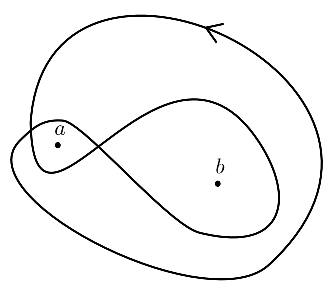
Study guide for midterm 1

Here are some topics you should make sure you understand for the first midterm.

- Geometric interpretation of complex multiplication and De Moivre's formula.
  - Can you determine all the 3-th roots of $-1+i$?
  - Can you write the map $f(x+iy) = (3x-2y) + i(2x+3y-1)$ using complex notation?
- Geometry of elementary functions.
  - Where does the imaginary axis get sent via the functions z^3+1, e^{z+1}, $\overline{(1+i)z}$ and $\cos z$?
  - What points in $\mathbb{C}$ get mapped to the positive real half-line via the functions z^4-1, e^{z+1}?
- Cauchy-Riemann equations and harmonic functions.
  - At what points is the function $f(x+iy) =$ (put your favorite function here) complex differentiable? At what points is it analytic?
  - Is the sum of harmonic functions harmonic? Is the product of harmonic functions harmonic? What about the product of harmonic conjugates?
  - Let $f(z)$ be an analytic function, and consider $g(z) = \overline{f(\bar{z})}$. Show that $g(z)$ is analytic.
- Limits, continuity and topology in $\mathbb{C}$.
  - Show that the continuous image of a connected set is connected.
  - Does the function $f(z) = z/\bar{z}$ have a limit when $z \to 0$?
- Multi-valued functions and their branches.
  - What are the possible values for $(1+i)^{(1+i)}$?
  - Can you give a region where the function $f(z) = \log(z^4-1)$ is analytic?
- Contour integrals and the fundamental theorem of calculus.
  - Compute $\int_\gamma (x^2-y^2)\,dz$, where γ is the unit circle.
  - Compute $\int_\gamma \sqrt{z}\,dz$, where γ is the upper half of the unit circle.
  - Find an upper bound for $\left|\int_\gamma dz/(2+z^2)\right|$ where γ is the upper half of the unit circle.
  - Argue that $\int_\gamma 1/z\,dz = 0$ whenever γ is a closed curve contained in the upper half-plane. Do not use Cauchy's theorem.
  - Show that the length of a curve is independent of the parametrization.
- Cauchy's theorem and the notion of homotopy.
  - Show that convex sets are simply connected.
  - Assuming the Deformation Theorem, prove the Homotopy Form of Cauchy's Theorem.
  - If γ is the curve depicted in the figure, compute

$$\int_\gamma \frac{e^z\,dz}{(z-a)(z-b)}$$



- Cauchy's integral formula and its consequences.
  - Prove that
    $$\int_{|z-1|=1} \frac{1}{z^2-1} dz = \pi i, \qquad \int_{|z+1|=1} \frac{1}{z^2-1} dz = -\pi i.$$
  - Suppose $p(z) = a_3 z^3 + a_2 z^2 + a_1 z + a_0$ is a polynomial of degree 3. If $|p(z)| \leq 1$ on the unit circle $\{z : |z| = 1\}$, then show that $|a_3| \leq 1$.
  - Prove that if f is an entire function which satisfies $|f(z)| \geq 1$ on the entire plane, then f is constant.
  - Prove that if an entire function has real part which is bounded above, then the function is constant.
  - Prove that if a real-valued function is harmonic on all of $\mathbb{C}$ and bounded, it must be constant.