} + C_2\, e^{x}$$

The initial conditions give:

$$\begin{aligned} 10 &= y(0) = C_1\, e^{6\times 0} + C_2\, e^{0} = C_1 + C_2 & \qquad C_1 + C_2 = 10 \\ 20 &= y'(0) = 6\, C_1\, e^{6\times 0} + C_2\, e^{0} = 6\, C_1 + C_2 & \qquad 6\, C_1 + C_2 = 20 \end{aligned}$$

Subtracting the first equation from the second equation we get

$$5\, C_1 = 10,$$

so

$$C_1 = 2.$$

Putting this in the first equation gives

$$2 + C_2 = 10,$$

so

$$C_2 = 8.$$

Hence the solution to the differential equation is:

$$\boxed{y = 2\, e^{6x} + 8\, e^{x}.}$$

(c) $y'' - 2y' + 2y = 0, \qquad y(0) = 1, \qquad y\left(\frac{\pi}{2}\right) = 0$

This is a second order linear homogeneous differential equation with constant coefficients. The auxiliary equation is

$$r^2 - 2r + 2.$$

To find the roots of the auxiliary equation we use the quadratic formula:

$$\begin{aligned} r &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \times 1 \times 2}}{2 \times 1} = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} \\ &= \frac{2 \pm \sqrt{4 \times (-1)}}{2} = \frac{2 \pm \sqrt{4}\sqrt{-1}}{2} = \frac{2 \pm 2\,i}{2} = 1 \pm i \end{aligned}$$

In other words, $r = \alpha \pm \beta\, i$, where $\alpha = 1$ and $\beta = 1$. Hence the general solution of the differential equation is

$$y = C_1\, e^x\, \cos x + C_2\, e^x\, \sin x$$

Now we use the initial conditions to find C_1 and C_2.

$$\begin{aligned} &y(0) = 1 \\ &C_1 \times e^0 \times \cos 0 \quad + \quad C_2 \times e^0 \times \sin 0 \quad = \quad 1 \\ &C_1 \times 1 \times 1 \quad + \quad C_2 \times 1 \times 0 \quad = \quad 1 \\ &C_1 = 1 \end{aligned}$$

$$\begin{aligned} &y(\tfrac{\pi}{2}) = 0 \\ &C_1 \times e^{\frac{\pi}{2}} \times \cos\tfrac{\pi}{2} \quad + \quad C_2 \times e^{\frac{\pi}{2}} \times \sin\tfrac{\pi}{2} \quad = \quad 0 \\ &C_1 \times e^{\frac{\pi}{2}} \times 0 \quad + \quad C_2 \times e^{\frac{\pi}{2}} \times 1 \quad = \quad 0 \\ &C_2 \times e^{\frac{\pi}{2}} = 0 \\ &C_2 = 0 \end{aligned}$$

Hence the solution to the differential equation is:

$$\boxed{y = e^x\, \cos x}$$
