Study guide for midterm 2

Here are some topics you should make sure you understand for the second midterm.

- Mean Value Property and Maximum Modulus Principle.
  - Find the maximum of $|e^z|$ on $|z| \leq 1$.
  - State and prove the Mean Value Property for analytic functions.
  - Prove that the solution to Dirichlet's problem is unique.
  - Show that a function which is harmonic on all of $\mathbb{C}$ and bounded must be constant.
  - Explain why the Minimum Modulus Principle does not hold for analytic functions. Can you give a counter-example?
- Analytic Convergence Theorem.
  - State the Analytic Convergence Theorem.
  - Give an example of a series of analytic functions which converges pointwise but not uniformly.
  - Show that $\sum_{n=1}^{\infty} e^{-n} \sin nz$ is analytic in the region $A = \{z : -1 < \text{Im}(z) < 1\}$.
- Series representations of analytic functions.
  - Assume that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ converges for $z = 1 + i$. Prove that $f(z)$ converges uniformly on the unit disk.
  - Find the radius of convergence of $\sum_{n=0}^{\infty} \frac{z^n}{(1+2n)^2}$ and $\sum_{n=0}^{\infty} 2^{n+(-1)^n} z^n$.
  - Find the Taylor series at the origin of e^{z^2} and $1/(z-1)(z-2)$, and compute their radius of convergence.
  - Find the Laurent series at the origin of $z^2 \sin(\frac{1}{z^2})$, $\frac{e^{z-1}}{z^2}$.
  - Find a Laurent series that converges to $\frac{1}{z^2-4}$ in the annulus $0 < |z-2| < 4$.
  - Find a Laurent series that converges to $\frac{z-2}{z^2-2z+2}$ in a deleted neighborhood of $1+i$.
- Singularities.
  - Define: isolated singularity, pole, removable singularity, essential singularity, residue.
  - For each of the following functions, determine the singularities and their type: $f_1(z) = \sin(\frac{1}{z^2})$, $f_2(z) = z\cot(z)$, $f_3(z) = \frac{z}{1-\cos z}$, $f_4(z) = \frac{\sin(z)}{1-e^{iz}}$.
  - For each removable singularity in the previous problem, compute the radius of convergence of the Taylor series.
  - Let $f(z) = \dfrac{1}{z^2}\dfrac{P(1/z)}{Q(1/z)}$, where P and Q are polynomials and $\deg Q \geq 2 + \deg P$. Show that $f(z)$ has a removable singularity at $z = 0$.
  - Assume that $f(z)$ has a zero of order n at $z = 0$. Show that $g(z) = f(z)/z^n$ has a removable singularity at $z = 0$. What is the value of $g(0)$?

- Residue calculus.
  - Compute: $\operatorname{Res}\left(\frac{z^{1/4}}{z+1}; -1\right)$, $\operatorname{Res}\left(\frac{\log z}{(z^2+1)^2}; i\right)$, $\operatorname{Res}\left(z \sec z; \frac{\pi}{2} + n\pi\right)$, $\operatorname{Res}\left(\frac{e^{zt}}{\sinh z}; \pi i\right)$.
  - Compute: $\displaystyle\int_{|z-2|=2} \frac{3z^2+2}{(z-1)(z^2+9)}\, dz, \quad \int_{|z|=2} \frac{dz}{z^3(z+4)}.$
  - Compute: $\displaystyle\int_{-\pi}^{\pi} \frac{d\theta}{5+4\sin\theta}, \quad \int_{-\infty}^{\infty} \frac{dx}{x^4+1}, \quad \int_0^{\infty} \frac{x^2\, dx}{(x^2+1)(x^2+4)}.$
  - Compute the Fourier transforms of $\displaystyle\frac{x}{1+x^2}$ and $\displaystyle\frac{x-2}{x^2-2x+2}$.
  - Compute: $\displaystyle\text{P.V.} \int_{-\infty}^{\infty} \frac{dx}{x(x^2-1)}.$