§1.2 #28

Prove the following identity:

$$\left(\sin\frac{\pi}{n}\right)\left(\sin\frac{2\pi}{n}\right)\cdots\left(\sin\frac{(n-1)\pi}{n}\right)=\frac{n}{2^{n-1}}$$

Step 1. We compute the product of the non-zero roots of the polynomial

$$f(z)=(1-z)^n-1.$$

Notice that w is a root of $f(z)$ if and only if $(1-w)^n=1$, i.e., $1-w$ is an n-th root of unity. Therefore $f(z)$ has exactly n roots, $z_0, z_1, \ldots, z_{n-1}$, and they verify:

$$1-z_k=e^{2\pi ki/n}.$$

Consider $\alpha_k=\frac{\pi k}{n}$. We have:

$$z_k=1-e^{2i\alpha_k}=e^{i\alpha_k}\left(e^{-i\alpha_k}-e^{i\alpha_k}\right)=-2i\ e^{i\alpha_k}\ \frac{e^{i\alpha_k}-e^{-i\alpha_k}}{2i}=-2i\ e^{i\alpha_k}\ \sin\alpha_k.$$

In last equality of the previous equation we used the definition of sin in terms of the exponential. Notice that $z_0=0$, so the product of the non-zero roots is:

$$\begin{aligned}z_1\cdot z_2\cdot\dots\cdot z_{n-1}&=(-2i\ e^{\alpha_1 i}\ \sin\alpha_1)(-2i\ e^{\alpha_2 i}\ \sin\alpha_2)\ldots(-2i\ e^{\alpha_{n-1} i}\ \sin\alpha_{n-1})\\&=(-1)^{n-1}\ 2^{n-1}\ i^{n-1}\ e^{i(\alpha_1+\alpha_2+\cdots+\alpha_{n-1})}\ (\sin\alpha_1)(\sin\alpha_2)\ldots(\sin\alpha_{n-1}).\end{aligned}$$

We have:

$$\begin{aligned}\alpha_1+\alpha_2+\cdots+\alpha_{n-1}&=\frac{\pi}{n}+\frac{2\pi}{n}+\ldots\frac{(n-1)\pi}{n}\\&=\frac{\pi}{n}(1+2+\cdots+(n-1))=\frac{\pi}{n}\frac{(n-1)n}{2}=(n-1)\frac{\pi}{2}.\end{aligned}$$

Hence:

$$e^{i(\alpha_1+\alpha_2+\cdots+\alpha_{n-1})}=e^{\frac{\pi i}{2}(n-1)}=\left(e^{\frac{\pi i}{2}}\right)^{n-1}=i^{n-1}.$$

In particular

$$\begin{aligned}(-1)^{n-1}\ 2^{n-1}\ i^{n-1}\ e^{i(\alpha_1+\alpha_2+\cdots+\alpha_{n-1})}&=(-1)^{n-1}\ 2^{n-1}\ i^{n-1}\ i^{n-1}\\&=(-1)^{n-1}\ 2^{n-1}\ (-1)^{n-1}=2^{n-1},\end{aligned}$$

and

$$\begin{aligned}z_1\cdot z_2\cdot\dots\cdot z_{n-1}&=2^{n-1}(\sin\alpha_1)(\sin\alpha_2)\ldots(\sin\alpha_{n-1})\\&=2^{n-1}\left(\sin\frac{\pi}{n}\right)\left(\sin\frac{2\pi}{n}\right)\cdots\left(\sin\frac{(n-1)\pi}{n}\right).\end{aligned}$$

Step 2. We expand the polynomial $f(z)$:

$$\begin{aligned} f(z) &= (1-z)^n - 1 \\ &= \left(1^n - \binom{n}{1}1^{n-1}z + \binom{n}{2}1^{n-2}z^2 - \dots + (-1)^n z^n\right) - 1 \\ &= -nz + \frac{n(n-1)}{2}z^2 - \dots + (-1)^n z^n. \end{aligned}$$

On the other hand, since $z_0, z_1, \dots, z_{n-1}$ are the roots of $f(z)$, we have

$$f(z) = a(z - z_0)(z - z_1)\dots(z - z_{n-1}),$$

where a is the leading coefficient of $f(z)$. In our case $a = (-1)^n$ and $z_0 = 0$, so we get:

$$\begin{aligned} f(z) &= (-1)^n z(z - z_1)\dots(z - z_{n-1}) \\ &= (-1)^n z\left(z^{n-1} - (z_1 + z_2 + \dots + z_{n-1})z^{n-2} + \dots + (-1)^{n-1}(z_1 z_2 \dots z_{n-1})\right) \\ &= -(z_1 z_2 \dots z_{n-1})z + \dots + (-1)^n z^n. \end{aligned}$$

Comparing the coefficients in front of z in our two expansions of $f(z)$, we get that

$$z_1 z_2 \dots z_{n-1} = n.$$

Combining this with Step 1 we obtain

$$2^{n-1}\left(\sin\frac{\pi}{n}\right)\left(\sin\frac{2\pi}{n}\right)\cdots\left(\sin\frac{(n-1)\pi}{n}\right) = n,$$

as required.