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The Hahn – Banach Theorem :

Extensions of Linear Forms and Separation of Convex Sets

Let E be a vector space over $\mathbb{R}$ and $F \subset E$ be a subspace. A function $f : F \to \mathbb{R}$ is linear if

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y) \ \forall \ \alpha, \beta \in \mathbb{R} \text{ and } \forall \ x, y \in F.$$

Theorem 1 (Hahn-Banach Theorem, Analytical Formulation) *Let E be a vector space and $p : E \to \mathbb{R}$ be such that*

1. p is positively homogeneous, i.e., $p(\lambda x) = \lambda p(x) \ \forall \ \lambda > 0, \ \forall \ x \in E$.

2. p is subadditive, i.e., $p(x + y) \leq p(x) + p(y) \ \forall \ x, y \in E$.

Let $G \subset E$ be a subspace and $g; G \to \mathbb{R}$ be a linear function such that $g(x) \leq p(x) \ \forall \ x \in G$. Then there exists $f : E \to \mathbb{R}$, a linear function, such that $f(x) = g(x) \ \forall \ x \in G$. Moreover, $f(x) \leq p(x) \ \forall x \in E$.

Proof : The proof uses Zorn's lemma:

Lemma 1 *Every nonempty, ordered, inductive set admits a maximal element.*

Recall that an inductive set is one such that every totally ordered subset has at least an upper bound. A subset Q of a set P is totally ordered if $a, b \in Q$ implies $a \leq b$ or $b \leq a$. An element $a \in P$ is an upper bound for Q if $x \leq a$ for all $x \in Q$. An element $m \in P$ is a maximal element if $x \in P$ and $m \leq x$ implies $m = x$.

Let us define a set

$$P = \{h : D(h) \to \mathbb{R} : D(h) \text{ is a subspace of } E, \ h(x) = g(x) \text{ if } x \in G,$$

$$D(h) \supset G, \ h(x) \leq p(x) \ \forall \ x \in D(h), \ h \text{ is linear}\}.$$

First note that $P \neq \emptyset$, since $g \in P$. We define an order relation on P by

$$h_1 \leq h_2 \ \Leftrightarrow \ D(h_1) \subset D(h_2) \text{ and } h_2(x) = h_1(x) \ \forall \ x \in D(h_1).$$

Next, we show that P is inductive. Let $Q = \{h_i : i \in I\}$ be a totally ordered subset of P. Let

$$D(h) = \bigcup_{i \in I} D(h_i),$$

and define $h : D(h) \to \mathbb{R}$ by $h(x) = h_i(x)$ if $x \in D(h_i)$ (it is left as an exercise to prove that h is well-defined by using the fact that Q is totally ordered). Also, $h \in P$ and h is an extension of every $h_i \in Q$. Thus h is an upper bound for Q. Zorn's lemma then implies that there exists $f \in P$ that is a maximal element of P. We claim that this f is the one required by the conclusion of the theorem. It remains only to show that $D(f) = E$. Note that $D(f) \subset E$. By contradiction, assume that there exists $x_0 \in E - D(f)$. We define $h : D(h) \to \mathbb{R}$, where $D(h) = D(f) + \mathbb{R}x_0 := \{x + tx_0 : x \in D(f), \ t \in \mathbb{R}\}$, by $h(x + tx_0) = f(x) + \alpha t$, with α to be chosen later such that $h \in P$.

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In the last installment, we showed that there is a maximal element of the set P by Zorn's Lemma. To show that this f is the one required by the statement of the Hahn-Banach theorem, we needed only to show that $D(f) = E$. To this end, we assumed that there was $x_0 \in E - D(f)$ and defined a function h on $D(f) + \mathbb{R}x_0$ by $h(x + tx_0) = f(x) + t\alpha$, where $\alpha \in \mathbb{R}$ is to be chosen later. Note that $f \leq h$ in the order defined previously. If we can show that $h \in P$, then we will have a contradiction with the maximality of f. Thus, we need to show that $h \leq p$ on $D(h)$, i.e. that

$$h(t + x_0) = f(x) + t\alpha \leq p(x + tx_0)\ \forall\ x \in D(f),\ \forall\ t \in \mathbb{R} \tag{1}$$

First, we claim that (1) holds if we can choose α such that

$$\begin{cases} f(x) + \alpha \leq p(x + x_0)\ \forall\ x \in D(f) \\ f(x) - \alpha \leq p(x - x_0)\ \forall\ x \in D(f) \end{cases} \tag{2}$$

The proof goes as follows: If $t = 0$, then (1) holds. If $t > 0$, then

$$p(x + tx_0) = p\left(t\left(\frac{1}{t}x + x_0\right)\right) = tp\left(\frac{1}{t}x + x_0\right) \geq t\left(f\left(\frac{1}{t}x\right) + \alpha\right) = f(x) + t\alpha.$$

If $t < 0$, we use the same argument with t replaced by $-t$. This establishes the first claim. Next, we claim that there exists $\alpha \in \mathbb{R}$ such that (2) holds. To see this, let $x, y \in D(f)$. Then

$$f(x) + f(y) = f(x + y) \leq p(x + y) = p(y - x_0 + x + x_0) \leq p(y - x_0) + p(x + x_0).$$

This implies that

$$f(y) - p(y - x_0) \leq p(x + x_0) - f(x)\ \forall\ y, x \in D(f),$$

so

$$\sup_{y \in D(f)} \{f(y) - p(y - x_0)\} \leq \inf_{x \in D(f)} \{p(x + x_0) - f(x)\}.$$

Thus any $\alpha \in \mathbb{R}$ such that

$$\sup_{y \in D(f)} \{f(y) - p(y - x_0)\} \leq \alpha \leq \inf_{x \in D(f)} \{p(x + x_0) - f(x)\}$$

will do.

Recall than if E is a normed vector space and if $f : E \to \mathbb{R}$ is linear, then f is also continuous if there exists $C > 0$ such that

$$|f(x)| \leq C\|x\|_E\ \forall\ x \in E.$$

We define

$$E' = \{f : E \to \mathbb{R} : f \text{ is linear, continuous}\}.$$

Then

$$\begin{aligned} \|f\|_{E'} = \sup\{|f(x)| : x \in E, \|x\|_E \leq 1\} &= sup_{\substack{x \in E \\ x \neq 0}} \frac{|f(x)|}{\|x\|_E} \\ &= \inf\{C > 0 : |f(x)| \leq C\|x\|\ \forall\ x \in E\} \end{aligned}$$

is a norm on E'.

Remark : In general, the sup in the definition of $\|\cdot\|_{E'}$ is not attained. It is always attained in reflexive spaces. R.C. James proved that if the sup is attained for all $f \in E'$, then the space E must be reflexive.

Corollary 1 *Let E be a normed vector space, and $G \subset E$ a subspace. If $g \in G'$, then there exists $f \in E'$ such that $\|f\|_{E'} = \|g\|_{g'}$ and f extends g.*

Proof : Define $p : E \to \mathbb{R}$ by $p(x) = \|g\|_{G'}\|x\|_E$. Then p is positive homogeneous and subadditive. We then apply the Hahn-Banach theorem with these choices of G, g, and p to find that there exists $f : E \to \mathbb{R}$ that is a linear extension of g such that

$$f(x) \leq \|g\|_{G'}\|x\|_E \ \forall \ x \in E.$$

Thus $f \in E'$, and since

$$\|f\|_{E'} \leq \|g\|_{G'} = \sup_{\substack{x\in G \\ \|x\|_G \leq 1}} |g(x)| = \sup_{\substack{x\in G \\ \|x\|_G \leq 1}} |f(x)| \leq \|f\|_{E'},$$

we also have that $\|f\|_{E'} = \|g\|_{G'}$.

Corollary 2 *Let E be a normed vector space, and let $x_0 \in E$. Then there exists $f_0 \in E'$ such that $\|f_0\|_{E'} = \|x_0\|_E$ and $f_0(x_0) = \|x_0\|_E^2$.*

Proof : Define $G = \mathbb{R}x_0$ and define $g : G \to \mathbb{R}$ by $g(tx_0) = t\|x_0\|_E^2$, $t \in \mathbb{R}$. Applying the first corollary, we find that there exists $f_0 \in E'$ such that f_0 extends f and $\|f_0\|_{E'} = \|g\|_{G'}$. Thus

$$\|f_0\|_{E'} = \|g\|_{G'} = \sup_{\substack{t\in\mathbb{R} \\ t\neq 0}} \frac{|g(tx_0)|}{\|fx_0\|_E} = \sup_{\substack{t\in\mathbb{R} \\ t\neq 0}} \frac{|t|\|x_0\|_E^2}{|t|\|x_0\|_E} = \|x_0\|_E.$$

Also, f_0 extends g, so $f(tx_0) = g(tx_0)$, and consequently, if $t = 1$, we have $f(x_0) - g(x_0) = \|x_0\|_E^2$.

The duality map between a normed vector space and its dual is defined by

$$E \ni x \mapsto \{f \in E' : f(x) = \|x\|_E^2, \ \|f\|_{E'} = \|x\|_E\}.$$

Remark : In general, the f_0 in the second corollary is not unique, so the duality map is multivalued. The functional f_0 is unique if E is strictly convex: $x \neq y$ and $\|x\|_E = \|y\|_E = 1$ imply that $\|tx + (1-t)y\|_E < 1$ for all $t \in (0, 1)$.

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Corollary 3 *For every $x \in E$, we have*

$$\|x\| = \sup_{\substack{f\in E' \\ \|f\|\leq 1}} |\langle f, x\rangle| = \max_{\substack{f\in E' \\ \|f\|\leq 1}} |\langle f, x\rangle|.$$

Proof : Choose $x \in E$. Then

$$|\langle f, x\rangle| \leq \|f\|\|x\| \ \Rightarrow \ \sup_{\substack{f\in E' \\ \|f\|\leq 1}} |\langle f, x\rangle| \leq \|x\|.$$

Conversely, there exists $f_0 \in E'$ such that $\|f_0\| = \|x\|$ and $\langle f, x\rangle = \|x\|^2$. We define a new functional $f_1 : E \to \mathbb{R}$ by

$$\langle f_1, y\rangle = \frac{\langle f_0, y\rangle}{\|x\|} \ \forall \ y \in E.$$

Then clearly $f_1 \in E'$ and

$$\langle f_1, x \rangle = \frac{\langle f_0, x \rangle}{\|x\|} = \|x\|.$$

Thus we have that

$$\|f_1\| = \sup_{\substack{y \in E \\ y \neq 0}} \frac{|\langle f_1, y \rangle|}{\|y\|} = \frac{|\langle f_1, x \rangle|}{\|x\|} = 1,$$

and then

$$\sup\{|\langle f, x \rangle| : f \in E', \ \|f\| \leq 1\} \geq \langle f_1, x \rangle = \|x\|.$$

This gives us the result.

The Geometric Versions of the Hahn – Banach Theorem

Separtaion of Convex Sets

Throughout what follows, E is a normed vector space with norm $\|\cdot\|_E$ (the subscript will be omitted unless it is unclear which norm is being used).

Definition 1 *Let $\alpha \in \mathbb{R}$ and $f : E \to \mathbb{R}$ be a linear (not necessarily continuous) functional, $f \not\equiv 0$. An (affine) hyperplane of equation $[f = \alpha]$ is a set of the form*

$$[f, \alpha] = \{x \in E : f(x) = \alpha\}.$$

Proposition 1 *A hyperplane $[f = \alpha]$ is closed if and only if f is continuous.*

The proof of this proposition can be found in the homework for the course, or in Brezis.

Definition 2 *Let $A, B \subset E$. We say that the hyperplane of equation $[f = \alpha]$ separates A and B if $f(x) \leq \alpha \ \forall \ x \in \alpha$ and $f(x) \geq \alpha \ \forall \ x \in B$.*

Definition 3 *Let $A, B \subset E$. We say that the hyperplane of equation $[f = \alpha]$ strictly separates A and B if there exists $\epsilon > 0$ such that $f(x) \leq \alpha - \epsilon \ \forall \ x \in A$ and $f(x) \geq \alpha + \epsilon \ \forall \ x \in B$.*

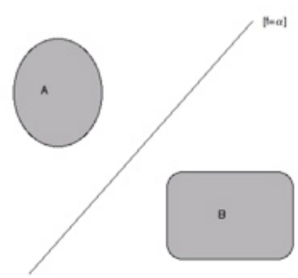

Figure 1: Sets A and B separated by a hyperplane.

Theorem 2 (Hahn-Banach, first geometric version) *Let $A, B \subset E$ be two nonempty, disjoint, and convex sets. Assume that A is open. Then there exists a closed hyperplane that separates A and B.*

Proof : The proof is based on two lemmas:

Lemma 2 *If $C \subset E$ is open, convex, $0 \in C$, then for every $x \in C$, we define*

$$p(x) = \inf \left\{ \alpha > 0 : \frac{1}{\alpha} x \in C \right\}.$$

Then

1. $p(x) = \lambda p(x) \ \forall \ \lambda > 0$ *and* $\forall \ x \in E$.
2. $p(x+y) \leq p(x) + p(y) \ \forall \ x, y \in E$.
3. $\exists \ M > 0$ *such that* $0 \leq p(x) \leq M\|x\| \ \forall \ x \in E$.
4. $C = \{x \in E : p(x) < 1\}$.

Lemma 3 *Let $C \subset E$ be nonempty, open, and convex. Let $x_0 \in E \backslash C$. Then there exists $f \in E'$ such that $f(x) < f(x_0)$ for all $x \in C$. In particular, the hyperplane $[f = f(x_0)]$ separates $\{x_0\}$ and C.*

We postpone the proofs of the lemmas until the end of the proof of the theorem.

First, define

$$C = A - B = \{x - y : x \in A, \ y \in B\}.$$

Then C is convex. Let $t \in [0, 1]$ and $x_1, x_2 \in A, \ y_1, y_2 \in B$. We have

$$t(x_1 - y_1) + (1-t)(x_2 - y_2) = tx_1 + (1-t)x_2 - [ty_1 + (1-t)y_2] \in A - B = C.$$

Also, C is open. We may express C as

$$C = A - B = \bigcup_{y \in B} (A - \{y\}).$$

We claim that $A - \{y\}$ is open for each $y \in B$. Let $x_0 \in A - \{y\}$. Then $x_0 + y \in A$. Since A is open, there exists $r > 0$ such that $B(x_0 + y, r) \subset A$. We want to show that $B(x_0, r) \subset A - \{y\}$, so we let $z \in B(x_0, r)$. We have that

$$\|(z + y) - (x_0 + y)\| = \|z - x_0\| < r,$$

so $z + y \in B(x_0 + y, r) \subset A$, which gives us that $z \in A - \{y\}$. This implies that C is open. It is easy to show that $0 \notin C$. If 0 were in C, then there exists $x \in A$ and $y \in B$ such that $0 = x - y$, so $x = y$, but this is a contradiction with the fact that A and B are disjoint. Lastly, we note that $C \neq \emptyset$. This is trivial from the definition of C and the fact that $A, B \neq \emptyset$.

Now we may apply (3) for $C = A - B$ and $x_0 = 0$. This gives us that there exists $f \in E'$ such that $f(z) < f(0) = 0$ for every $z \in C$. Also, each $z \in C$ may be written as $x - y$ for some $x \in A$ and $y \in B$. By the linearity of f, we then have that $f(x) < f(y)$ for every $x \in A$ and $y \in B$. This shows that

$$\sup_{x \in A} f(x) \leq \inf_{y \in B} f(y).$$

If we choose α such that

$$\sup_{x \in A} f(x) \leq \alpha \leq \inf_{y \in B} f(y),$$

then $f(x) \leq \alpha \leq f(y)$ for all $x \in A$ and $y \in B$. This tells us that the hyperplane $[f = \alpha]$ separates A and B. Since f is continuous, $[f = \alpha]$ is closed.

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Proof of Lemma 1 : 1. We defined the function

$$p(\lambda x) = \inf\left\{\alpha > 0 : \frac{1}{\alpha}x \in C\right\},$$

and we want to show that

$$\inf\left\{\alpha > 0 : \frac{1}{\alpha}\lambda x \in C\right\} = \lambda \inf\left\{\alpha > 0 : \frac{1}{\alpha}x \in C\right\}.$$

Choose $\alpha > 0$ such that $\alpha^{-1}\lambda x \in C$. Then $\alpha = \lambda(\alpha\lambda^{-1})$, and $(\alpha\lambda^{-1})^{-1}x \in C$. Thus

$$\alpha\lambda \in \{\beta > 0 : \beta^{-1}x \in C\}$$

$$\Rightarrow \ \alpha\lambda \geq \inf\{\beta > 0 : \beta^{-1}x \in C\}$$

$$\Rightarrow \ \alpha = \lambda(\alpha\lambda^{-1}) \geq \lambda \inf\{\beta > 0 : \beta^{-1}x \in C\}$$

$$\Rightarrow \ \inf\{\alpha > 0 : \alpha^{-1}\lambda x \in C\} \geq \lambda \inf\{\beta > 0 : \beta^{-1}x \in C\}.$$

Let $\beta > 0$ be such that $\beta^{-1}x \in C$. Then $\beta = \lambda^{-1}(\beta\lambda)$. Since $\beta^{-1}x \in C$, we have that $(\beta\lambda)^{-1}\lambda x \in C$ and

$$\beta\lambda \in \{\alpha > 0 : \alpha^{-1}\lambda x \in C\}.$$

This in turn gives us that

$$\beta\lambda \geq \inf\{\alpha > 0 : \alpha^{-1}\lambda x \in C\}$$

$$\Rightarrow \ \beta \geq \frac{1}{\lambda} \inf\{\alpha > 0 : \alpha^{-1}\lambda x \in C\}$$

$$\Rightarrow \ \inf\{\beta > 0 : \beta^{-1}x \in C\} \geq \frac{1}{\lambda} \inf\{\alpha > 0 : \alpha^{-1}\lambda x \in C\}.$$

This completes the proof of part 1.

2. We have that

$$p(x+y) = \inf\{\alpha > 0 : \alpha^{-1}(x+y) \in C\}.$$

Let $\alpha > 0$ be such that $\alpha^{-1}x \in C$, and let $\beta > 0$ be such that $\beta^{-1}y \in C$. We seek γ such that $\gamma \leq \alpha + \beta$ such that $\gamma^{-1}(x+y) \in C$, so we look for t such that

$$\gamma^{-1}(x+y) = t\alpha^{-1}x + (1-t)\beta^{-1}y.$$

This implies that

$$t\alpha^{-1} = \gamma^{-1} \text{ and } (1-t)\beta^{-1} = \gamma^{-1}.$$

Thus

$$t = 1 - \beta\gamma^{-1} \ \Rightarrow \ \alpha\gamma^{-1} = 1 - \beta\gamma^{-1} \ \Rightarrow \ (\alpha+\beta)\gamma^{-1} = 1 \ \Rightarrow \ \gamma = \alpha + \beta.$$

Therefore, we have that

$$(\alpha+\beta)^{-1}(x+y) = \frac{\alpha}{\alpha+\beta}(\alpha^{-1}x) + \left(1 - \frac{\alpha}{\alpha+\beta}\right)(\beta^{-1}y) \in C.$$

We then calculate that

$$\begin{aligned} \alpha+\beta &\geq \inf\{\gamma > 0 : \gamma^{-1}(x+y) \in C\} \\ \Rightarrow \ \inf\{\alpha > 0 : \alpha^{-1}x \in C\} + \inf\{\beta > 0 : \beta^{-1}y \in C\} &\geq p(x+y) \\ \Rightarrow \ p(x)+p(y) &\geq p(x+y). \end{aligned}$$

3. We know that $0 \in C$ and C is open, so there exists $r > 0$ such that $\overline{B(0,r)} \subset C$, so for every $x \in E$, we have that

$$\left(\frac{1}{r}\|x\|\right)^{-1} x = r\frac{x}{\|x\|} \in \overline{B(0,r)} \in C.$$

This gives us that

$$\begin{aligned} \frac{1}{r}\|x\| &\in \{\alpha > 0 : \alpha^{-1}x \in C\} \\ \Rightarrow \ \frac{1}{r}\|x\| &\geq \inf\{\alpha > 0 : \alpha^{-1}x \in C\} \\ \Rightarrow \ 0 \leq p(x) &\leq M\|x\|, \text{ where } M = \frac{1}{r}. \end{aligned}$$

4. If $x \in C$, there exists $\epsilon > 0$ such that $x + \epsilon x \in C$. This gives us that

$$(1+\epsilon)x \in C \ \Rightarrow \ \left(\frac{1}{(1+\epsilon)}\right)^{-1} x \in C,$$

so

$$\frac{1}{1+\epsilon} \in \{\alpha > 0 : \alpha^{-1}x \in C\} \ \Rightarrow \ 1 > \frac{1}{1+\epsilon} \geq p(x) \ \Rightarrow \ x \in \{x \in E : p(x) < 1\}.$$

Now choose $x \in E$ such that $p(x) < 1$. This implies that

$$\inf\{\alpha > 0 : \alpha^{-1}x \in C\} < 1,$$

so there exists β such that $\inf\{\alpha > 0 : \alpha^{-1}x \in C\} < \beta < 1$ and $\beta^{-1}x \in C$. We then have that

$$x = \beta(\beta^{-1}x) + (1-\beta)\cdot 0 \in C.$$

Proof of Lemma 2 : Without loss of generality, assume that $0 \in C$ (if not, we translate). Let

$$p(x) = \inf\{\alpha > 0 : \alpha^{-1}x \in C\}, \ G = \mathbb{R}x_0,$$

and define $g : G \to \mathbb{R}$ by $g(tx_0) = t$. We show that $g(tx_0) \leq p(tx_0)$. If $t < 0$, then

$$p(tx_0) = \inf\{\alpha > 0 : \alpha^{-1}tx_0 \in C\} = tp(x_0) \geq t = g(tx_0).$$

If $t = 0$, then $g(0x_0) = g(0)$ and $p(0x_0) = p(0)$ and $p(0+0) \le p(0) + p(0)$, so $p(0) \ge 0 = g(0)$. If $t < 0$, then

$$p(tx_0) = \inf\{\alpha > 0 : \alpha^{-1} t x_0 \in C\} \ge 0,$$

and $p(tx_0) = t < 0$. We have then that $g : G \to \mathbb{R}$ is linear, $G \subset E$ is a subspace, and $g(x) \le p(x)$ for every $x \in G$. By the Hahn-Banach theorem (analytical version), there exists a linear functional f such that f extends g and $f(x) \le p(x)$ for every $x \in E$. Also, since $f(x) \le p(x) \le M\|x\|$, f is continuous. We have that $f(x_0) = g(x_0) = 1$, and $p(x) < 1$ for every $x \in C$. This means that $f(x) \le p(x) < 1 = f(x_0)$ for all $x \in C$.

Theorem 3 (Hahn-Banach Theorem, second geometric version) *Suppose $A, B \subset E$ are nonempty, convex, and disjoint. Assume in addition that A is closed and B is compact. Then there exists a closed hyperplane strictly separating A and B.*

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Theorem 4 (Hahn-Banach Theorem, second geometric version) *Let E be a normed vector space, and let $A, B \neq \emptyset$ be two disjoint, convex sets. Assume that A is closed and B is compact. Then the sets A and B can be strictly separated by a closed hyperplane.*

Proof : Define

$$A_\epsilon = A + B(0,\epsilon) \text{ and } B_\epsilon = B + B(0,\epsilon).$$

Then $A_\epsilon, B_\epsilon \neq \emptyset$, A_ϵ and B_ϵ are both open, and A_ϵ and B_ϵ are both compact. The convexity follows since if $\theta \in [0,1]$ and $x, y \in A + B(0,\epsilon)$, then we can write

$$x = a + z, \text{ where } a \in A,\ z \in B(0,\epsilon),\ y = b + t, \text{ where } b \in A,\ t \in B(0,\epsilon).$$

Then

$$\theta x + (1-\theta)y = \theta a + (1-\theta)b + \theta z + (1-\theta)t \in A + B(0,\epsilon),$$

by the convexity of A and the ball. It still remains to show that $A_\epsilon \cap B_\epsilon = \emptyset$ for ϵ sufficiently small. If not, then there exists a sequence $\{\epsilon_n\}$ such that $\epsilon_n \to 0$ and corresponding sequences $\{a_n\}$ and $\{b_n\}$ such that $a_n + \theta_n = b_n + t_n$, with $\theta_n, t_n \in B(0,\epsilon_n)$. This implies that

$$\|a_n - b_n\| = |\theta_n - t_n\| < 2\epsilon_n \to 0.$$

Since B is compact, (up to a subsequence) we have that there exists $b \in B$ such that $b_n \to b$. This gives us that

$$\|a_n - b\| \le \|a_n - b_n\| + \|b_n - b\| < 2\epsilon_n + \|b_n - b\| \to 0.$$

We now have that $a_n \to b$, and A is closed, so $b \in A$, which implies that $A \cap B \neq \emptyset$, which is a contradiction.

We are now able to apply the first geometric version of the Hahn-Banach theorem. There exists a hyperplane $[f = \alpha]$ which separates A_ϵ and B_ϵ for ϵ sufficiently small, so

$$f(x + \epsilon z) \le \alpha \le f(y + \epsilon z)\ \forall\ x \in A,\ y \in B,\ z \in B(0,1).$$

This gives us that

$$f(x) + \epsilon f(z) \le \alpha \le f(y) + \epsilon f(z),$$

so $f(x) \leq \alpha - \epsilon f(z)$, and if we pass to the supremum over $z \in B(0,1)$, then we find that $f(x) \leq \alpha - \epsilon\|f\|$. Similarly, we have that $f(y) \geq \alpha - \epsilon f(z)$, and this is true for all $z \in B(0,1)$. If we interchange z and $-z$, then this becomes $f(y) \geq \alpha + \epsilon f(z)$. Again we pass to the supremum over $z \in B(0,1)$ to find that $f(y) \geq \alpha + \epsilon\|f\|$. These inequalities hold for all $x \in A$, $y \in B$, and $0 < \epsilon \ll 1$, so we have that $[f = \alpha]$ separates A and B strictly.

Corollary 4 *Let F be a subspace of a normed vector space E such that $\bar{F} \neq E$. Then there exists $f \in E'$, $f \not\equiv 0$ such that $\langle f, x\rangle = 0$ for all $x \in F$.*

This is useful in proving density results. We simply show that if $f \in E'$ is such that $f = 0$ on F, then $f = 0$ on E as well.

Proof : let $x_0 \in E\backslash\bar{F}$ We would like to apply the second geometric version of the Hahn-Banach theorem to $A = \bar{F}$, $B = \{x_0\}$. The set A is convex since F is a subspace. Then there exists a closed hyperplane $[f = \alpha]$ which separates $\bar{F}$ and $\{x_0\}$ strictly. In other words,

$$f(x) < \alpha < f(x_0) \ \forall \ x \in F.$$

Let $x \in F$ be arbitrary, and let $\lambda \in \mathbb{R}$. Then $\lambda f(x) = f(\lambda x) < \alpha$, and this is true for all real λ, so we must have that $f(x) = 0$.

Remark : 1. In general, two convex, nonempty, disjoint sets cannot be separated (without additional assumptions)

2. One can construct examples of sets A and B that are convex, nonempty, disjoint, and both closed so that A and B cannot be separated by a closed hyperplane (compactnes of B in the second version is essential).

3. If $\dim(E) < \infty$, then one can always separate any two nonempty, disjoint, convex subsets of E.

4. The first geometric version of the Hahn-Banach theorem generalizes to topological spaces. The second version generalizes to locally convex spaces.

Definition 4 *A point x in a subset K of a normed vector space E is an extremal point of K if*

$$\left\{\begin{array}{l} x = tx_0 + (1-t)x_1 \\ t \in (0,1) \\ x_0, x_1 \in K \end{array}\right. \quad \Rightarrow \ x_0 = x_1 = x.$$

Theorem 5 (Krein-Milman) *If K is a convex, compact subset of a normed vector space E, then K coincides with the closed convex hull of its extremal points.*

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Lemma 4 (Baire's Lemma) *Let X be a complete metric space, and consider a sequence $\{X_n\}$ of closed sets such that $int(X_n) \neq \emptyset$ for each $n \in \mathbb{N}$. Then*

$$int\left(\bigcup_{i=1}^{\infty} X_n\right) = \emptyset.$$

Proof : Let $\mathcal{O}_n = X \backslash X_n$, which is open for each $n \in \mathbb{N}$. We have then that

$$\overline{\mathcal{O}_n} = \overline{X \backslash X_n} = \bigcup_{\substack{F \supset X \backslash X_n \\ F \text{ closed}}} F = \bigcap_{\substack{V \subset X \backslash (X \backslash X_n) \\ V \text{ open}}} X \backslash V = X \backslash \bigcup_{\substack{V \subset X_n \\ V \text{ open}}} V = X \backslash \operatorname{int}(X_n) = X.$$

Thus, $\mathcal{O}_n$ is dense in X. Similarly,

$$\operatorname{int}\left(\bigcup_{n=1}^{\infty} X_n\right) = X \backslash \overline{\left(X \backslash \bigcup_{n=1}^{\infty} X_n\right)} = X \backslash \overline{\bigcap_{n=1}^{\infty} (X \backslash X_n)} = X \backslash \overline{\bigcap_{n=1}^{\infty} \mathcal{O}_n}.$$

We want to show that

$$\overline{\bigcup_{n=1}^{\infty} \mathcal{O}_n} = X,$$

i.e., that the intersection of the $\mathcal{O}_n$ is dense in X. Let

$$G = \bigcap_{n=1}^{\infty} \mathcal{O}_n.$$

We want to show that $\omega \cap G \neq \emptyset$ for each nonempty, open $\omega \subset X$. Let ω be such a set. Then there exists $x_0 \in \omega$, and there exists $r_0 > 0$ such that $\overline{B(x_0, r_0)} \subset \omega$.

Step 1: Since $\mathcal{O}_1$ is open and dense in X, there exists $x_1 \in B(x_0, r_0) \cap \mathcal{O}_1$. Since this set is open as well, there exists $r_1 > 0$ such that

$$\overline{B(x_1, r_1)} \subset (B(x_0, r_0) \cup \mathcal{O}_1) \text{ and } 0 < r_1 < \frac{r_0}{2}.$$

Step 2: Now choose $x_2 \in B(x_1, r_1) \cap \mathcal{O}_2$. There exists $r_2 > 0$ such that

$$\overline{B(x_2, r_2)} \subset B(x_1, r_1) \cap \mathcal{O}_2, \text{ and } 0 < r_2 < \frac{r_1}{2}.$$

Inductively, we can construct a sequence $\{x_n\}$ of points and radii $\{r_n\}$ such that

$$\overline{B(x_{n+1}, r_{n+1})} \subset B(x_n, r_n) \cap \mathcal{O}_{n+1} \text{ and } 0 < r_{n+1} < \frac{r_n}{2} < \cdots < \frac{r_0}{2^{n+1}}.$$

We claim that $\{x_n\}$ is a Cauchy sequence. To see this, note that

$$d(x_{n+p}, x_n) \leq d(x_{n+p}, x_{n+p-1}) + \cdots + d(x_{n+1}, x_n) \leq r_{n+p-1} + \cdots + r_n$$

$$< \frac{r_0}{2^{n+p-1}} + \cdots \frac{r_0}{2^n} = \frac{r_0}{2^n}\left(1 + \frac{1}{2} + \cdots + \left(\frac{1}{2}\right)^{p-1}\right) \leq \frac{r_0}{2^{n-1}} \to 0 \text{ as } n \to \infty.$$

Now, since X is a complete metric space, $\{x_n\}$ is convergent, so there exists an element $l \in X$ such that $x_n \to l$ as $n \to \infty$. By construction, we have that $x_{n+p} \in B(x_n, r_n)$, so when we pass to the limit, we have that $l \in \overline{B(x_n, r_n)}$. We know that $B(x_n, r_n) \subset \mathcal{O}_n \cap \omega$, so $l \in \mathcal{O}_n$ for each n. Also,

$$l \in \bigcap_{n=1}^{\infty} \mathcal{O}_n = G \to l \in \omega \cap G.$$

This completes the proof.

Corollary 5 *Let X be a complete metric space, and let $\{X_n\}$ be a sequence of closed sets such that*

$$\bigcup_{n=1}^{\infty} X_n = X.$$

Then there exists $n_0 \in \mathbb{N}$ such that int$(X_{n_0}) \neq \emptyset$.

Proof : If we assume that $\text{int}(X_n) = \emptyset$ for every $n \in \mathbb{N}$, then

$$\text{int}\left(\bigcup_{n=1}^{\infty} X_n\right) = \emptyset.$$

This implies that $\text{int}(X) = \emptyset$. But

$$\text{int}(X) = \bigcup_{\substack{V \subset X \\ V \text{ open}}} V \supset X \backslash \{x_0\},$$

and this set is open (unless X has only one point, in which cast the result is trivial).

Notation : We denote by

$$\mathcal{L}(E, F)\{T : E \to F : T \text{ is linear and continuous}\},$$

endowed with the norm

$$\|T\|_{\mathcal{L}(E,F)} = \sup_{\substack{x \in E \\ \|x\| \leq 1}} \|Tx\|.$$

We also write $\mathcal{L}(E) = \mathcal{L}(E, E)$.

Theorem 6 (Banach-Steinhaus Theorem/Principle of Uniform Boundedness) *Let E and F be Bancach spaces, and let $\{T_i\}_{i \in I} \subset \mathcal{L}(E, F)$. Assume that*

$$\sup_{i \in I} \|T_i x\| < \infty \ \forall x \in E. \tag{3}$$

Then $\sup_{i \in I} \|T_i\| < \infty$, i.e., there exists $C > 0$ such that $\|T_i x\| \leq C\|x\|$ for every $x \in E$ and for every $i \in I$. We simply take $C = \sup_{i \in I} \|T_i\|$.

Proof : For each $n \in \mathbb{N}$, define

$$X_n = \{x \in E : \|T_i x\| \leq n \ \forall i \in I\}.$$

Since $T_i \in \mathcal{L}(E, F)$, X_n is closed for each n. Then (3) implies that

$$\bigcap_{n=1}^{\infty} X_n = E,$$

and the corollary to Baire's Theorem tells us that there exists $n_0 \in \mathbb{N}$ such that $\text{int}(X_{n_0}) \neq \emptyset$. Thus, there exist x_0 and $r > 0$ such that $B(x_0, r) \subset X_{n_0}$. In turn, this gives us that $\|T_i x\| < n_0$ for each $i \in I$ and for each $x \in B(x_0, r)$. We can rewrite this as

$$\|T_i(x_0 + rz)\| = \|T_i x_0 + rT_i z \leq n_0 \ \Rightarrow \ \|T_i z\| \leq \frac{1}{r}(n_0 + \|T_i x_0\|) \ \forall \ i \in I, \ \forall \ z \in B(0, 1)$$

by the reverse triangle inequality. When we pass to the supremum over z, we find that

$$\|T_i\| \leq \frac{1}{r}(n_0 + \|T_i x_0\|) \ \forall i \in I.$$

Next, we pass to the supremum over $i \in I$ to find that

$$\sup_{i \in I} \|T_i\| \leq \frac{1}{r}(n_0 + \sup_{i \in I} \|T_i x_0\|) < \infty.$$

Corollary 6 *If E and F are Banach spaces and $\{T_n\} \subset \mathcal{L}(E, F)$ is such that for each $x \in E$ $T_n x \to l = Tx$, then*

1. $\sup_{n \in \mathbb{N}} \|T_n\| < \infty$,
2. $T \in \mathcal{L}(E, F)$,
3. $\|T\| \leq \liminf_{n \to \infty} \|T_n\|$.

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Corollary 7 *Let $\{T_n\} \subset \mathcal{L}(E, F)$. Suppose that there exists $l \in F$ such that $T_n x \to l$ as $n \to \infty$ for each $x \in E$. Define $T : E \to F$ by $Tx = l$. Then*

1. $\sup_{n \in \mathbb{N}} \|T_n\| < \infty$,
2. $T \in \mathcal{L}(E, F)$,
3. $\|T\| \leq \liminf_{n \to \infty} \|Tn\|$.

Proof : 1. The Banach-Steinhaus theorem tells us that

$$\sup_{n \in \mathbb{N}} \|T_n x\| < \infty \ \Rightarrow \ \sup \|T_n\| < \infty.$$

2. There exists $C > 0$ such that $\|T_n x\| \leq C\|x\|$ for every $x \in E$ and for every $n \in \mathbb{N}$. If we let $n \to \infty$ in this inequality, we find that $\|Tx\| \leq C\|x\|$ for every $x \in E$, so T is continuous. The computation

$$T(\alpha x + \beta y) = \lim_{n \to \infty} T_n(\alpha x + \beta y) = \lim_{n \to \infty} \alpha T_n(x) + \lim_{n \to \infty} \beta T_n(y) = \alpha T(x) + \beta T(y)$$

shows that T is linear.

3. Since $T_n \in \mathcal{L}(E, F)$, we have that $\|T_n x\| \leq \|T_n\|\|x\|$. This gives us that

$$\liminf_{n \to \infty} \|T_n x\| \leq \liminf_{n \to \infty} \|T_n\|\|x\| \ \Rightarrow \ \|Tx\| \leq \liminf_{n \to \infty} \|T_n\|\|x\|.$$

We then have that

$$\|T\| = \sup_{\substack{x \in E \\ \|x\| \leq 1}} \|Tx\| \leq \liminf_{n \to \infty} \|T_n\|\|x\| \leq \liminf_{n \to \infty} \|T_n\|.$$

Corollary 8 *Let G be a Banach space, and $B \subset G$. Assume that $f(B) = \{\langle f, b \rangle : b \in B\}$ is bounded for every $f \in G'$ (weakly bounded). Then B is bounded in G, i.e., there exists $C > 0$ such that $\|b\| < C$ for every $b \in B$.*

Proof : We apply the Banach-Steinhaus theorem with $E = G'$, $F = \mathbb{R}$, $I = B$, and $T_b : G' \to \mathbb{R}$ defined by $T_b(f) = \langle f, b\rangle$. Then

$$\sup_{b\in B} \|T_b f\| = \sup_{b\in B} |\langle f, b\rangle| < \infty,$$

so there exists $C > 0$ such that $\|T_b f\| \leq C\|f\|$ for every $f \in G'$ and for every $b \in B$. Then

$$|\langle f, b\rangle| \leq C\|f\| \ \forall\ f \in G',\ \forall\ b \in B.$$

This gives us that

$$\|b\| = \sup_{\substack{f\in G' \\ \|f\|\leq 1}} |\langle f, b\rangle| \leq C \ \forall\ b \in B.$$

Corollary 9 *Let G be a Banach space, and let $B' \subset G'$, the dual space of G. If $B'(x) = \{\langle f, x\rangle : f \in B'\}$ is bounded for every $x \in B$ (B' is weakly star bounded), then B' is bounded.*

Proof : This is similar to the previous corollary. We let $T_f : G \to \mathbb{R}$, where $T_f(g) = \langle f, g\rangle$, and $I = B'$, $E = G$, and $F = \mathbb{R}$. We then have that

$$\sup_{f\in B'} \|T_f f\| = \sup_{f\in B'} |\langle f, g\rangle| < \infty,$$

so the Banach-Steinhaus theorem implies that there exists $C > 0$ such that $\|T_f\| \leq C\|g\|$ for every $f \in B'$ and for every $g \in G$. If $f \in B'$, then

$$\|f\| = \sup_{\substack{g\in G \\ \|g\|\leq 1}} |\langle f, g\rangle| = \sup_{\substack{g\in G \\ \|g\|\leq 1}} \|T_f\| \leq C.$$

Theorem 7 (Open Mapping Theorem, S. Banach) *Let E and F be Banach spaces, and let $T \in \mathcal{L}(E, F)$, with T surjective. Then there exists $C > 0$ such that $T(B_E(0, 1)) \supset B_F(0, C)$.*

Remark : The statement $T(B_E(0, 1)) \supset B_F(0, C)$ implies that T maps open sets to open sets. Let $U \subset E$ be open. We want to show that $T(U)$ is open in F. Let $y_0 \in T(U)$. Then $y_0 = T(x_0)$ for some $x_0 \in U$. Since U is open, there exists $r > 0$ such that $B(x_0, r) = x_0 + B(0, r) \subset U$. We then have that

$$T(B(x_0, r)) = T(x_0 + B(0, r)) = T(x_0) + T(B(0, r)) = y_0 + T(B(0, r)).$$

The statement in the Theorem then gives that

$$T(B(0, r)) \supset rB(0, C) = B(0, rC).$$

Thus,

$$B(y_0, rC) \subset T(B(x_0, r)).$$

Proof of Open Mapping Theorem : Step 1: There exists $C > 0$ such that $\overline{T(B(0, 1))} \supset B(0, 2C)$. Let

$$X_n = n\overline{T(B(0, 1))},$$

which is a closed set for each n. We claim that

$$\bigcup_{n\in\mathbb{N}} X_n = F.$$

Let $y \in F$. Then $y = Tx$ for some $x \in E$. We have then that

$$Tx = \|x\| T\left(\frac{x}{\|x\|}\right) \in ([\|x\|]+1)\overline{T(B(0,1))} = X_{[\|x\|]+1},$$

where the square brackets denote the integer part of a real number. Now Baire's Theorem implies that there exists n_0 such that $\text{int}X_{n_0} \neq \emptyset$. Thus, there exists $y_0 \in T(B(0,1))$ and $C > 0$ such that $B(y_0, 4C) \subset \overline{T(B(0,1))}$. Then we also have that

$$y_0 + B(0,4C) \subset \overline{T(B(0,1))} \text{ and } -y_0 \in \overline{T(B(0,1))}$$

by the linearity of T and the symmetry of the unit ball. Thus,

$$-y_0 + y_0 + B(0,4C) \subset \overline{T(B(0,1))} + \overline{T(B(0,1))} \subset 2\overline{T(B(0,1))},$$

since $\overline{T(B(0,1))}$ is convex. This is true since if $a, b \in T(B(0,1))$ and $\theta \in [0,1]$, then $a = Tx$ and $b = Ty$ for some $x, y \in B(0,1)$. Then

$$\theta a + (1-\theta)b = \theta Tx + (1-\theta)Ty = T(\theta x + (1-\theta)y) \in T(B(0,1)),$$

since $B(0,1)$ is convex. Since this holds in $T(B(0,1))$, we may pass to the limit and conclude that it also holds in $\overline{T(B(0,1))}$. This gives us that

$$B(0,4C) \subset 2\overline{T(B(0,1))} \;\Rightarrow\; B(0,2C) \subset \overline{T(B(0,1))}.$$

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Recall that we are working on the proof of the Open Mapping theorem, and last time we showed that there exists $C > 0$ such that

$$\overline{T(B(0,1))} \supset B(0,2C). \tag{4}$$

We want to show that if $y \in F$ with $\|y\| < C$, then there exists $x \in B_E(0,1)$ such that $y = Tx$. Let $y \in B_F(0,C) \subset \overline{T(B(0,1/2))}$. The inclusion is true by the linearity of T and the symmetry of the ball, along with (4). Then there exists $z_1 \in E$ such that $\|z_1\| < 1/2$ and $\|y - Tz_1\| < C/2$. This implies that $y - TZ_1 \in B(0, C/2) \subset \overline{T(B(0,1/4))}$. This, in turn, implies that there exists $z_2 \in E$ such that $\|z_2\| < 1/2^2$ and $\|y - Tz_1 - Tz_2\| < C/2^2$. Inductively, we can find a sequence $\{z_n\} \subset E$ such that

$$\|z_n\| < \frac{1}{2^n} \text{ and } \|y - (Tz_1 + \cdots + Tz_n)\| < \frac{C}{2^n},\; n \in \mathbb{N}.$$

Define $x_n = z_1 + \cdots + z_n$. We claim that $\{x_n\}$ is Cauchy. This is verified by the calculation

$$\|x_{n+p} - x_n\| = \|z_{n+p} + \cdots z_{n+1}\| \leq \frac{1}{2^{n+p}} + \frac{1}{2^{n+p-1}} + \cdots + \frac{1}{2^{n+1}} \to 0 \text{ as } n \to \infty.$$

for fixed p. Since E is a Banach space, there exists $x \in E$ such that $x_n \to x$. We also have that

$$\|x\| = \lim_{n\to\infty} \|x_n\| \leq \lim_{n\to\infty} (\|z_1\| + \|z_2\| + \cdots + \|z+n\|) \leq 1$$

and

$$\|y - Tx\| = \lim_{n\to\infty} \|y - Tx_n\| \le \lim_{n\to\infty} \frac{C}{2^n} \;\Rightarrow\; y = Tx.$$

Corollary 10 *Suppose that E and F are Banach spaces, that $T \in \mathcal{L}(E, F)$, and that T is bijective. Then the inverse operator T^{-1} is continuous.*

Proof : The proof uses a scaling argument. We claim that if $\|Tx\|_F < C$, then $\|x\|_E < 1$ (here the C is the one guaranteed by the Open Mapping theorem). Indeed, if

$$z = Tx \in B(0, C) \subset T(B(0, 1)),$$

then there exists $x' \in B(0, 1)$ such that $Tx = Tx'$. But since T is one-to-one, we must then have $x = x'$. Take $\lambda \in \mathbb{R}$. Then

$$\|T(\lambda x)\|_F < C \;\Rightarrow\; \|\lambda x\|_E < 1,$$

and therefore

$$|\lambda|\|Tx\| < C \;\Rightarrow\; |\lambda|\|x\| < 1,$$

$$|\lambda| < \frac{C}{\|Tx\|} \;\Rightarrow\; |\lambda| < \frac{1}{\|x\|}.$$

Since this is true for all λ real, we have that

$$\frac{C}{\|Tx\|} \le \frac{1}{\|x\|} \;\Rightarrow \frac{1}{C}\|Tx\| \;\Rightarrow\; \|T^{-1}y\| \le \frac{1}{C}\|y\| \;\forall\; y \in F.$$

Corollary 11 *Suppose that $(E, \|\cdot\|_1)$ and $(E, \|\cdot\|_2)$ are both Banach spaces. Assume that there exists $C > 0$ such that $\|x\|_2 \le \|x\|_1$ for every $x \in E$. Then there exists $C' > 0$ such that*

$$\|x\|_1 \le C'\|x\|_2 \;\forall\; x \in E.$$

Note that $C_1\|x\|_2 \le \|x\|_1 \le C_2\|x\|_2$ tells us that the norms are equivalent.

Proof : We take $T = id$, where $id : E \to E$ is defined by $id(x) = x$. Since $id \in \mathcal{L}(E, F)$ (since $\|x\|_2 \le \|x\|_1$) and it is bijective, We immediately have that the inverse of the identity operator is continuous as well, which is the desired result.

Theorem 8 (Closed Graph Theorem) *Suppose that E and F are Banach spaces and that $T : E \to F$ is linear. Then $G = \{(x, Tx) : x \in E\}$ is closed if and only if T is continuous.*

Proof : Define $\|\cdot\|_2$ to be the usual norm which makes E a Banach space, and define $\|x\|_1 = \|x\|_2 + \|Tx\|_F$. It can easily be verified that $\|\cdot\|_1$ is also a norm on E, and that $\|x\|_2 \le \|x\|_1$ for every $x \in E$. We claim that $(E, \|\cdot\|_1)$ is a Banach space. Let $\{x_n\}$ be such that

$$\|x_n - x_m\|_2 + \|Tx_n - Tx_m\|_F \to 0 \text{ as } m, n \to \infty.$$

This tells us that each term must go to zero separately. Thus $\{x_n\}$ converges to some $x \in E$ and $\{Tx_n\}$ converges to some $y \in F$, since both are Bancach spaces when endowed with these norms. We also know that $(x, y) \in G(T)$ since the graph is closed. This implies that $y = Tx$, and therefore

$$\|x_n - x\|_1 = \|x_n - x\|_2 + \|Tx_n - Tx\|_F \to 0 \text{ as } n \to \infty,$$

so $(E, \|\cdot\|_1)$ is a Banach space. The previous corollary tells us that there exists $C' > 0$ such that $\|x\| \leq C'\|x\|_2$, and therefore,

$$\|Tx\|_F \leq \|x\|_2 + \|Tx\|_F \leq C'\|x\|_2,$$

so T is continuous. The converse is trivial.

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Theorem 9 *Suppose that $A : D(A) \subset E \to F$ is a linear operator between two Banach spaces E and F, and $D(A)$ is closed. Then A^* is a closed operator.*

Proof : Let $\{u_n\} \subset D(A^*)$ be such that $u_n \to u$, and suppose that $A^*u_n \to f$ in E'. We want to show that

$$\begin{cases} u \in \{D(A^*) = \{u \in F' : \langle u, Av\rangle \leq C\|v\|\} \\ f = A^*u \end{cases}.$$

Since $u_n \to u$ and $A^*u_n \to f$, we have that $\langle A^*u_n, v\rangle_{E',E} \to \langle f, v\rangle_{E',E}$, and $\langle u_n, Av\rangle_{F',F} \to \langle u, Av\rangle_{F',F}$, for every $v \in D(A)$. Thus

$$|\langle u, Av\rangle| = |\langle f, v\rangle| \leq \|f\|\|v\|.$$

If we take $\|f\|$ to be the C in the definition of $D(A^*)$, then $u \in D(A^*)$. Also, since $\langle f, v\rangle = \langle A^*u, v\rangle$ for every $v \in D(A)$ and since $\overline{D(A)} = E$, we have that

$$\langle f, v\rangle = \langle A^*u, v\rangle \ \forall\ v \in E.$$

Thus $f = A^*u$.

Proposition 2 *Suppose that $A : D(A) \subset E \to F$ is linear. Let $X = E \times F$, $G = G(A)$, and $L = E \times \{0\}$. Then*

1. $N(A) \times \{0\} = G \cap L$.
2. $E \times R(A) = G + L$.
3. $\{0\} \times N(A^*) = G^\perp \cap L^\perp$.
4. $R(A^*) \times F' = G^\perp + L^\perp$.

Proposition 3 *Suppose $A : D(A) \subset E \to F$ is a closed operator, and that $D(A)$ is dense in E. Then*

(i) $N(A) = R(A^*)^\perp$.

(ii) $N(A^*) = R(A)^\perp$.

(iii) $N(A)^\perp \supset \overline{R(A^*)}$.

(iv) $N(A^*)^\perp = \overline{R(A)}$.

Proof of (i) : If we can show that

$$N(A) \times \{0\} = G \cap L = (G^\perp + L^\perp)^\perp = (R(A^*) \times F')^\perp = R(A^*) \times \{0\},$$

then we have shown that $N(A) = R(A^*)^\perp$. We must verify $G \cap L = (G^\perp + L^\perp)^\perp$, and that $(R(A^*) \times F')^\perp = R(A^*)^\perp \times \{0\}$. First, to see that $G \cap L \subset (G^\perp + L^\perp)^\perp$, take $(x, y) \in G \cap L$ and $f_1 + f_2 \in G^\perp + L^\perp$. We then have that $\langle f_1 + f_2, (x, y)\rangle = \langle f_1, (x, y)\rangle + \langle f_2, (x, y)\rangle = 0 + 0 = 0$. Next, to see that $(G^\perp + L^\perp)^\perp \subset G \cap L$, we note that $G^\perp \subset G^\perp + L^\perp$, which then implies that $(G^\perp)^\perp \supset (G^\perp + L^\perp)^\perp$. But we also have that $(G^\perp)^\perp = \overline{G} = G$ since A is a closed operator, so $(G^\perp + L^\perp)^\perp \subset G$.

In order to verify that $(R(A^*) \times F')^\perp = R(A^*) \times \{0\}$, note that (up a bijective mapping) we may write $(E \times F)' = E' \times F'$. Suppose that $(x, y) \in (R(A^*) \times F')^\perp$. Then $\langle (u, v), (x, y)\rangle = 0$ for all $(u, v) \in R(A^*) \times F$. In particular, if we fix v, then we see that $\langle u, x\rangle = 0$ for all $u \in R(A^*)$, so $x \in R(A^*)^\perp$. Also, if we fix u, then $\langle v, y\rangle = 0$ for all $v \in F'$, so $v = 0$ (by the corollary to the analytical version of the Hahn-Banach theorem). This shows that $(R(A^*) \times F')^\perp \subset R(A^*)^\perp \times \{0\}$. To see the other inclusion, let $x \in R(A^*)^\perp$. Then if $(u, v) \in R(A^*) \times F'$, we have that $\langle (u, v), (x, 0)\rangle = 0$, so $R(A^*)^\perp \times \{0\} \subset (R(A^*) \times F')^\perp$.

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Theorem 10 (Characterization of Bounded Operators) *Let E and F be Banach spaces, and suppose that $A : D(A) \subset E \to F$ is a closed linear operator and $\overline{D(A)} = E$. Then the following are equivalent:*

(i) A is bounded,

(ii) $D(A) = E$ (closed graph theorem),

(iii) A^* is bounded,

(iv) $D(A^*) = F'$.

Weak Topologies

Throughout the following, E will be a Banach space, $f \in E'$, and $\varphi_f : E \to \mathbb{R}$ is defined by $\varphi_f(x) = \langle f, x\rangle$.

Definition 5 *The weak topology $\sigma(E, E')$ is the finest (the most economical; having the least open sets) on E for which all the maps $\{\varphi_f\}_{f \in E'}$ are continuous.*

Proposition 4 *The weak topology $\sigma(E, E')$ is separated (T2).*

Proof : : Let $x_1, x_2 \in E$ with $x_1 \neq x_2$. We use the Hahn-Banach theorem, second geometric version, with $A = \{x_1\}$ and $B = \{x_2\}$. Then there exists $f \in E'$ and $\alpha \in \mathbb{R}$ such that

$$\langle f, x\rangle < \alpha < \langle f, x_2\rangle.$$

We need to find disjoint open sets $\mathcal{O}_1$ and $\mathcal{O}_2$ in $\sigma(E, E')$ such that $x \in \mathcal{O}_1$ and $x_2 \in \mathcal{O}_2$. We take

$$\mathcal{O}_1 = \{x : \langle f, x\rangle < \alpha\} = \varphi_f^{-1}((-\infty, \alpha))$$

and

$$\mathcal{O}_2 = \{x : \langle f, x\rangle > \alpha\} = \varphi_f^{-1}((\alpha, \infty)).$$

Proposition 5 *Let $x_0 \in E$. The family $V = \{x : |\langle f, x - x_0\rangle| < \epsilon,\ i \in I\}$, with $f \in E'$, $i \in I$, a finite set, and $\epsilon > 0$ forms a basis for x_0 in the weak topology $\sigma(E, E')$.*

Proof : We have that
$$x_0 \in V = \bigcap_{i\in I} \varphi_{f_i}^{-1}((\langle f_i, x_0\rangle - \epsilon, \langle f_i, x_0\rangle + \epsilon)),$$
and V is open in $\sigma(E, E')$. Let U be an open set in $\sigma(E, E')$ which contains x_0. From the way $\sigma(E, E')$ was constructed (details were omitted), U contains a set of the form
$$W = \bigcap_{i\in I} \varphi_{f_i}^{-1}(\mathcal{O}_i),$$
with I a finite set and $\mathcal{O}_i$ a neighborhood of $\langle f_i, x_0\rangle$. Choose $\epsilon > 0$ small enough such that
$$(\langle f_i, x_0\rangle - \epsilon, \langle f_i, x_0\rangle + \epsilon) \subset \mathcal{O}_i \ \forall\ i \in I.$$
This implies that the corresponding V is a subset of W, so $x_0 \in V \subset W \subset U$.

Proposition 6 *Suppose that E is a Banach space, $\{x_n\} \subset E$. Then*

(i) $x_n \rightharpoonup x \ \Leftrightarrow\ \langle f, x_n\rangle \to \langle f, x\rangle \ \forall\ f \in E'$,

(ii) $x_n \to x \ \Rightarrow\ x_n \rightharpoonup x$.

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Theorem 11 *Suppose that E is a Banach space. Then the following are true:*

(i) $x_n \rightharpoonup x$ weakly in E if and only if $\langle f, x_n\rangle \to \langle f, x\rangle$ for all $f \in E'$.

(ii) $x_n \to x$ strongly in E implies that $x_n \rightharpoonup x$ weakly in E.

(iii) $x_n \rightharpoonup x$ weakly in E implies that $\{x_n\}$ is bounded and
$$\liminf_{n\to\infty} \|x_n\| \geq \|x_n\|.$$

(iv) If $f_n \to f$ strongly in E' and $x_n \rightharpoonup x$ weakly in E, then $\langle f_n, x_n\rangle \to \langle f, x\rangle$.

Proof : (i)($\Rightarrow$) Since $f \in E'$, f is continuous with respect to the weak topology $\sigma(E, E')$. Thus $x_n \rightharpoonup x$ implies that $\langle f, x_n\rangle \to \langle f, x\rangle$.

($\Leftarrow$) Let U be a weak neighborhood of x in $\sigma(E, E')$. Without loss of generality, we may assume that
$$U = \{y \in E : |\langle f_i, y - x\rangle| < \epsilon, i \in I\},$$
where I is a finite set and $f_i \in E'$ for all $i \in I$. Since $\langle f_i, x_n\rangle \to \langle f_i, x\rangle$ for each $i \in I$, we have that for each $i \in I$ there exists $N_i \in \mathbb{N}$ such that $|\langle f_i, x_n - x\rangle| \leq \epsilon$ for each $n \geq N_i$. Let $N = \max_{i\in I} N_i$. Then $x_n \in U$ for every $n \geq N$, which implies that $x_n \rightharpoonup x$ weakly.

(ii) Since $x_n \to x$ strongly in E, we have that
$$|\langle f, x_n\rangle - \langle f, x\rangle| = |\langle f, x_n - x\rangle| \leq \|f\|\|x_n - x\| \to 0.$$

(iii) Recall one of the corollaries of the Banach-Steinhaus theorem:

If $B \subset E$ and $f(B) = \{\langle f, x\rangle : x \in B\}$ is bounded for every $f \in E'$, then $\{x_n\}$ is bounded in E, i.e., there exists $C > 0$ such that $\|x_n\| \leq C$ for all n.

Here we take $B = \{x_n\}$. Since $\langle f, x_n\rangle \to \langle f, x\rangle$ for every $f \in E'$, we have that $f(B)$ is bounded for every $f \in E'$. Thus $\{x_n\}$ is bounded in E. Also, we have that

$$\liminf_{n\to\infty} |\langle f, x_n\rangle| \leq \liminf_{n\to\infty} \|f\|\|x_n\| \ \Rightarrow\ |\langle f, x\rangle| \leq \|f\| \liminf_{n\to\infty} \|x_n\|.$$

We then pass to the supremum over $f \in E'$ with $\|f\| \leq 1$ to find that

$$\|x\| \leq \liminf_{n\to\infty} \|x_n\|.$$

(iv) We have that

$$|\langle f, x_n\rangle - \langle f, x\rangle| \leq |\langle f_n, x_n\rangle - \langle f, x_n\rangle| + |\langle f, x_n\rangle - \langle f, x\rangle| \leq \|f_n - f\|\|x_n\| + |\langle f, x_n\rangle - \langle f, x\rangle|.$$

The first term goes to zero by the convergence $f_n \to f$ strongly in E', and the fact that $\{x_n\}$ is bounded since it converges weakly. The second term goes to zero by the definition of weak convergence.

Proposition 7 *Let E be a Banach space with* $\dim(E) = n < \infty$*. Then a set is open in the strong topology of E if and only if it is open in $\sigma(E, E')$. In particular, in finite dimensional spaces, weak convergence is equivalent to strong convergence.*

Proof : It is trivial that if a set is weakly open, then it is also strongly open. Conversely, let U be open in the strong topology, and let $x_0 \in U$. We want to find f_i, $i = 1, 2, \ldots, n$ and $\epsilon > 0$ such that

$$V = \{x \in E : |\langle f_i, x - x_0\rangle| < \epsilon, i \in I\} \subset U.$$

There exists $r > 0$ such that $B(x_0, r) \subset U$. Let $f_i \in E'$ be such that $\langle f, x\rangle = x_i$, where $x = (x_1, x_2, \ldots, x_n)$. Take $\epsilon = r/nC$, where C is such that

$$\|x\|_2 \leq C\|x\|_1 \ \forall\ x \in E,$$

which is possible since all norms on E are equivalent. Then we have that

$$\|x - x_0\|_2 \leq \|x - x_0\|_1 = \sum_{j=1}^{n} |x_j - x_{0j}| \leq C \sum_{j=1}^{n} |\langle f_i, x - x_0| \leq Cn\epsilon = r.$$

Thus $V \subset B(x_0, r) \subset U$.

Remark : If $\dim(E) = \infty$, then one can always find sets which are open (closed) in the strong topology, but are not open (closed) in the weak topology.

Example : Suppose that E is a Banach space and $\dim(E) = \infty$. Then if $S = \{x \in E : \|x\| = 1\}$, S is not closed in the weak topology $\sigma(E, E')$. In fact, the weak closure of S is

$$\overline{S}^{\sigma(E,E')} = \{x \in E : \|x\| \leq 1\}.$$

Example : Let E be a Banach space with $\dim(E) = \infty$. Then the open unit ball

$$B_E(0,1) = \{x \in E : \|x\| \leq 1\}$$

is not open in the weak topology. In fact,

$$\operatorname{int}_{\sigma(E,E')}(B_E(0,1)) = \emptyset.$$

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Example : Let E be a Banach space with $\dim(E) = \infty$. Then

$$S = \{x \in E : \|x\| = 1\}$$

is not weakly closed. In fact, we have

$$\overline{S}^{\sigma(E,E')} = \{x \in E : \|x\| \leq 1\}.$$

Let $x_0 \in E$ be such that $\|x_0\| \leq 1$. We want to show that $x_0 \in \overline{S}^{\sigma(E,E')}$. Consider a neighborhood V of x_0 in $\sigma(E,E')$. We'll show that $V \cap S \neq \emptyset$. Without loss of generality, we may assume that

$$V = \{x \in E : |\langle f_i, x\rangle| < \epsilon,\ i \in I\},$$

where I is a finite set (of n elements, say) and $f_i \in E'$ for all i. We claim that there exists $y_0 \in E\backslash\{0\}$ such that $\langle f_i, y_0\rangle = 0$ for every $i \in I$. If this were not so, then the map $\varphi : E \to \mathbb{R}^n$ defined by

$$\varphi(z) = (\langle f_1, z\rangle, \langle f_2, z\rangle, \ldots, \langle f_n, z\rangle)$$

is injective, which in turn implies that $\dim(E) \leq n$, which is a contradiction. Now define $g : [0,\infty) \to [0,\infty)$ by $g(t) = \|x_0 + ty_0\|$. Then g is continuous, $g(0) = \|x_0\| < 1$, and $\lim_{t\to\infty} g(t) = \infty$. The intermediate value theorem tells us that there exists $t_0 \in [0,\infty)$ such that $g(t_0) = 1$, so $x_0 + t_0y_0 \in S$ and $x_0 + t_0y_0 \in V$ as well. This shows that $\{x \in e : \|x\| \leq 1\} \subset \overline{S}^{\sigma(E,E')}$. It remains to show that $\{x \in E : \|x\| \leq 1\}$ is weakly closed. The following lemma contains this result:

Lemma 5 *Let C be a convex set in E. Then C is strongly closed if and only if C is weakly closed.*

Proof : ($\Leftarrow$) Easy.

($\Rightarrow$) Let C be a strongly closed, convex subset of E. We want to show that $E\backslash C$ is open in $\sigma(E,E')$. Let $x_0 \in E\backslash C$. The Hahn-Banach theorem, second geometric version, tells us that there exists $f \in E'$ and $\alpha \in \mathbb{R}$ such that

$$\langle f, x_0\rangle < \alpha < \langle f, y\rangle\ \forall\ y \in C.$$

Let $V = \{x \in E : \langle f, x\rangle < \alpha\}$, which is open in $\sigma(E,E')$, and $x_0 \in V \subset E\backslash C$, so $E\backslash C$ is open in $\sigma(E,E')$.

Note : Included in the proof of the example above is the fact that every open set in the weak topology contains an entire line.

Example : The set $U = \{x \in E : \|x\| < 1\}$ is not open in $\sigma(E,E')$ if $\dim(E) = \infty$. Indeed, $\operatorname{int}_{\sigma(E,E')}(U) = \emptyset$. This follows from the note above.

Proposition 8 *If $I : E \to \mathbb{R} \cup \{\infty\}$ is convex and lower semi-continuous (for every $\alpha \in \mathbb{R}$, the set $\{x \in E : I(x) \leq \alpha\}$ is closed, in particular, $u_n \to u$ strongly implies $I(u) \leq \liminf_{n\to\infty} I(u_n)$), then I is weakly lower semi-contiunuous ($\{x \in E : I(x) \leq \alpha\}$ is closed in $\sigma(E, E')$ for every $\alpha \in \mathbb{R}$).*

Proof : Let $\alpha \in \mathbb{R}$ and consider the set $\{x \in E : I(x) \leq \alpha\}$. This set is convex and strongly closed, which implies that it is weakly closed. This implies that I is weakly lower semi-continuous.

Recall that we proved before that $\|u\| \leq \liminf_{n\to\infty} \|u_n\|$ whenever $u_n \rightharpoonup u$ weakly in E. A simpler proof is to use the previous proposition and take $I(u) = \|u\|$, which is convex and continuous.

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Theorem 12 *Let E and F be Banach spaces, and let $T \in \mathcal{L}(E, F)$. Then T is also continuous from $(E, \sigma(E, E'))$ to $(F, \sigma(F, F'))$ and conversely.*

Proof : Let us consider $T : (E, \sigma(E, E')) \to (F, \sigma(F, F'))$. Let U be $\sigma(F, F')$-open. We want to show that $T^{-1}(U)$ is $\sigma(E, E')$-open. We may write

$$U = \bigcup_{arb.} \bigcap_{i\in I} \varphi_{f_i}^{-1}(\mathcal{O}_i),$$

where the $\mathcal{O}_i$ are open intervals in $\mathbb{R}$, I is a finite set, and $f_i \in F'$ for all i, by the construction of $\sigma(F, F')$. We then have that

$$T^{-1}(U) = T^{-1}\left(\bigcup_{arb.} \bigcap_{i\in I} \varphi_{f_i}(\mathcal{O}_i)\right) = \bigcup_{arb} \bigcap_{i\in I} T^{-1}(\varphi_{f_i}^{-1}(\mathcal{O}_i)) = \bigcup_{arb.} \bigcap_{i\in I} (\varphi_{f_i} \circ T)^{-1}(\mathcal{O}_i).$$

Also, we have that

$$(\varphi_{f_i} \circ T)(x) = \varphi_{f_i}(T(x)) = \langle f_i, Tx\rangle_{F',F},$$

and $\varphi_{f_i} \circ T : E \to \mathbb{R}$ is linear and continuous, since

$$|(\varphi_{f_i} \circ T)(x)| \leq \|f_i\|\|Tx\| \leq \|f_i\|\|T\|\|x\|.$$

Thus, the set $(\varphi_{f_i} \circ T)^{-1}(\mathcal{O}_i)$ is in the subbasis for $\sigma(E, E')$, and therefore $T^{-1}(U)$ is open in $\sigma(E, E')$. Conversely, if $T : (E, \sigma(E, E')) \to (F, \sigma(F, F'))$ is continuous and linear, then $G(T)$ is $\sigma(E, E') \times \sigma(F, F')$-closed, which implies that it is strongly closed. This implies that T is strongly continuous.

The Weak $*$ Topology on $\mathbf{E'}$

If E is a Banach space, then E' is its dual, and $E'' = (E')'$ is the bidual, which is endowed with the norm

$$\|\xi\|_{E''} = \sup_{\|f\|_{E'}\leq 1} |\langle \xi, f\rangle|.$$

Define $J : E \to E''$ by $\langle Jx, f\rangle_{E',E''} = \langle f, x\rangle_{E',E}$ for every $x \in E$ and $f \in E'$. This is the cannonical mapping from E to its bidual. The mapping J is also a linear isometry, since

$$\|Jx\|_{E''} = \sup_{\|f\|_{E'}\leq 1} |\langle Jx, f\rangle| = \sup_{\|f\|_{E'}\leq 1} |\langle f, x\rangle| = \|x\|_E,$$

by a corollary to the Hahn-Banach theorem.

Remark : J is not surjective in general. If J happens to be surjective, then E is said to be a reflexive space.

The Banach space E' comes with the strong topology, which it inherits from its norm, and a weak topology $\sigma(E', E'')$. We can define another topology on E' as follows. For $x \in E$, consider the family of mappings $\varphi_x : E' \to \mathbb{R}$ defined by $\varphi_x(f) = \langle f, x\rangle_{E',E}$. Then the weak* topology on E', denoted $\sigma(E', E)$ is the topology which has the smallest number of open sets among all topologies that make the mappings $\{\varphi_x\}_{x\in E}$ continuous.

Remark : We have that $E \subset E''$ in the sense of the canonical mapping. Also, the weak* topology has fewer open (closed) sets than the weak topology. Also note that A weak* open in E' implies A weakly open in E, which implies A strongly open.

Remark : Less and less open sets implies more and more compact sets.

Theorem 13 (Banach-Alaouglu) *The unit ball $B_E(0, 1) = \{f \in E' : \|f\|_E \leq 1\}$ is weakly* compact.*

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Notation : If f_n converges to f in the weak* topology $\sigma(E', E)$, then we write $f_n \overset{*}{\rightharpoonup} f$.

Proposition 9 *The weak* topology $\sigma(E', E)$ is separated.*

Proposition 10 *Let $f_0 \in E'$. A basic system of neighborhoods for f_0 in the weak* topology is given by the sets*

$$\{f \in E' : |\langle f - f_0, x_i\rangle| < \epsilon,\ i \in I\},$$

where I is a finite set, $x_i \in E$ for each i, and $\epsilon > 0$.

Proposition 11 *(i) $f_n \overset{*}{\rightharpoonup} f$ implies $\langle f_n, x\rangle \to \langle f, x\rangle$ for each $x \in E$.*

(ii) $f_n \to f$ strongly implies $f_n \rightharpoonup f$ weakly in $\sigma(E', E'')$, which implies $f_n \overset{*}{\rightharpoonup} f$ weakly in $\sigma(E', E)$.*

(iii) $f_n \overset{*}{\rightharpoonup} f$ weakly implies that $\{f_n\}$ is bounded in E' and $\|f\|_{E'} \leq \liminf_{n\to\infty} \|f_n\|_{E'}$.*

(iv) If $f_n \overset{*}{\rightharpoonup} f$ weakly in E' and $x_n \to x$ strongly in E, then $\langle f_n, x_n\rangle \to \langle f, x\rangle$.*

Remark : If $\dim(E) < \infty$, then $\sigma(E', E) = \sigma(E', E'') = (E', \|\cdot\|_{E'})$.

Theorem 14 (Banach-Alaouglu) *The closed unit ball*

$$\overline{B}_{E'}(0, 1) = \{f \in E : \|f\| \leq 1\}$$

is weakly compact.*

Proof : Let

$$Y = \mathbb{R}^E = \{\omega : \omega = (\omega_x)_{x\in E},\ \omega_x \in \mathbb{R}.\},$$

endowed with the product topology (the topology with the smallest number of open sets among all those that make all maps $Y \ni \omega \mapsto \omega_x \in \mathbb{R}$ $(x \in E)$ continuous). Define $\Phi : E' \to Y$ by $\Phi(f) = (\langle f, x\rangle)_{x\in E}$. We claim that Φ is a homeomorphism onto $\Phi(E')$, where E' is considered with the weak topology and Y has the product topology (proof to come soon (hopefully)). Secondly, we claim that

$$\Phi(\overline{B}_{E'}(0, 1) := K$$

$$= \{\omega \in Y : |\omega_x| \le \|x\| \ \forall\ x \in E,\ \omega_{x+y} = \omega_x + \omega_y,\ \omega_{\lambda x} = \lambda\omega_x \ \forall\ x, y \in E,\ \lambda \in \mathbb{R}\}.$$

If $f \in \overline{B}_{E'}(0,1)$, then $\Phi(f) = (\langle f, x\rangle)_{x\in E}$. This implies that

$$|\langle f, x\rangle| \le \|f\|\|x\| \le \|x\|.$$

Also, we have that

$$\Phi(f)_{x+y} = \langle f, x+y\rangle = \langle f, x\rangle + \langle f, y\rangle = \Phi(f)_x + \Phi(f)_y$$

and

$$\Phi(f)_{\lambda x} = \langle f, \lambda x\rangle = \lambda\langle f, x\rangle = \lambda\Phi(f)_x.$$

On the other hand, if $\omega \in K$, we define $f \in E'$ by $\langle f, x\rangle = \omega_x$. It is clear that f defined in this way is continuous and $\|f\| \le 1$. We then note that

$$K = \left(\prod_{x\in E}[-\|x\|, \|x\|]\right) \cap \left(\bigcap_{\substack{x\in E\\ y\in E}}\{\omega \in Y : \omega_{x+y} - \omega_x - \omega_y = 0\}\right) \cap$$

$$\left(\bigcap_{\substack{x\in E\\ \lambda\in\mathbb{R}}}\{\omega \in Y : \omega_{\lambda x} - \lambda\omega_x = 0\}\right).$$

We can think of this as the intersection of three sets, the first of which is compact by Aleksandrov's theorem, and the last two are closed by the continuity of the component mappings in the product topology. Since the intersection of a compact set with a closed set is compact, we have that K is compact. Then we simply note that

$$\overline{B}_{E'}(0,1) = \Phi^{-1}(K).$$

Functional Analysis 3-3-06

Reflexive Spaces

We define the mapping $J : E \to E'$ by $\langle Jx, f\rangle_{E'',E} = \langle f, x\rangle_{E',E}$. This mapping is an isometry, injective, and linear.

Definition 6 *A Banach space E is reflexive if $J(E) = E''$.*

Remark : The elements of E'' will be identified (via J) with elements of E when E is reflexive.

Remark : The use of J in the definition is important. There exist non-reflexive spaces E for which one can construct a surjective isometry between E and E'' (example given by R.C. James in the 1950's).

Theorem 15 (Characterization of Reflexive Spaces, Kakutani) *E is reflexive if and only if $\overline{B}_E(0,1)$ is weakly compact.*

Proof : ($\Rightarrow$) First we claim that $J(\overline{B}_E(0,1)) = \overline{B}_{E''}(0,1)$. To see this, suppose that $x_0 \in \overline{B}_E(0,1)$. Then

$$\|Jx_0\|_{E''} = \sup_{\substack{f\in E' \\ \|f\|_{E'}\leq 1}} |\langle Jx_0, f\rangle| = \sup_{\substack{f\in E' \\ \|f\|_{E'}\leq 1}} |\langle f, x_0\rangle| = \|x_0\|_E \leq 1.$$

Conversely, if $\|\xi\|_{E''} \leq 1$ and E is reflexive, then there exists x_0 such that $Jx_0 = \xi$. Then the same calculation as above shows that $\|x_0\| \leq 1$. Next we claim that $J^{-1} : (E'', \sigma(E'', E')) \to (E, \sigma(E, E'))$ is continuous. If so, then $\overline{B}_E(0,1)$ is the image of the (weakly*) compact set $\overline{B}_{E''}(0,1)$ through a continuous mapping, so $\overline{B}_E(0,1)$ is compact in $\sigma(E, E')$.

In order to prove the claim, let

$$\mathcal{O} = \bigcup_{arb}\bigcap_{i\in I} \varphi_{f_i}^{-1}(\mathcal{O}_i),$$

where the $\mathcal{O}_i$ are open sets in $\mathbb{R}$, $\varphi_f : E \to \mathbb{R}$ is defined by $\varphi_f(x) = \langle f, x\rangle$, and I is a finite set. We then have that

$$(J^{-1})^{\leftarrow}(\mathcal{O}) = (J^{-1})^{\leftarrow}\left(\bigcup_{arb}\bigcap_{i\in I} \varphi_{f_i}^{\leftarrow}(\mathcal{O}_i)\right) = \bigcup_{arb}\bigcap_{i\in I}(J^{-1})^{\leftarrow}(\varphi_{f_i}^{\leftarrow}(\mathcal{O}_i))$$

$$= \bigcup_{arb}\bigcap_{i\in I}(\varphi_{f_i} \circ J^{-1})^{\leftarrow}(\mathcal{O}_i).$$

We calculate that

$$(\varphi_{f_i} \circ^{-1})(\xi) = \varphi_{f_i}(J^{-1}\xi) = \langle f_i, J^{-1}\xi\rangle = \langle J(J^{-1}(\xi), f_i\rangle = \langle \xi, f_i\rangle.$$

This is of the form φ_x, and the preimages of open sets under these mappings form the subbasis for the weak* topology. Thus each of the sets above are open in the weak* topology.

The proof of the converse is omitted (see text).

Proposition 12 *Suppose that E is a reflexive Banach space, and let M be a closed subspace of E. Then M is also reflexive.*

Proof : We have that

$$\overline{B}_M(0,1) = \{x \in M : \|x\|_E \leq 1\} = \overline{B}_E(0,1) \cap M.$$

We have that $\overline{B}_E(0,1)$ is weakly compact and M is strongly closed and convex, so it is also weakly closed. Thus their intersection is weakly compact by Kakutani. (Here we are using the restricted topology on M.)

Corollary 12 *A Banach space E is reflexive if and only if E' is reflexive.*

Proof : ($\Rightarrow$) If E is reflexive, then $\sigma(E', E)'' ='' \sigma(E', E'')$. Since $\overline{B}_{E'}(0,1)$ is weak* compact, it is also weakly compact, and therefore E' is reflexive. The details of the converse are left as an exercise.

Functional Analysis 3-6-06

Lemma 6 *Suppose that E is a reflexive Banach space and that $A \subset E$ is closed, bounded, and convex. Then A is $\sigma(E, E')$-compact.*

Proof : Since A is closed and convex, it is $\sigma(E, E')$ closed. Since A is bounded, there exists $n \in \mathbb{N}$ such that $A \subset \overline{B}_E(0, n)$, which is compact in the topology $\sigma(E, E')$. Since A is a closed subset of a compact set, it is also $\sigma(E, E')$-compact.

Theorem 16 *Let E be a reflexive Banach space, and suppose that $A \subset E$ is nonempty, closed, and convex. Let $\varphi : A \to R \cup \{\infty\}$ be proper ($\varphi \not\equiv \infty$), lower semi-continuous, convex, and coercive ($\lim_{\|x\| \to \infty} \varphi(x) = \infty$). Then φ attains a minimum.*

Proof : Let $a \in A$ be such that $\varphi(a) < \infty$. Define

$$\tilde{A} = \{x \in A : \varphi(x) \leq \varphi(a)\} \neq \emptyset.$$

Then $\tilde{A}$ is convex, closed, and bounded (by coercivity of φ). The previous lemma then implies that $\tilde{A}$ is weakly compact. Also, φ is weakly lower semi-continuous, since it is strongly lower semicontinuous and convex. Thus there exists $x_0 \in \tilde{A}$ such that $\varphi(x_0) \leq \varphi(x)$ for every $x \in \tilde{A}$. Let $x \in A \backslash \tilde{A}$. Then $\varphi(x_0) \leq \varphi(a) < \varphi(x)$.

Separable Spaces

Definition 7 *Let (E, d) be a metric space. We say that E is separable if there exists $D \subset E$ that is both countable and dense in E.*

Proposition 13 *Let E be a Banach space. Then E' separable implies that E is separable.*

Remark : The converse is false. For example, take $E = L^1(\Omega)$.

Proof : Let $\{f_n\} \subset E'$ be dense in E'. Recall that the norm in E' is defined by

$$\|f_n\|_{E'} = \sup_{\|x\| \leq 1} \langle f_n, x \rangle,$$

so there exists $x_n \in E$ with $\|x_n\| \leq 1$ such that

$$\frac{1}{2}\|f_n\| \leq \langle f_n, x_n \rangle \leq \|f_n\|.$$

Define

$$L_0 = \left\{ \sum_{i=1}^{m} q_i x_i : q_i \in \mathbb{Q},\ m \in \mathbb{N} \right\} = \text{span}\langle \{x_n\} \rangle.$$

Note that L_0 is countable. We claim that L_0 is dense in E. To see this, define

$$L = \left\{ \sum_{i=1}^{m} r_i x_i : r_i \in \mathbb{R},\ m \in \mathbb{N} \right\}.$$

Then L_0 is dense in L. We want to show that L is dense in E, so we show that if $f \in E'$ and $\langle f, x \rangle = 0$ for every $x \in L$, then $f \equiv 0$. Without loss of generality, we may assume that $f_n \to f$. Let $\epsilon > 0$ and choose N large enough so that $\|f - f_n\| < \epsilon$ for $n \geq N$. We then calculate that

$$\frac{1}{2}\|f_n\| \leq \langle f_n, x_n \rangle = \langle f_n - f, x_n \rangle + \langle f, x_n \rangle \leq \|f_n - f\| \|x_n\| \leq \|f_n - f\| < \epsilon.$$

Now let $\epsilon \to 0$.

Corollary 13 *A Banach space E is reflexive and separable if and only if E' is reflexive and separable.*

Proof : ($\Leftarrow$) Follows from above.

($\Rightarrow$) We have that $E'' = J(E)$, and J is a bijective isometry. Thus if E is reflexive and separable, then E'' is reflexive and separable, and thus E' is by the above proposition.

Theorem 17 *Let E be a separable Banach space. Then $\overline{B}_{E'}(0,1)$ with the weak* topology is metrizable. Conversely, if E is a Banach space such that $\overline{B}_{E'}(0,1)$ in the weak* topology is metrizable, then E is separable.*

Corollary 14 *Suppose E is a separable Banach space and $\{f_n\} \subset E'$ is bounded. Then there exists a subsequence $\{f_{n_k}\}$ and $f \in E'$ such that $f_{n_k} \overset{*}{\rightharpoonup} f$ weakly*.*

Theorem 18 (Eberlein-Smulian) *Let E be a reflexive Banach space. Then the restriction of the weak topology $\sigma(E, E')$ on $\overline{B}_E(0,1)$ is metrizable. Conversely, if E is a Banach space such that the restriction of $\sigma(E, E')$ to $\overline{B}_E(0,1)$ is metrizable, then E is reflexive.*

Corollary 15 *If E is a reflexive Banach space and $\{x_n\} \subset E$ is bounded, then there exists a subsequence $\{x_{n_k}\}$ and an element $x \in E$ such that $x_{n_k} \rightharpoonup x$ weakly.*

Functional Analysis 3-7-06

The Lᵖ Spaces

Let $\Omega \subset \mathbb{R}^N$ be open.

Definition 8 *We define*

$$L^1(\Omega) = \left\{ f : \Omega \to \mathbb{R} : f \text{ is measurable}, \int_\Omega |f(x)|\, dx < \infty \right\},$$

with the norm

$$\|f\|_{L^1(\Omega)} = \int_\Omega |f(x)|\, dx.$$

We also recall the following results:

Theorem 19 (The Monotone Convergence Theorem, Beppo-Levi) *Suppose that $\{f_n\} \subset L^1(\Omega)$ and that $f_n \le f_{n+1}$ a.e. in Ω, and that*

$$\sup_{n \in \mathbb{N}} \int_\Omega |f_n|\, dx < \infty.$$

Then $f_n(x) \to f(x)$ for a.e. $x \in \Omega$. Moreover, $f \in L^1(\Omega)$ and $\|f_n - f\|_{L(\Omega)} \to 0$ as $n \to \infty$.

Theorem 20 (Lebesgue Dominated Convergence Theorem) *Suppose $\{f_n\} \subset L^1(\Omega)$, and assume that*

(i) $f_n(x) \to f(x)$ for a.e. $x \in \Omega$

(ii) There exists $g \in L^1(\Omega)$ such that $|f_n(x)| \le g(x)$ for a.e. $x \in \Omega$ and every n.

Then $f \in L^1(\Omega)$ and $\|f_n - f\|_{L^1(\Omega)} \to 0$.

Theorem 21 (Fatou's Lemma) *Suppose that* $\{f_n\} \subset L^1(\Omega)$*,* $f_n \geq 0$ *a.e. in* Ω *for each* n*, and that*

$$\sup_{n\in\mathbb{N}} \int_\Omega f_n \; dx < \infty.$$

Let

$$f(x) = \liminf_{n\to\infty} f_n(x).$$

Then $f \in L^1(\Omega)$ *and*

$$\int_\Omega f(x) \; dx \leq \liminf_{n\to\infty} \int_\Omega f_n(x) \; dx.$$

Theorem 22 *Let*

$$C_c(\Omega) = \{\textit{continuous } f : \Omega \to \mathbb{R} : \exists \; K \textit{ compact s.t. } f \equiv 0 \textit{ on } \Omega \backslash K\}.$$

We have that $C_c(\Omega)$ *is dense in* $L^1(\Omega)$ *(*$\forall \; f \in L^(\Omega)$ *and* $\forall \; \epsilon > 0$*,* $\exists \; f_1 \in C_c(\Omega)$ *such that* $\|f - f_1\|_{L^1(\Omega)} < \epsilon$*).*

In what follows, let $\Omega_1 \subset \mathbb{R}^{N_1}$ and $\Omega_2 \subset \mathbb{R}^{N_2}$ be open, and let $F : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be measurable.

Theorem 23 (Tonelli) *If*

$$\int_{\Omega_2} |F(x,y)| \; dy < \infty \textit{ and } \int_{\Omega_1} \left(\int_{\Omega_2} |F(x,y)| \; dy \right) \; dx < \infty,$$

then $F \in L^1(\Omega_1 \times \Omega_2)$*.*

Theorem 24 (Fubini) *Let* $F \in L^1(\Omega_1 \times \Omega_2)$*. Then*

(i) $F(x, \cdot) \in L^1(\Omega_2)$ *and*

$$\int_{\Omega_1} \left(\int_{\Omega_2} |F(x,y) \; dy \right) \; dx < \infty.$$

(ii) $F(\cdot, y) \in L^1(\Omega_1)$ *and*

$$\int_{\Omega_2} \int_{\Omega_1} |F(x,y)| \; dx \; dy < \infty.$$

Moreover,

$$\int_{\Omega_1} \left(\int_{\Omega_2} F(x,y) \; dy \right) \; dx = \int_{\Omega_2} \left(\int_{\Omega_1} F(x,y) \; dx \right) \; dy = \int_{\Omega_1 \times \Omega_2} F(x,y) \; dx \, dy.$$

Definition 9 *For* $1 < p < \infty$ *we define*

$$L^p(\Omega) = \{\textit{measurable } f : \Omega \to \mathbb{R} : |f|^p \in L^1(\Omega)\},$$

endowed with the norm

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p \; dx \right)^{\frac{1}{p}}.$$

For $p=\infty$, we define

$$L^\infty(\Omega) = \{\text{measurable } f : \Omega \to \mathbb{R} : \exists\ C > 0 \text{ st } |f(x)| \le C \text{ for a.e. } x \in \Omega\},$$

with the norm

$$esssup_\Omega f = \|f\|_{L^\infty(\Omega)} = \inf\{C > 0 : |f(x)| \le C \text{ for a.e. } x \in \Omega\}.$$

Proposition 14 *If $f \in L^\infty(\Omega)$, then $|f(x)| \le \|f\|_{L^\infty(\Omega)}$ for a.e. $x \in \Omega$.*

Proof : Let C_n be such that

$$\|f\|_{L^\infty(\Omega)} \le C_n \le \|f\|_{L^\infty(\Omega)} + \frac{1}{n}, \tag{5}$$

and $|f(x)| \le C_n$ for every $x \in \Omega\backslash E_n$, with $\mathcal{L}^N(E_n) = 0$ for every n. Let

$$E = \bigcup_{n\in\mathbb{N}} E_n.$$

Then $\mathcal{L}(E) = 0$. If we let $n \to \infty$ in (5), then

$$|f(x)| \le \|f\|_{L^\infty(\Omega)}\ \forall\ x \in \Omega\backslash E.$$

Theorem 25 (Fischer-Riesz) *For $1 \le p \le \infty$, $(L^p(\Omega), \|\cdot\|_{L^p(\Omega)})$ is a Banach space.*

Remark : The proof uses Hölder's inequality: for $1 \le p \le \infty$, define p', the conjugate exponent of p such that $1/p + 1/p' = 1$ $(\infty' = 1,\ 1' = \infty)$. Then

$$\int_\Omega |fg|\ dx \le \|f\|_{L^p(\Omega)}\|g\|_{L^{p'}(\Omega)}.$$

Proposition 15 *Let $\{f_n\} \subset L^p(\Omega)$ be such that $\|f_n - f\|_{L^p(\Omega)} \to 0$. Then there exists a subsequence $\{f_{n_k}\} \subset \{f_n\}$ and there exists $g \in L^p(\Omega)$ such that*

$f_{n_k} \to g(x)$ for a.e. $x \in \Omega$,

$|f_{n_k}(x)| \le g(x)$ for a.e. $x \in \Omega$ and for every $n \in \mathbb{N}$.

Remark : $f_n \to f$ in $L^p(\Omega)$ does not in general imply that $f_n(x) \to f(x)$ for a.e. $x \in \Omega$ for the whole sequence.

The Study of $\mathbf{L^p(\Omega)}$ for$\mathbf{1 < p < \infty}$

Theorem 26 *$L^p(\Omega)$ is reflexive for $1 < p < \infty$.*

Proof : Step 1: Suppose that $2 \le p < \infty$. We'll show first in this case that $L^p(\Omega)$ is a uniformly convex Banach space.

Definition 10 *A Banch space E is uniformly convex if for every $\epsilon > 0$ there exists $\delta > 0$ such that $x, y \in E$ with $\|x\|, \|y\| \le 1$ and $\|x - y\| > \epsilon$ imply*

$$\left\|\frac{x+y}{2}\right\| < 1 - \delta.$$

If $\epsilon > 0$ and $f, g \in L^p(\Omega)$ with $\|f\|_{L^p(\Omega)}, \|g\|_{L^p(\Omega)} \leq 1$ and $\|f - g\|_{L^p(\Omega)} > \epsilon$, then we employ Clarkson's inequality:

If $f, g \in L^p(\Omega)$ and $2 \leq p < \infty$, then

$$\left\|\frac{f+g}{2}\right\|_{L^p(\Omega)}^p + \left\|\frac{f-g}{2}\right\|_{L^p(\Omega)}^p \leq \frac{1}{2}\left(\|f\|_{L^p(\Omega)}^p + \|g\|_{L^p(\Omega)}^p\right).$$

In our case, we have

$$\left\|\frac{f+g}{2}\right\|_{L^p(\Omega)}^p \leq 1 - \left(\frac{\epsilon}{2}\right)^p,$$

so we take

$$\delta = \min\left\{\frac{1}{2}, \left(1 - \left(\frac{\epsilon}{2}\right)^p\right)^{\frac{1}{p}}\right\}.$$

Then we have that

$$\left\|\frac{f+g}{2}\right\|_{L^p(\Omega)} \leq \left(1 - \left(\frac{\epsilon}{2}\right)^p\right)^{\frac{1}{p}} \;\Rightarrow\; \left\|\frac{f+g}{2}\right\|_{L^p(\Omega)} < 1 - \delta.$$

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Theorem 27 *$L^p(\Omega)$ is reflexive for $1 < p < \infty$.*

Proof : Step 1: We want to show first that $L^p(\Omega)$ is reflexive if $p \geq 2$. Last time we showed that for $p \geq 2$, $L^p(\Omega)$ is uniformly convex. (E is uniformly convex if for every $\epsilon > 0$ there exists $\delta > 0$ such that $x, y \in E$ with $\|x\|, \|y\| < 1$ and $\|x - y\| > \epsilon$ implies that $\|(x+y)/2\| < 1 - \delta$.)

Lemma 7 (Milman-Pettis) *Any uniformly convex Banach space is reflexive.*

Proof : If E is uniformly convex, we want to show that $J(E) = E''$. ($\langle Jx, f\rangle = \langle f, x\rangle$.) Since J is an isometry, we only need to show that $J(\overline{B}_E(0,1)) = B_{E''}(0,1)$. Note thtat the set on the right hand side is strongly closed in E''. Let $\xi \in \overline{B}_E(0,1)$ have $\|\xi\|_{E''} = 1$. We'll show that $\xi \in \overline{J(\overline{B}_E(0,1))} = J(\overline{B}_E(0,1))$, or, for every $\epsilon > 0$, there exists $x \in \overline{B}_E(0,1)$ such that $\|Jx - \xi\| < \epsilon$. Let $\epsilon > 0$ and let $\delta > 0$ be from the definition of uniform convexity of E. Let $f \in E'$ be such that $\|f\|_{E'} \leq 1$ and

$$\langle \xi, f\rangle_{E'',E} > 1 - \frac{\delta}{2}. \tag{6}$$

Consider

$$V = \left\{\eta \in E'' : |\langle \eta - \xi, f\rangle| < \frac{\delta}{2}\right\}.$$

This is a neighborhood of ξ in the weak topology $\sigma(E'', E')$. We need the following

Lemma 8 (Goldstine) *$J(\overline{B}_E(0,1))$ is dense in $\overline{B}_{E''}(0,1)$ with respect to $\sigma(E'', E')$*

This lemma tells us that $V \cap J(\overline{B}_E(0,1)) \neq \emptyset$, so there exists $x \in \overline{B}_E(0,1)$ such that $J(x) \in V$, i.e.

$$|\langle J(x) - \xi, f\rangle| < \frac{\delta}{2} \;\Leftrightarrow\; |\langle Jx, f\rangle - \langle \xi, f\rangle| < \frac{\delta}{2} \;\Leftrightarrow\; |\langle f, x\rangle - \langle \xi, f\rangle| < \frac{\delta}{2}. \tag{7}$$

We claim that $\xi \in J(x) + \epsilon \overline{B}_{E''}(0,1)$. If not, then

$$\xi \in E'' \backslash (J(x) + \epsilon \overline{B}_{E''}(0,1)) =: W,$$

which is an open set. Also, since $\overline{B}_{E''}(0,1)$ is weakly closed, it is also closed in $\sigma(E'', E')$, so W is a weak* neighborhood of ξ. Using Goldstine's lemma again, we find that there exists $\hat{x} \in B_E(0,1)$ such that $J(\hat{x}) \in V \cap W$. The same computation as in (7) gives that

$$|\langle f, \hat{x} \rangle - \langle \xi, f \rangle| < \frac{\delta}{2} \tag{8}$$

Combining (7) and (8) gives that

$$2\langle \xi, f \rangle \leq \langle f, x + \hat{x} \rangle + \delta \leq \|x + \hat{x}\| + \delta,$$

and along with (6), we have that

$$\left\| \frac{x + \hat{x}}{2} \right\| \geq 1 - \delta.$$

Our choice of δ implies that $\|x - \hat{x}\| \leq \epsilon$, so $\|Jx - J\hat{x}\| \leq \epsilon$, and therefore, $J\hat{x} \notin W$, which is a contradiction with $J\hat{x} \in V \cap W$.

Step 2: Now we show that $L^p(\Omega)$ is reflexive for $1 < p \leq 2$. Consider $T : L^p(\Omega) \to (L^{p'}(\Omega))'$ $(1/p + 1/p' = 1)$ defined by

$$\langle Tu, f \rangle = \int_\Omega f(x)u(x)\ dx$$

for every $u \in L^p(\Omega)$ and every $f \in L^{p'}(\Omega)$. Then $|\langle Tu, f \rangle| \leq \|f\|_{L^{p'}(\Omega)} \|u\|_{L^p(\Omega)}$, which implies that $\|Tu\|_{(L^{p'}(\Omega))'} \leq \|u\|_{L^p(\Omega)}$. We claim that actually $\|Tu\|_{(L^{p'}(\Omega))'} = \|u\|_{L^p(\Omega)}$. To see this, we take

$$f_0(x) = \begin{cases} |u(x)|^{p-2}u(x) & \text{if } u(x) \neq 0 \\ 0 & \text{otherwise} \end{cases} \in L^{p'}(\Omega),$$

since

$$\int_\Omega |f_0|^{p'}\ dx = \int_\Omega |u(x)|^{(p-1)p'}\ dx = \int_\Omega |u(x)|^p\ dx < \infty.$$

We calculate that

$$\|f_0\|_{L^{p'}(\Omega)} = \left(\int_\Omega |u|^p\ dx \right)^{\frac{1}{p'}} = \left(\left(\int_\Omega |u|^p\ dx \right)^{\frac{1}{p}} \right)^{\frac{p}{p'}} = \|u\|^{p-1}_{L^p(\Omega)},$$

which gives that

$$\frac{\langle Tu, f_0 \rangle}{\|f_0\|_{L^{p'}(\Omega)}} = \frac{\int_\Omega |u(x)|^p\ dx}{\|f_0\|_{L^{p'}(\Omega)}} = \frac{\|u\|^p_{L^p(\Omega)}}{\|f_0\|_{L^{p'}(\Omega)}} = \|u\|_{L^p(\Omega)}.$$

Now consider $T : L^p(\Omega) \to T(L^p(\Omega)) \subset (L^{p'}(\Omega))'$. Since $T(L^p(\Omega))$ is a closed subspace and $1 < p \leq 2$, we have that $p' \geq 2$, so $(L^{p'}(\Omega))'$ is reflexive, and then $L^p(\Omega)$ is reflexive as well.

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Remark : If $1 < p \le 2$, then $L^p(\Omega)$ is uniformly convex (and therefore reflexive), as can be seen by Clarkson's second inequality:

$$\left\| \frac{f+g}{2} \right\|_{L^p(\Omega)}^{p'} + \left\| \frac{f-g}{2} \right\|_{L^p(\Omega)}^{p'} \le \left[\frac{1}{2} \left(\|f\|_{L^p(\Omega)}^p + \|g\|_{L^p(\Omega)}^p \right) \right]^{\frac{1}{p-1}}.$$

Riesz′ Representation Theorem

Theorem 28 *Let $1 < p < \infty$, and let $\varphi \in (L^p(\Omega))'$. Then there exists a unique element $u \in L^{p'}(\Omega)$ (1/p+1/p'=1) such that*

$$\langle \varphi, f \rangle_{(L^p(\Omega))', L^p(\Omega)} = \int_\Omega fu \; dx.$$

Moreover, $\|\varphi\|_{(L^p(\Omega))'} = \|u\|_{L^{p'}(\Omega)}$.

Proof : Define $T : L^{p'}(\Omega) \to (L^p(\Omega))'$ by

$$\langle Tu, f \rangle = \int_\Omega uf \; dx$$

for $u \in L^{p'}(\Omega)$. First, we observe that T is an isometry, i.e.

$$\|Tu\|_{L^p(\Omega))'} = \|u\|_{L^{p'}(\Omega)}.$$

The Hölder inequality gives one inequality, and the method followed in the proof of reflexivity of L^p spaces gives the opposite inequality. Next, we want to show that T is surjective, or that

$$E := T(L^{p'}(\Omega)) = (L^p(\Omega))'.$$

Since E is closed (think about Cauchy sequences), we will show that E is dense in $(L^{p'}(\Omega))'$. Let $\varphi \in (L^p(\Omega))''$ be such that $\langle \varphi, f \rangle = 0$ for every $f \in E$. We want to show that $\varphi \equiv 0$. Let $J(h) = \varphi$. Then since $\langle \varphi, Tu \rangle = 0$ for every $u \in L^{p'}(\Omega)$, we have that

$$\int_\Omega hu \; dx = 0 \; \forall \; u \in L^{p'}(\Omega) \tag{9}$$

Since $h \in L^p(\Omega)$, we have that $u = |h|^{p-2}h \in L^{p'}(\Omega)$, and therefore

$$\int_\Omega |h|^p \; dx = 0 \;\; \Rightarrow \;\; h = 0 \text{ a.e. } x \in \Omega.$$

Theorem 29 *$C_c(\Omega)$ is dense in $L^p(\Omega)$ for $1 \le p < \infty$.*

Proof : The case $p = 1$ was stated before. Let $1 < p < \infty$. We want to show that if $\varphi \in (L^p(\Omega))'$ with $\langle \varphi, f \rangle = 0$ for every $f \in C_c(\Omega)$, then $\varphi \equiv 0$. According to the representation theorem proved above, we must decide if $u \in L^{p'}(\Omega)$ and

$$\int_\Omega uf \; dx = 0 \; \forall \; f \in C_c(\Omega)$$

implies that $u = 0$ for a.e. $x \in \Omega$

Lemma 9 (The Fundamental Theorem of the Calculus of Variations) *Let $u \in L^1_{loc}(\Omega)$ ($\forall\ K \subset\subset \Omega,\ u\chi_K \in L^1(\Omega)$). If*

$$\int_\Omega uf\ dx = 0\ \forall\ f \in C_c(\Omega),$$

then $u = 0$ for a.e. $x \in \Omega$.

Theorem 30 *If $1 < p < \infty$, then $L^p(\Omega)$ is separable.*

Proof : Let $\epsilon > 0$ and $f \in L^p(\Omega)$. Also, let $f_1 \in C_c(\Omega)$ be such that $\|f - f_1\|_{L^p(\Omega)} < \epsilon$. Let Ω' be open and be such that $\text{supp}(f_1) \subset \Omega' \subset \Omega$. Let $\mathcal{R}$ be the family of all open cubes with vertices that are rational numbers. Since f is uniformly continuous, there exists $\delta > 0$ such that if $x, y \in \Omega'$ are such that $|x - y| < \delta$, then $|f_1(x) - f_1(y)| < \epsilon/(2|\Omega|^{\frac{1}{p}})$. Consider an arbitrary covering of $\text{supp}(f_1)$ with cubes from $\mathcal{R}$ which have diameters less than δ. Extract a subset $\mathcal{R}' \subset \mathcal{R}$, which is a finite subcovering of $\text{supp}(f_1)$. Let

$$E = \left\{ \sum_{i=1}^m q_i \chi_{Q_i} : i = 1, 2, \ldots, m,\ , \in \mathbb{N},\ Q_i \in \mathcal{R} \right\}.$$

We claim that there exists $f_2 \in E$ such that $\|f_1 - f_2\|_{L^p(\Omega)} < \epsilon$. If this is the case, then we are done. To this end, we seek f_2 such that

$$|f_2(x) - f_1(x)| < \frac{\epsilon}{|\Omega'|^{\frac{1}{p}}} \tag{10}$$

for a.e. $x \in \Omega'$. This will prove the claim. Let us write $\mathcal{R}' = \{Q_1, \ldots, Q_m\}$, and

$$f_2(x) = \sum_{i=1}^m q_i \chi_{Q_i}(x),$$

where the q_i are chosen that

$$\left| q_i - \max_{\overline{Q}_i} f_1 \right| < \frac{\epsilon}{2|\Omega'|^{\frac{1}{p}}}.$$

Let $f(x_0) = \max_{\overline{Q}_i} f_1(x)$. Then if $x \in Q_i$,

$$|f_2(x) - f_1(x)| = |f_2(x) - f_1(x_0)| + |f_1(x_0) - f_1(x)| < \frac{\epsilon}{2|\Omega'|^{\frac{1}{p}}} + \frac{\epsilon}{2|\Omega'|^{\frac{1}{p}}},$$

because of the choice of q_i and the uniform continuity of f_1.

Functional Analysis 3-20-06

The Study of $\mathbf{L^1(\Omega)}$

Recall that for $1 < p < \infty$, $(L^p(\Omega))' = L^{p'}(\Omega)$.

Theorem 31 (The dual of $L^1(\Omega)$ is $L^\infty(\Omega)$) *Let $\varphi \in (L^1(\Omega))'$. Then there exists a unique $u \in L^\infty(\Omega)$ such that*

$$\langle \varphi, f \rangle_{(L^1)', L^1} = \int_\Omega f(x)u(x)\ dx\ \forall\ f \in L^1(\Omega).$$

Moreover, $\|\varphi\|_{(L^1(\Omega))'} = \|u\|_{L^\infty(\Omega)}$.

Proof : Without loss of generality, assume that $0 \in \Omega$. Define $w \in L^2(\Omega)$ by $w(x) = \alpha_n$ if $x \in \Omega \cap \{x \in \mathbb{R}^n : n < |x| \leq n+1\}$, where

$$\alpha_n = \frac{1}{(n+1)\sqrt{|B(0,n+1) - B(0,n)|}}.$$

Then

$$\int_\Omega |w|^2 \, dx \leq \sum_{n=0}^{\infty} \int_{\{x\in\mathbb{R}^n : n<|x|\leq n+1\}} \alpha_n^2 \, dx$$

$$= \sum_{n=0}^{\infty} \alpha_n^2 |B(0,n+1)\backslash B(0,n)| = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = \frac{\pi^2}{6}.$$

Notice that for any compact set $K \subset \Omega$, there exists and $\epsilon(K) > 0$ such that $w(x) \geq \epsilon(K)$ for a.e. $x \in K$. We define a function by

$$L^2(\Omega) \ni f \overset{\Psi}{\mapsto} \langle \varphi, wf \rangle_{(L^1)',L^1}.$$

We claim that $\Psi \in (L^2(\Omega))'$. To see this, note that

$$|\Psi(f)| \leq \|\varphi\|_{(L^1(\Omega))'} \|wf\|_{L^1(\Omega)} \leq \|\varphi\|_{(L^1(\Omega))'} \|w\|_{L^2(\Omega)} \|f\|_{L^2(\Omega)}.$$

This being the case, the Riesz Representation theorem tell us that there exists $v \in L^2(\Omega)$ such that

$$\langle \Psi, f \rangle_{(L^2)',L^2} = \langle \varphi, wf \rangle_{(L^1)',L^1} = \int_\Omega vf \, dx \ \forall \ f \in L^2(\Omega) \tag{11}$$

Define a function u by

$$u(x) = \frac{v(x)}{w(x)}$$

for a.e. $x \in \Omega$. We now claim that this is the u that we desire. First, we will show that $u \in L^\infty(\Omega)$. Let $C > \|\varphi\|_{(L^1(\Omega))'}$. We'll show that $|u(x)) \leq C$ for a.e. $x \in \Omega$. Let

$$A = \{x \in \Omega : |u(x)| > C\}.$$

We want to show that $|A| = 0$. To obtain a contradiction, assume that $|A| > 0$. Consider a set $\tilde{A} \subset A$ such that $0 < |\tilde{A}| < |A|$. Let $f = \chi_{\tilde{A}} \text{sgn}(u) \in L^2(\Omega)$. By (11), we have that

$$\int_{\tilde{A}} uw \ \text{sgn}(u) \, dx = \int_{\tilde{A}} |u| w \, dx \leq \|\varphi\|_{L^1(\Omega))'} \int_{\tilde{A}} |wf| \, dx = \|\varphi\|_{(L^1(\Omega))'} \int_{\tilde{A}} |w| \, dx.$$

Thus

$$C \int_{\tilde{A}} w \, dx \leq \|\varphi\|_{L^1(\Omega))'} \int_{\tilde{A}} w \, dx,$$

and since

$$\int_{\tilde{A}} w \, dx \neq 0,$$

we have that $C \leq \|\varphi\|_{(L^1(\Omega))'}$, which is a contradiction. Thus $|A| = 0$ and $u \in L^\infty(\Omega)$ and $\|u\|_{L^\infty(\Omega)} \leq C$. Now let $C \to \|\varphi\|_{(L^1(\Omega))'}$. Let $g \in C_c(\Omega)$, and define $f = g/w$. We test with f in (11) to find that

$$\langle \varphi, g \rangle_{(L^1)', L^1} = \int_\Omega uw\frac{g}{w}\ dx = \int_\Omega ug\ dx\ \forall\ g \in C_c(\Omega),$$

and therefore we also have that

$$\langle \varphi, g \rangle_{(L^1)', L^1} = \int_\Omega ug\ dx\ \forall\ g \in L^1(\Omega)$$

by the density of $C_c(\Omega)$ and the continuity of φ. Using the Hölder inequality and passing to the sup over all $g \in L^1(\Omega)$ with $\|g\|_{L^1(\Omega)} \leq 1$, we find that $\|\varphi\|_{(L^1(\Omega))'} \leq \|u\|_{L^\infty(\Omega)}$. All that remains is the uniqueness claim. Assume that given $\varphi \in (L^1(\Omega))'$, there exist $u_1, u_2 \in L^\infty(\Omega)$ such that

$$\langle \varphi, g \rangle_{(L^1)', L^1} = \int_\Omega u_1 g\ dx = \int_\Omega u_2 g\ dx.$$

Then

$$\int_\Omega (u_1 - u_2) g\ dx = 0\ \forall\ g \in L^1(\Omega),$$

and the Fundamental Theorem of the Calculus of Variations implies that $u_1 - u_2 = 0$ for a.e. $x \in \Omega$.

Theorem 32 *$L^1(\Omega)$ is not reflexive.*

Proof : Without loss of generality, assume that $0 \in \Omega$. Let us suppose that $L^1(\Omega)$ is reflexive. Consider

$$f_n = \frac{1}{\left|B\left(0, \frac{1}{n}\right)\right|}\chi_{B\left(0,\frac{1}{n}\right)} \in L^1(\Omega).$$

Then

$$\|f_n\|_{L^1(\Omega)} = \int_{B\left(0,\frac{1}{n}\right)} \frac{1}{\left|B\left(0, \frac{1}{n}\right)\right|}\ dx = 1.$$

Thus $\{f_n\}$ is bounded in $L^1(\Omega)$. The Eberlein-Smulyan theorem implies that there exists a subsequence $\{f_{n_k}\}$ and a $f \in L^1(\Omega)$ such that $f_{n_k} \rightharpoonup f$ weakly in $L^1(\Omega)$. Our representation theorem also tells us that

$$\int_\Omega f_{n_k} g\ dx \to \int_\Omega fg\ dx\ \forall\ g \in L^\infty(\Omega).$$

Take $g \in C_c(\Omega\backslash\{0\})$. Then for k large, $\text{supp}(g) \cap \text{supp}(f_{n_k}) = \emptyset$. This implies that

$$\int_\Omega fg\ dx = 0\ \forall\ g \in C_c(\Omega).$$

The Fundamental Theorem of the Calculus of Variations implies that $f = 0$ for a.e. $x \in \Omega\backslash\{0\}$, so $f = 0$ a.e. in Ω. If we take $g \equiv 1$, then

$$\int_\Omega f\ dx = \lim_{k\to\infty} \|f_{n_k}\|_{L^1(\Omega)} = 1,$$

which is a contradiction.

Functional Analysis 3-21-06

The Study of $\mathbf{L}^\infty(\boldsymbol{\Omega})$

So far we know that $L^\infty(\Omega) = (L^1(\Omega))'$. This gives us the following immediate results.

1. $\overline{B}_{L^\infty(\Omega)}(0,1)$ is compact with respect to the weak* topology $\sigma(L^\infty(\Omega), L^1(\Omega))$ by Banach-Alaouglu.

2. $L^1(\Omega)$ is separable, so if a sequence $\{f_n\} \subset L^\infty(\Omega)$ is bounded, then there exists a subsequence $\{f_{n_k}\}$ and an $f \in L^\infty(\Omega)$ such that $f_{n_k} \rightharpoonup f$ weakly* in $L^\infty(\Omega)$.

3. $L^\infty(\Omega)$ is not reflexive.

Also, $L^\infty(\Omega) = (L^1(\Omega))'$ implies that $(L^\infty(\Omega))' = (L^1(\Omega))'' \supset L^1(\Omega)$ strictly. In other words, the dual of $L^\infty(\Omega)$ contains $L^1(\Omega)$ strictly.

Remark : There exist linear and continuous finctionals on $L^\infty(\Omega)$, say $\varphi \in (L^\infty(\Omega))'$, for which there is no function $u \in L^1(\Omega)$ such that

$$\langle \varphi, f\rangle_{(L^\infty)', L^\infty} = \int_\Omega uf \; dx \;\forall\; f \in L^\infty(\Omega).$$

Proof : Assume without loss of generality that $0 \in \Omega$. Define $\varphi_0 \in (C_c(\Omega))'$ by $\varphi_0(f) = f(0)$. Then $|\varphi_0(f)| \leq \|f\|_{L^\infty(\Omega)}$. Since $C_c(\Omega)$ is a subspace of $L^\infty(\Omega)$, we may apply the Hahn-Banach theorem to extend φ_0 to $\varphi \in (L^\infty(\Omega))'$. Then

$$\langle \varphi, f\rangle_{(L^\infty)', L^\infty} = f(0) \;\forall\; f \in C_c(\Omega).$$

Let us assume that there exists $u \in L^1(\Omega)$ such that

$$\langle \varphi, f\rangle_{(L^\infty)', L^\infty} = \int_\Omega uf \; dx \;\forall\; f \in L^\infty(\Omega).$$

For any $f \in C_c(\Omega\backslash\{0\})$ (extended to $C_c(\Omega)$ by $f(0) = 0$), we have

$$\int_\Omega uf \; dx = \langle \varphi, f\rangle_{(L^\infty)', L^\infty} = f(0) = 0.$$

This implies that $u = 0$ for a.e. $x \in \Omega$ by the Fundamental Theorem of the Calculus of Variations, so $\varphi \equiv 0$ on $L^\infty(\Omega)$. Since $\varphi = \varphi_0$ on $C_c(\Omega)$, we must have that $f(0) = 0$ for all $f \in C_c(\Omega)$, which is an obvious contradiction.

Theorem 33 *$L^\infty(\Omega)$ is not separable.*

The proof of the theorem will follow from this lemma:

Lemma 10 *Let E be a Banach space. Assume that there exists a sequence $\{\mathcal{O}_i\}_{i\in I}$ of subsets of E such that*

(i) $\mathcal{O}_i \neq \emptyset$ and is open for every $i \in I$.

(ii) $\mathcal{O}_i \cap \mathcal{O}_j = \emptyset$ if $i \neq j$.

(iii) I is not countable.

Then E is not separable.

Proof : (of lemma) Assume to the contrary that there is a sequence $\{u_n\}_{n\in\mathbb{N}} \subset E$ that is dense in E. Then assumption (i) implies that for each i, there exists $n(i)$ such that $u_{n(i)} \in \mathcal{O}_i$. Consider the mapping

$$I \ni i \mapsto n(i) \in \mathbb{N}.$$

This map is injective because the $\mathcal{O}_i$ are disjoint. This implies that $\text{card}(\mathbb{N}) \leq \text{card}(I)$, which is a contradiction.

Pf (of theorem) For each $a \in \Omega$, take $r_a < \text{dist}(a, \mathbb{R}^n \backslash \Omega)$, and take $u_a = \chi_{B(a,r_a)}$. Define $\mathcal{O}_a = B_{L^\infty}(u_a, 1/2)$. Then the $\mathcal{O}_a$ are open, nonempty, and $\{a \in \Omega\}$ is uncountable. We claim that if $a \neq b$, then $\mathcal{O}_a \cap \mathcal{O}_b = \emptyset$. If not, let $a \neq b$ be such that $f \in \mathcal{O}_a \cap \mathcal{O}_b$. Then $\|f - u_a\|_{L^\infty(\Omega)} < 1/2$ and $\|f - u_b\|_{L^\infty(\Omega)} < 1/2$. Thus the triangle inequality implies that $\|u_a - u_b\|_{L^\infty(\Omega)} < 1$. On the other hand, we have that

$$(u_a - u_b)(x) = \begin{cases} 1 & \text{if } x \in B(a, r_a) \backslash B(b, r_b) \\ -1 & \text{if } x \in B(b, r_b) \backslash B(a, r_a) \\ 0 & \text{elsewhere} \end{cases} .$$

This implies that $B(a, r_a) = B(b, r_b)$.

In conclusion, we have the following table.

Space	Reflexive	Separable	Dual
$L^1(\Omega)$	No	Yes	$L^\infty(\Omega)$
$L^p(\Omega)$ $1 < p < \infty$	Yes	Yes	$L^{p'}(\Omega)$ $1/p + 1/p' = 1$
$L^\infty(\Omega)$	No	No	Contains $L^1(\Omega)$ strictly.

Functional Analysis 3-22-06

Convolution and Mollification

Theorem 34 *Let $\Omega = \mathbb{R}^N$, and let $1 \leq p \leq \infty$. Then if $f \in L^1(\mathbb{R}^N)$ and $g \in L^p(\mathbb{R}^N)$, for a.e. $x \in \mathbb{R}^N$, the map*

$$x \mapsto f(x - y)g(y)$$

is measurable, and if the convolution product is given by

$$(f * g)(x) = \int_{\mathbb{R}^N} f(x - y)g(y)\ dy,$$

then $f * g \in L^p(\mathbb{R}^N)$. Moreover, $\|f * g\|_{L^p(\mathbb{R}^N)} \leq \|f\|_{L^1(\mathbb{R}^N)} \|g\|_{L^p(\mathbb{R}^N)}$.

Proof : Case 1: $p = \infty$. We have that

$$\left| \int_{\mathbb{R}^N} f(x - y)g(y)\ dy \right| \leq \int_{\mathbb{R}^N} |f(x - y)||g(y)|\ dy$$

$$\leq \|g\|_{L^\infty(\mathbb{R}^N)} \int_{\mathbb{R}^N} |f(y)|\ dy = \|g\|_{L^\infty(\mathbb{R}^N)} \|f\|_{L^1(\mathbb{R}^N)}.$$

Thus

$$\|f * g\|_{L^\infty(\mathbb{R}^N)} \leq \|g\|_{L^\infty(\mathbb{R}^N)} \|f\|_{L^1(\mathbb{R}^N)}.$$

Case 2: $p = 1$. Set $F(x,y) = f(x-y)g(y)$. Then

$$\int_{\mathbb{R}^N} |F(x,y)|\ dx = \int_{\mathbb{R}^N} |f(x-y)||g(y)|\ dx = |g(y)| \int_{\mathbb{R}^N} |f(z)|\ dz < \infty.$$

This implies that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |F(x,y)|\ dx\, dy = \int_{\mathbb{R}^N} |g(y)| \int_{\mathbb{R}^N} |f(z)|\ dz\, dy < \infty,$$

so we may apply Fubini's theorem to compute that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |F(x,y)|\ dy\, dx = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |F(x,y)|\ dx\, dy \leq \|f\|_{L^1(\mathbb{R}^N)}\|g\|_{L^1(\mathbb{R}^N)},$$

and therefore,

$$\|f*g\|_{L^1(\mathbb{R}^N)} = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |f(x-y)g(y)|\ dy\, dx \leq \|f\|_{L^1(\mathbb{R}^N)}\|g\|_{L^1(\mathbb{R}^N)}.$$

Case 3: We now take $1 < p < \infty$. In this case we have that $|g|^p \in L^1(\mathbb{R}^N)$ and $|f| \in L^1(\mathbb{R}^N)$. From case 2, we see that for a.e. $x \in \mathbb{R}^N$, the mapping $[y \mapsto |f(x-y)||g(y)|^p]$ is integrable. Therefore,

$$[y \mapsto |f(x-y)|^{\frac{1}{p}}|g(y)|] \in L^p(\mathbb{R}^N) \text{ and } [y \mapsto |f(x-y)|^{\frac{1}{p'}}] \in L^{p'}(\mathbb{R}^N),$$

and then Hölder's inequality implies that

$$[y \mapsto |f(x-y)|^{\frac{1}{p'}}|f(x-y)|^{\frac{1}{p}}|g(y)|] \in L^1(\mathbb{R}^N) \ \Rightarrow\ [y \mapsto |f(x-y)||g(y)|] \in L^1(\mathbb{R}^N).$$

For a.e. $x \in \mathbb{R}^N$, we have that

$$\begin{aligned} |(f*g)(x)| &\leq \int_{\mathbb{R}^N} |f(x-y)||g(y)|\ dy = \int_{\mathbb{R}^N} |f(x-y)|^{\frac{1}{p'}}|f(x-y)|^{\frac{1}{p}}|g(y)|\ dy \\ &\leq \||f(x-\cdot)|^{\frac{1}{p'}}\|_{L^{p'}(\mathbb{R}^N)}\||f(x-\cdot)|^{\frac{1}{p}}|g|\|_{L^p(\mathbb{R}^N)} \\ &= \left\{\int_{\mathbb{R}^N} |f(x-y)|\ dy\right\}^{\frac{1}{p'}} \left\{\int_{\mathbb{R}^N} |f(x-y)||g(y)|^p\ dy\right\}^{\frac{1}{p}} = \|f\|_{L^1(\mathbb{R}^N)}^{\frac{1}{p'}} \left\{(|f|*|g|^p)(x)\right\}^{\frac{1}{p}}. \end{aligned}$$

Thus,

$$|(f*g)(x)|^p \leq \|f\|_{L^1(\mathbb{R}^N)}^{\frac{p}{p'}}(|f|*|g|^p)(x).$$

The second factor here is integrable by case 2. Again we calculate that

$$\|f*g\|_{L^p(\mathbb{R}^N)}^p \leq \|f\|_{L^1(\mathbb{R}^N)}^{\frac{p}{p'}}\||f|*|g|^p\|_{L^1(\mathbb{R}^N)} \leq \|f\|_{L^1(\mathbb{R}^N)}^{\frac{p}{p'}}\|f\|_{L^1(\mathbb{R}^N)}\|g\|_{L^p(\mathbb{R}^N)}^p,$$

and taking the pth root of both sides, we obtain

$$\|f*g\|_{L^p(\mathbb{R}^N)} \leq \|f\|_{L^1(\mathbb{R}^N)}^{\frac{1}{p'}}\|f\|_{L^1(\mathbb{R}^N)}^{\frac{1}{p}}\|g\|_{L^p(\mathbb{R}^N)} = \|f\|_{L^1(\mathbb{R}^N)}\|g\|_{L^p(\mathbb{R}^N)}.$$

If f is a continuous function, then $\text{supp}(f)$ is the compliment of the largest open set on which f vanishes, or $\text{supp}(f) = \overline{\{x \in \Omega : f(x) \neq 0\}}$.

Definition 11 *Let $f : \Omega \to \mathbb{R}$ be measurable, and let $\{\omega_i\}_{i\in I}$ be the family of all open sets such that $f = 0$ a.e in ω_i $(i \in I)$. Let*

$$\omega = \bigcup_{i\in I} \omega_i,$$

which is open, and $f = 0$ a.e. in ω. We call supp$(f) = \Omega\backslash\omega$.

Proof that $f = 0$ a.e. in ω: For each $n \in \mathbb{N}$, define $K_n = \{x \in \omega : \text{dist}(x, \mathbb{R}^N\backslash\omega) \geq \frac{1}{n} \text{ and } |x| \leq n\}$. Then

$$\omega = \bigcup_{n\in\mathbb{N}} K_n.$$

Since for each n, the set K_n is compact, and

$$K_n \subset \bigcup_{i\in I} \omega_i,$$

for each $n \in \mathbb{N}$, there exists a finite set $I_N \subset I$ such that

$$K_n \subset \bigcup_{i\in I_n} \omega_i.$$

If we also define

$$J = \bigcup_{n\in\mathbb{N}} I_n,$$

then

$$\omega = \bigcup_{i\in J} \omega_i,$$

so $f = 0$ a.e. in Ω.

Functional Analysis 3-24-06

Recall that last time we gave the definition of support for measurable functions.

Remark : 1. If $f_1, f_2 : \Omega \to \mathbb{R}$ are measurable and $f_1 = f_2$ for a.e. $x \in \Omega$, then $\text{supp}(f_1) = \text{supp}(f_2)$. In particular, we can talk about $\text{supp}(f)$ for $f \in L^p(\Omega)$ without referring to a particular representation of f.

2. If $f : \Omega \to \mathbb{R}$ is continuous, then $\text{supp}(f) = \overline{\{x : f(x) \neq 0\}}$.

Proposition 16 *If $f \in L^1(\mathbb{R}^N)$ and $g \in L^p(\mathbb{R}^N)$, and if $1 \leq p \leq \infty$, $\text{supp}(f*g) \subset \overline{\text{supp}(f) + \text{supp}(g)}$.*

Proof : We have that

$$(f * g)(x) = \int_{\mathbb{R}^N} f(x-y)g(y)\ dy = \int_{(\text{supp}(g))\cap(\{x\}-\text{supp}(f))} f(x-y)g(y)\ dy.$$

If $x \notin \text{supp}(f) + \text{supp}(g)$, then $(\text{supp}(g)) \cap (\{x\} - \text{supp}(f)) = \emptyset$, so $(f * g)(x) = 0$ for a.e. $x \in \mathbb{R}^N\backslash\overline{\text{supp}(f) + \text{supp}(g)}$. In particular this is true if $x \in \text{int}(\mathbb{R}^N\backslash\text{supp}(f) + \text{supp}(g)) = \mathbb{R}^N\backslash\overline{(\text{supp}(f) + \text{supp}(g))}$. The equality here follows from DeMorgan's law:

$$\mathbb{R}^N\backslash\overline{A} = \mathbb{R}^N\backslash \bigcap_{\substack{F\supset A \\ F \text{ closed}}} F = \bigcup_{\substack{F\supset A \\ F \text{ closed}}} \mathbb{R}^N\backslash F = \bigcup_{\substack{V\subset\mathbb{R}^N\backslash A \\ V \text{ open}}} V = \text{int}(\mathbb{R}^N\backslash A).$$

Thus there exists $i_0 \in I$ such that $\omega_{i_0} = \mathbb{R}^N \backslash \overline{(\text{supp}(f) + \text{supp}(g))}$, where the ω_i $(i \in I)$ are those open sets in the definition of supp$(f * g)$. We now have that

$$\text{supp}(f * g) = \mathbb{R}^N \backslash \left(\bigcup_{i \in I} \omega_i \right) = \bigcap_{i \in I} (\mathbb{R}^N \backslash \omega_i) \subset \mathbb{R}^N \backslash \omega_{i_0} = \overline{\text{supp}(f) + \text{supp}(g)}.$$

Mollifiers

Definition 12 *A sequence $\{\rho_n\} \subset C_c^\infty(\mathbb{R}^N)$ is called a mollifying sequence if*

(i) supp$(\rho_n) \subset \overline{B}(0, \frac{1}{n})$,

(ii) $\rho_n \geq 0$ in $\mathbb{R}^N$,

(iii)

$$\int_{\mathbb{R}^N} \rho_n \; dy = 1$$

for all n.

Remark : Such objects do exist. For example, take

$$\rho(x) = \begin{cases} e^{\frac{1}{|x|^2 - 1}} & |x| < 1 \\ 0 & \text{elsewhere} \end{cases},$$

and let

$$\rho_n(x) = \frac{1}{\int_{\mathbb{R}^N} \rho(x) \; dx} n^N \rho(nx).$$

Lemma 11 *Let $f \in C(\mathbb{R}^N)$. Then $\rho_n * f \to f$ uniformly on compact subsets of $\mathbb{R}^N$.*

Proof : Let $K \subset\subset \mathbb{R}^N$ be compact. Then $f|_K$ is uniformly continuous, so for every $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x - y) - f(x)| < \epsilon$ whenever $x \in K$ and $y \in b(0, \delta)$. We want to show that

$$\sup_{x \in K} |(\rho_n * f)(x) - f(x)| \to 0 \text{ as } n \to \infty.$$

We calculate that

$$\begin{aligned}
(\rho_n * f)(x) - f(x)| &= \left| \int_{\mathbb{R}^N} f(x - y) \rho_n(y) \; dy - f(x) \right| \\
&= \left| \int_{\mathbb{R}^N} f(x - y) \rho_n(y) \; dy - \int_{\mathbb{R}^N} f(x) \rho_n(y) \; dy \right| \\
&\leq \int_{B(0, \frac{1}{n}} |f(x - y) - f(x)) |\rho_n(y)| \; dy < \epsilon \int_{B(0, \frac{1}{n})} \rho_n \; dy = \epsilon,
\end{aligned}$$

if n is chosen large enough so that $\delta > 1/n$. Thus, for every compact set K and for every $\epsilon > 0$, there exists $N(K, \epsilon)$ such that

$$\sup_{x \in K} |(\rho_n * f)(x) - f(x)| < \epsilon$$

for every $n \geq N(K, \epsilon)$.

Theorem 35 *let $f \in L^p(\mathbb{R}^N)$ and $1 \leq p < \infty$. Then $(\rho_n * f) \to f$ strongly in $L^p(\mathbb{R}^N)$.*

Proof : let $\epsilon > 0$. Since $C_c(\mathbb{R}^N)$ is dense in $L^p(\mathbb{R}^N)$, there exists $f_1 \in C_c(\mathbb{R}^N)$ such that $\|f - f_1\|_{L^p(\mathbb{R}^N)} < \epsilon$. By the previous lemma, $\rho_n * f \to f_1$ uniformly on compact subsets of $\mathbb{R}^N$. We want to show that

$$\int_{\mathbb{R}^N} |\rho_n * f_1 - f_1|^p \, dx \to 0.$$

In order to show this, we seek to replace $\mathbb{R}^N$ with a compact set K. Thus, we check that $\text{supp}(\rho_n * f_1)$ is compact. We have that

$$\text{supp}(\rho_n * f_1) \subset \overline{\text{supp}(\rho_n) + \text{supp}(f_1)}$$
$$\subset \overline{\text{supp}(\rho_n)} + \overline{\text{supp}(f_1)} \subset \overline{B(0,1)} + \text{supp}(f_1) \subset \overline{B(0,M)}$$

for some M large enough. We then have that $\text{supp}(\rho_n * f_1 - f_1)\overline{B(0.M)}$, and therefore

$$\int_{\overline{B(0,M)}} |\rho_n * f_1 - f_1|^p \, dx \to 0.$$

Now we use the triangle inequality to compute that

$$\|\rho_n * f - f\|_{L^p(\mathbb{R}^N)} = \|\rho_n * f - \rho_n * f_1 + \rho_n * f_1 - f_1 + f_1 - f\|_{L^p(\mathbb{R}^N)}$$
$$\leq \|\rho_n * (f - f_1)\|_{L^p(\mathbb{R}^N)} + \|\rho_n * f_1 - f_1\|_{L^p(\mathbb{R}^N)} + \|f_1 - f\|_{L^p(\mathbb{R}^N)} \to 0.$$

Theorem 36 *Let $p \in [1, \infty)$. Then $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$.*

Proof : Let $f \in L^p(\Omega)$, and let $f - 1 \in C_c(\Omega)$ be such that for given $\epsilon > 0$, $\|f = f_1\|_{L^p(\Omega)} < \epsilon$. Define

$$\bar{f}_1(x) = \begin{cases} f_1(x) & x\imath\Omega \\ 0 & \text{elsewhere} \end{cases}.$$

Then $\rho_n * \bar{f}_1 \to \bar{f}_1$ strongly in $L^p(\mathbb{R}^N)$. Define $u_n = \rho_n * \bar{f}_1\big|_\Omega$. Then

$$\text{supp}(u_n) \subset \overline{\text{supp}(\rho_n) + \text{supp}(\bar{f}_1)} \subset \overline{\text{supp}(\rho_n)} + \overline{\text{supp}(\bar{f}_1)}.$$

The first set in this chain of inclusion is closed, and the last is compact, so we conclude that u_n has compact support. We now have that $\|u_n - f_1\|_{L^p(\Omega)} \to 0$, so

$$\|u_n - f\|_{L^p(\Omega)} \leq \|u_n - f_1\|_{L^p(\Omega)} + \|f_1 - f\|_{L^p(\Omega)},$$

which implies that

$$\limsup_{n\to\infty} \|u_n - f\|_{L^p(\Omega)} < \epsilon.$$

We now let ϵ decrease to zero.

Functional Analysis 3-27-06

Weak Compactness in $\mathbf{L^p(\Omega)}$, $\mathbf{1 \leq p \leq \infty}$

Recall that if $1 < p < \infty$, then $L^p(\Omega)$ is reflexive, so Eberlein-Smulijan applies, and if $\{u_n\}$ is bounded in $L^p(\Omega)$, then there exists a subsequence $\{u_{n_k}\} \subset \{u_n\}$ and $u \in L^p(\Omega)$ such that $u_{n_k} \rightharpoonup u$ weakly in $L^p(\Omega)$ (i.e. $\{u_n\}$ is weakly relatively compact in $L^p(\Omega)$). If $p = \infty$, then $\{u_n\}$

bounded in $L^\infty(\Omega)$ implies that there exists a subsequence $\{u_{n_k}\} \subset \{u_n\}$ and $u \in L^\infty(\Omega)$ such that $u_{n_k} \overset{*}{\rightharpoonup} u$ weakly* in $L^\infty(\Omega)$. If $p = 1$, then we have no such thing. There are two reasons. First, $L^1(\Omega)$ is not reflexive, and second, $L^1(\Omega)$ is not the dual of any separable Banach space. So how do we fix this? One answer is to immerse $L^1(\Omega)$ into the larger space $M(\Omega)$ of Radon measures. Recall that $(C_c(\Omega))' = M(\Omega)$. Define $T : L^1(\Omega) \to M(\Omega)$ by

$$\langle Tf, \varphi \rangle = \int_\Omega f(x)\varphi(x)\ dx.$$

Then

$$\|Tf\|_{M(\Omega)} = \sup_{\substack{\varphi \in C_c(\Omega) \\ \|\varphi\|_\infty \le 1}} |\langle Tf, \varphi \rangle| = \sup_{\substack{\varphi \in C_c(\Omega) \\ \|\varphi\|_\infty \le 1}} \left| \int_\Omega f\varphi\ dx \right| \le \|f\|_{L^1(\Omega)}.$$

In fact, the supremum is attained, so $\|Tf\|_{M(\Omega)} = \|f\|_{L^1(\Omega)}$, and therefore T is an isometry. Thus, via T, $L^1(\Omega)$ can be identified with a subspace of $M(\Omega)$, and if $\{u_n\}$ is bounded in $L^1(\Omega)$, then there exists a subsequence $\{u_{n_k}\} \subset \{u_n\}$ and $\mu \in M(\Omega)$ such that $u_{n_k} \overset{*}{\rightharpoonup} \mu$ in $\sigma(M(\Omega), C_c(\Omega))$, i.e.

$$\int_\Omega u_{n_k}\varphi\ dx \to \langle \mu, \varphi \rangle\ \forall\ \varphi \in C_c(\Omega).$$

This is often called convergence in the sense of measures. We can also write the line above out as

$$\int_\Omega u_{n_k}\varphi\ dx \to \int_\Omega \varphi\ d\mu.$$

Weak Compactness in $\mathbf{L^1(\Omega)}$

Theorem 37 (Dunford-Pettis) *Let $\{u_n\} \subset L^1(\Omega)$ be a bounded sequence. Then the following are equivalent:*

(i) $\{u_n\}$ is weakly relatively compact

(ii) $\{u_n\}$ is equiintegrable ($\forall\ \epsilon > 0\ \exists\ \delta > 0$ s.t. $\sup_{n\in\mathbb{N}} \int_A |u_n|\ dx < \epsilon\ \forall\ A \in \mathcal{B}(\Omega)$ if $|A| < \delta$).

Theorem 38 (De la Valleé-Poussin) *Let $\{u_n\} \subset L^1(\Omega)$ be bounded. Then $\{u_n\}$ is equiintegrable if and only if there exists $\Theta : [0,\infty) \to [0,\infty)$ such that*

$$\frac{\Theta(t)}{t} \to \infty \text{ as } t \to \infty$$

and

$$\sup_n \int_\Omega |\Theta(u_n)|\ dx < \infty$$

(i.e. $\{\Theta(u_n)\}$ is bounded in $L^1(\Omega)$).

Theorem 39 (Vitali's Convergence Theorem) *A sequence $u_n \rightharpoonup u$ weakly in $L^p(\Omega)$ with $1 \le p < \infty$ if and only if*

(i) $u_n \to u$ in measure ($\forall\ \epsilon > 0\ |\{x \in \Omega : |u_n(x) - u(x)| > \epsilon\}| \to 0$ as $n \to \infty$).

(ii) $\{|u_n|^p\}$ is equiintegrable.

Theorem 40 (Chacon's Biting Lemma) *let $\{u_n\}$ be bounded in $L^1(\Omega)$. Then there exists a subsequence $\{u_{n_k}\} \subset \{u_n\}$, $u \in L^1(\Omega)$, and a family of open subsets $\{E_i\}_{i\in\mathbb{N}}$ such that $E_i \supset E_{i+1}$ and $|E_i| < 1/i$ so that $u_{n_k} \rightharpoonup u$ in $L^1(\Omega \backslash E_i)$ for all $i \in I$.*

The sets E_i in the previous theorem are called bites, and this convergence is called biting convergence. A good reference on convergence in $L^1(\Omega)$ is Giaquinta, Modica, and Soucek.

Functional Analysis 3-28-06

Hilbert Spaces

Definition 13 *Let H be a vector space. A mapping $(\cdot,\cdot) : H \times H \to \mathbb{R}$ is a scalar product if*

(i) $(\cdot,\cdot)$ is bilinear,

(ii) $(\cdot,\cdot)$ is symmetric ($(u,v) = (v,u)\ \forall\ u, v \in H$)

(ii) $(\cdot,\cdot)$ is positive ($(u,u) \geq 0\ \forall\ u \in H$ and $(u,u) = 0$ iff $u = 0$).

Recall that we have the Cauchy-Schwartz inequality $|(u,u)| \leq \sqrt{(u,u)}\sqrt{(v,v)}$ for all $u, v \in H$. This follows from the fact that $(u - tv, u - tv) \geq 0$ for all $t \in \mathbb{R}$.

Proposition 17 *The mapping $|\cdot| : H \to \mathbb{R}$ defined by $|u| = \sqrt{(u,u)}$ is a norm on H.*

With the notation of the norm, the Cauchy-Schwartz inequality becomes $|(u,v)| \leq |u||v|$ for every $u, v \in H$.

Remark : It is not difficult to show that the parallelogram identity

$$\left|\frac{u+v}{2}\right|^2 + \left|\frac{u-v}{2}\right|^2 = \frac{1}{2}(|u|^2 + |v|^2)\ \forall\ u, v \in H \tag{12}$$

holds in any inner product space. In fact, we have that

Theorem 41 (Freschet-Von Neumann-Jordan) *If $(E, |\cdot|)$ is a Banach space such that (12) holds, then there exists a scalar product on E such that $|u| = \sqrt{(u,u)_E}$.*

Definition 14 *Let H be a vector space on which we have defined a scalar product $(\cdot,\cdot)$ and the associated norm $|\cdot|$. If $(H, |\cdot|)$ is a Banach space, then it is called a Hilbert space.*

Theorem 42 *Any Hilbert space is uniformly convex ($\forall\ \epsilon > 0\ \exists\ \delta > 0$ st $u, v \in H$, $|u|, |v| \leq 1$, $|u - v| > \epsilon\ \Rightarrow\ |(u+v)/2| < 1 - \delta$).*

Proof : let $\epsilon > 0$ be given, and let $|u|, |v| \leq 1$ with $|u - v| > \epsilon$. Then the parallelogram law gives that

$$\left|\frac{u+v}{2}\right|^2 = \frac{1}{2}(|u|^2 + |v|^2) - \left|\frac{u-v}{2}\right|^2 \leq 1 - \frac{\epsilon^2}{4},$$

so we take $\delta = 1 - \sqrt{1 - \epsilon^2/4}$. Then

$$\left|\frac{u+v}{2}\right| < 1 - \delta.$$

Corollary 16 *Any Hilbert space is reflexive.*

Proof : Use the Milman-Pettis theorem and the previous theorem.

Example : Take $L^2(\Omega) = H$ with

$$(u,v)_{L^2(\Omega)} = \int_\Omega u(x)v(x)\ dx$$

as the scalar product. This gives us that $L^2(\Omega)$ is a Hilbert space for the associated norm.

Remark : A sequence $u_n \rightharpoonup u$ weakly in $L^2(\Omega)$ if and only if

$$\int_\Omega u_n v\ dx \to \int_\Omega uv\ dx\ \forall\ v \in L^2(\Omega),$$

or, in other words, $(u_n, v)_{L^2(\Omega)} \to (u,v)_{L^2(\Omega)}$. This is how we'll check for weak convergence in any Hilbert space.

Theorem 43 (The Projection Theorem) *Let H be a Hilbert Space, and $K \subset H$ be closed convex and nonempty. Then for every $f \in H$, there exists a unique $u \in K$ such that*

$$|f-u| = \min_{v\in K} |f-v|. \tag{13}$$

Moreover, u is characterized by

$$(f-u, v-u) \leq 0\ \forall\ v \in K. \tag{14}$$

Terminology : We write $u = P_K f$, the projection of f onto K.

Proof : Define $\varphi : K \to \mathbb{R}$ by $\varphi(v) = |f-v|$. Then φ is continuous and coercive ($\varphi(v) \to \infty$ as $|v| \to \infty$). By a previous theorem, there exists $u \in K$ such that $\varphi(u) = \min_K \varphi$. Thus we have established existence.

"(13)⇒(14)" Let $w \in K$ be arbitrary. Then $(1-t)u + tw \in K$ for all $t \in [0,1]$. This gives us that

$$\begin{aligned} |f-u|^2 \leq |f-[(1-t)u+tw]|^2 &= |(f-u)-t(w-u)|^2 \\ &= |f-u|^2 + t^2|w-u|^2 - 2t(f-u,w-u), \end{aligned}$$

so

$$(f-u,w-u) \leq \frac{t}{2}|w-u|^2.$$

Now let $t \to 0$, and recall that w was arbitrary.

"(14)⇒(13)" Let u be such that (14) holds. Let $v \in K$. Then

$$|f-u|^2 - |f-v|^2 = (f-u,f-u) - (f-v,f-v) = (f-u,v-u+f-v) - (f-u+u-v,f-v)$$

$$= (f-u,v-u) + (f-u,f-v) - (f-u,f-v) + (f-v,v-u)$$

$$= (f-u,v-u) + (f-u+u-v,v-u) = 2(f-u,v-u) - |v-u|^2 \leq 0.$$

Thus $|f-u|^2 - |f-v|^2 \leq 0$ for every $v \in K$, so u satisfies (13).

As for the uniqueness, assume that there exists $u_1 = P_K f$ and $u_2 = P_K f$. Then

$$\begin{array}{lcl} (f - u_1, v - u_1) \le 0 & \Rightarrow & (f - u_1, u_2 - u_1) \le 0 \\ (f - u_2, v - u_2) \le 0 & & (f - u_2, u_1 - u_2) \le 0 \end{array}.$$

Adding these gives us that $|u_1 - u_2|^2 \le 0$, so we conclude that $u_1 = u_2$.

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Corollary 17 *Under the same assumptions as in the projection theorem, we have that*

$$|P_K f_1 - P_K f_2| \le |f_1 - f_2|$$

for all $f_1, f_2 \in H$.

Proof : Let us write $u_1 = P_K f_1$ and $u_2 = P_K f_2$. Then by the projection theorem, we have that

$$(f_1 - u_1, v - u_1) \le 0 \text{ and } (f_2 - u_2, v - u_2) \le 0 \ \forall\ v \in K.$$

In particular, we have that

$$(f_1 - u_1, u_2 - u_1) \le 0 \text{ and } (f_2 - u_2, u_1 - u_2) \le 0,$$

and adding these inequalities gives us that

$$(f_1 - u_1 - f_2 + u_2, u_2 - u_1) \le 0 \ \Rightarrow\ |u_1 - u_2|^2 \le (f_1 - f_2, u_1 - u_2) \le |f_1 - f_2||u_1 - u_2|$$

$$\Rightarrow\ |u_2 - u_1| \le |f_1 - f_2|.$$

Corollary 18 *If M is a closed subspace of H, then for every $f \in H$, the element $P_M f$ is characterized by $(f - u, v) = 0$ for all $v \in M$. Moreover, $P_M : H \to M$ is linear.*

Proof : The projection theorem tells us that $(f - u, v - u) \le 0$ for all $v \in M$, where $u = P_M f$. Let $v \in M$ and $t \in \mathbb{R}$. Then $tv \in M$, so

$$(f - u, tv - u) \le 0 \ \forall\ v \in M,\ t \in \mathbb{R}. \tag{15}$$

Assume that there exists $v_0 \in M$ such that $(f - u, v_0) \ne 0$. We consider separate cases. If $(f - u, v_0) < 0$, then (15) implies that $(f - u, tv_0 - u) \le 0$ for all $t \in \mathbb{R}$, so

$$t \ge \frac{(f - u, u)}{(f - u, v_0)}.$$

Now take $t \to -\infty$ to get a contradiction. The other case is handled similarly. See page 81 of Rudin for a proof of the linearity.

The Dual of a Hilbert Space

Theorem 44 *Let H be a Hilbert space. For every $\varphi \in H'$, there exists a unique $u \in H$ such that*

$$\langle \varphi, v \rangle_{H', H} = (u, v).$$

Moreover, $\|\varphi\|_{H'} = |u|$. Thus, we may identify H' with H.

Remark : Note that $u_n \rightharpoonup u$ weakly in H if and only if $(u_n, v) \to (u, v)$ for every $v \in H$.

Proof : Define $T : H \to H'$ by $\langle Tf, v\rangle = (f, v)$. Then T is linear, and

$$\|Tf\|_{H'} = \sup_{\substack{v\in H\\|v|\le 1}} |\langle Tf, v\rangle| = \sup_{\substack{v\in H\\|v|\le 1}} |(f, v)| \le \sup_{\substack{v\in H\\|v|\le 1}} |f| = |f|.$$

We can actually attain this bound if we take $v = f/|f|$. It remains to show that T is surjective, i.e. that $\overline{T(H)} = H'$. Note that $T(H)$ is a closed subspace of H', since H is reflexive. Thus we claim that $\overline{T(H)} = H'$. Let $f \in H''$ be such that

$$\langle F, f\rangle = 0 \;\forall\; f \in T(H) \tag{16}$$

Since H is reflexive, we may think of F as being an element of H. Thus (16) implies that

$$\langle Tu, F\rangle = 0 \;\forall\; u \in H,$$

and therefore,

$$(u, F) = 0 \;\forall\; u \in H.$$

Thus $F = 0$, which completes the proof.

Definition 15 *Let $a : H \times H \to \mathbb{R}$ be bilinear.*

(i) The mapping a is said to be continuous if there exists $C > 0$ such that $|a(u, v)| \le C|u||v|$ for every $u, v \in H$.

(ii) The mapping a is said to be coercive if there exists $\alpha > 0$ such that $|a(u, u)| \ge \alpha|u|^2$ for every $u \in H$.

Theorem 45 (Stampaccia) *Let a be bilinear continuous and coercive, and let $K \subset H$ be nonempty, closed and convex. Then for every $\varphi \in H'$, there exists a unique $u \in K$ such that*

$$a(u, v - u) \ge \langle \varphi, v - u\rangle \;\forall\; v \in K. \tag{17}$$

Moreover, if a is symmetric, then u is characterized by

$$\begin{cases} u \in K \\ \frac{1}{2}a(u, u) - \langle \varphi, u\rangle = \min_{v\in K}\left\{\frac{1}{2}a(v, v) - \langle \varphi, v\rangle\right\} \end{cases}.$$

Remark : (**??**) is a typical variational inequality. See the book by Kinderlehrer and Stampaccia (SIAM Classics) for more information.

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Theorem 46 (Stampaccia's Theorem) *Suppose that H is a Hilbert Space and that $K \subset H$ is nonempty, convex, and closed. Let $a : H \times H \to \mathbb{R}$ be a bilinear, continuous, and coercive form. Then for every $\varphi \in H'$, there exists a unique $u \in K$ such that*

$$a(u, v - u) \ge \langle \varphi, v - u\rangle \;\forall v \in K. \tag{18}$$

If a is symmetric, then the solution u to (18) is characterized by

$$\frac{1}{2}a(u, u) - \langle \varphi, u\rangle = \min_{v\in K}\left\{\frac{1}{2}a(v, v) - \langle \varphi, v\rangle\right\}.$$

Proof : By the Riesz-Freschet representation theorem, there exists a unique$f \in H$ such that $\langle \varphi, v\rangle = (f, v)$ for all $v \in H$. Also, for every $u \in H$, the map $v \mapsto a(u, v)$ is in H', so there exists $Au \in H$ such that $a(u, v) = (Au, v)$ for every $v \in H$. The mapping $A : H \to H$ is a linear and continuous operator, since

$$|Au|^2 = (Au, Au) = |a(u, Au)| \leq C|u||Au|$$

and

$$(A(\alpha u + \beta w), v) = a(\alpha u + \beta w, v) = \alpha a(u, v) + \beta a(w, v) = (\alpha Au + \beta Aw, v)$$
$$\Rightarrow \ (A(\alpha u + \beta w) - \alpha Au - \beta Aw, v) = 0 \ \forall \ v \in H.$$

We also have that

$$(Au, u) = a(u, u) \geq \alpha|u|^2 \ \forall \ u \in H.$$

Now (18) is equivalent to

$$(Au, v - u) \geq (f, v - u) \ \Leftrightarrow \ (f - Au, v - u) \leq 0 \ \forall \ v \in K.$$

Take $\rho > 0$ to be determined later. If

$$(\rho f - \rho Au + u - u, v - u) \leq 0 \ \forall \ v \in K, \tag{19}$$

then we will be done. We want to show that for ρ small enough, there exists $u \in K$ such that (19) holds. To accomplish that, we show that for some ρ small enough, there exists a unique $u \in K$ such that

$$P_K(\rho f - \rho Au + u) = u. \tag{20}$$

Define $S : H \to K$ by $Sv = P_K(\rho f - \rho Av + v)$. We seek to show that S has a unique fixed point. Recall

Theorem 47 (Banach Fixed Point Theorem) *Let (X, d) be a metric space. If $S : X \to X$ is a strict contraction ($d(Sv_1, Sv_2) \leq Cd(v_1, v_2)$ for all $v_1, v_2 \in X$ with $C \in (0, 1)$), then S has a unique fixed point.*

We then calculate that

$$|Sv_1 - Sv_2|^2 = |P_K(\rho f - \rho Av_1 + v_1) - P_K(\rho f - \rho Av_2 + v_2)|^2 \leq |\rho(Av_2 - Av_1) + v_1 - v_2|^2$$
$$= \rho^2|Av_2 - Av_1|^2 - 2\rho(A(v_2 - v_1), v_2 - v_1) + |v_1 - v_2|^2$$
$$\leq \rho^2 C^2|v_2 - v_1|^2 - 2\rho\alpha|v - 2 - v_1|^2 + |v_1 - v_2|^2 = |v_1 - v_2|^2(1 - \rho(2\alpha - \rho C^2)).$$

Thus, we choose $0 < \rho < 2\alpha/C^2$ so that S is a strict contraction. Now we apply the fixed point theorem to find that S has a unique fixed point u, so (20) holds. Next, assume that a is symmetric. Then $a(\cdot, \cdot)$ defines a new scalar product on H and a new associated norm $\sqrt{a(\cdot, \cdot)}$ on H, which is equivalent to the original norm by the continuity and coercivity of a. If $\varphi \in H'$, there exists a unique $g \in H$ such that $\langle \varphi, v\rangle = a(g, v)$ for every $v \in H$. The first part of the theorem now implies that

$$a(u, v - u) \geq a(g, v - u) \ \Rightarrow \ a(g - u, v - u) \leq 0 \ \forall \ v \in K.$$

Thus $u = P_K g$ with respect to the a scalar product. The projection theorem now tells us that u is characterized by

$$a(g - u, g - u) = \min_{v \in K} a(g - v, g - v).$$

Expanding, we have

$$a(g,g) + a(u,u) - 2a(g,u) = \min_{v\in K} \{a(g,g) + a(v,v) - 2a(g,v)\}$$

$$\Rightarrow \ \frac{1}{2}a(u,u) - \langle \varphi, u\rangle = \min_{v\in K} \left\{ \frac{1}{2}a(v,v) - \langle \varphi, v\rangle \right\}.$$

Theorem 48 (Lax-Milgram) *Suppose that $a : H \times H \to \mathbb{R}$ is bilinear, continuous, and coercive. Then for every $\varphi \in h'$, there exists a unique $u \in H$ such that $a(u,v) = \langle \varphi, v\rangle$ for every $v \in H$. Moreover, if a is symmetric, u is characterized by*

$$\frac{1}{2}a(u,u) - \langle \varphi, u\rangle = \min_{v\in H} \left\{ \frac{1}{2}a(u,v) - \langle \varphi, v\rangle \right\}.$$

The equation above is the Euler-Lagrange equation for $F(u) = 1/2a(u,u) - \langle \varphi, u\rangle$.

Proof : By Stampaccia's theorem (take $K = H$), there exists a unique $u \in H$ such that

$$a(u, v-u) \geq \langle \varphi, v-u\rangle \ \forall \ v \in H.$$

Since H is a vector space, we have that

$$a(u, tv-u) \geq \langle \varphi, tv-u\rangle \ \forall \ v \in H, \ t \in \mathbb{R}.$$

If the quantity $a(u,v) - \langle \varphi, v\rangle$ is not zero, then we may solve the above inequality for t, and then let t approach either positive or negative infinity to obtain a contradiction. Thus $a(u,v) = \langle \varphi, v\rangle$.

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Hilbert Sums, **Hilbert Bases**

Definition 16 *Let $\{E_n\} \subset H$ be a sequence of closed subspaces. We say that*

$$H = \bigoplus_{n\in\mathbb{N}} E_n$$

(H is a Hilbert sum of the subspaces E_n) if

(i) The sets E_n are pairwise orthogonal ($(u,v) = 0$ whenever $u \in E_n$ and $v \in E_m$ with $m \neq n$)

(ii) The vector space (over $\mathbb{R}$) generated by $\{E_n\}$ is dense in H.

Theorem 49 *Let H be a Hilbert space such that*

$$H = \bigoplus_{n\in\mathbb{N}} E_n.$$

Let $u \in H$, and let $u_n = P_{E_n} u$. Then

(i)

$$u = \sum_{n=1}^{\infty} u_n$$

(ii) (Parseval's Identity)

$$|u|^2 = \sum_{n=1}^{\infty} |u_n|^2.$$

Conversely, given a sequence $\{u_n\} \subset H$ *such that each* $u_n \in E_n$ *and* $\sum_{n=1}^{\infty} |u_n|^2 < \infty$*, then*

$$u = \sum_{n=1}^{\infty} u_n$$

converges, and $u_n = P_{E_n} u$*.*

Definition 17 *A family* $\{e_n\}_{n\in\mathbb{N}} \subset H$ *is said to be a Hilbert Basis for the Hilbert space* H *if*

(i)

$$\bigoplus_{n\in\mathbb{N}} \langle e_n \rangle.$$

(Here $\langle\cdot\rangle$ *denotes span)*

(ii) $|e_n| = 1$ *for each* $n \in \mathbb{N}$

Corollary 19 *(of Theorem (49))If* $\{e_n\}$ *is a Hilbert basis of* H*, then each* $u \in H$ *can be written as*

$$u = \sum_{n=1}^{\infty} (u, e_n) e_n,$$

and

$$|u| = \sum_{n=1}^{\infty} |(u, e_n)|^2.$$

Theorem 50 *Any separable Hilbert space admits a Hilbert basis.*

Proof : let $\{f_n\}$ be a dense subset of H, and let F_k be the vector space generated by $f_1, f_2, \ldots, f_k$. For each k, we may pick an orthonormal basis for F_K, and then complete it as a basis for F_{k+1}. It is then easy to see that the result is a Hilbert basis for H.

Compact Operators

Throughout the following, let E and F be Banach spaces, and denote by B_E the set $\{x \in E : \|x\|_E \leq 1\}$.

Definition 18 *A linear and continuous operator* $T : E \to F$ *is said to be compact if* $T(B_E)$ *is relatively compact in* F*. (i.e.* $\overline{T(B_E)}$ *is compact.)*

We denote by $\mathcal{K}(E, F)$ the set of all compact operators from E to F.

Proposition 18 $\mathcal{K}(E, F)$ *is a closed subspace of* $\mathcal{L}(E, F)$ *(considered with the norm* $\|\cdot\|_{\mathcal{L}(E,F)}$*).*

Proof : Let T_1 and T_2 be elements of $\mathcal{K}(E, F)$. Then $T_1(B_E)$ and $T_2(B_E)$ are relatively compact, so

$$(T_1 + T_2)(B_E) \subset T_1(B_E) + T_2(B_E)$$

is relatively compact in F (this follows from the continuity of the addition operator). Let $\{T_n\} \subset \mathcal{K}(E,F)$ be such that $\|T_n - T\|_{\mathcal{L}(E,F)} \to 0$, where $T \in \mathcal{L}(E,F)$. We want to show that $T \in \mathcal{K}(E,F)$. It suffices to show that for every $\epsilon > 0$, there exists a finite set I such that

$$T(B_E) \subset \bigcup_{i\in I} B(f_i, \epsilon),$$

where $f_i \in F$ for every i. Let $\epsilon > 0$. Choose $n \gg 1$ such that $\|T_n - T\|_{\mathcal{L}(E,F)} < \epsilon/2$. For such n, there exists a finite set I such that

$$T_n(B_E) \subset \bigcup_{i\in I} B\left(f_i, \frac{\epsilon}{2}\right).$$

If $x \in B_E$, then there exists $i_0 \in I$ such that $\|T_n x - f_{i_0}\| < \epsilon/2$. We then have that

$$\|Tx - f_{i_0}\| \leq \|Tx - T_n x\| + \|T_n x - f_{i_0}\| \leq \|T_n - T\|\|x\| + \frac{\epsilon}{2} < \epsilon.$$

Thus, $Tx \in B(f_{i_0}, \epsilon)$ and

$$T(B_E) \subset \bigcup_{i\in I} B(f_i, \epsilon).$$

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Definition 19 *We say that an operator $T \in \mathcal{L}(E,F)$ has finite rank if dim$(R(T)) < \infty$.*

Note that it is clear that a continuous operator with finite rank is compact.

Corollary 20 *Let $\{T_n\}$ be a sequence of continuous operators with finite rank from E to F and Let $T \in \mathcal{L}(E,F)$ be such that $\|T_n - T\|_{\mathcal{L}(E,F)} \to 0$. Then $T \in \mathcal{K}(E,F)$*

Remark : The celebrated approximation problem of Banach and Grothendieck concerns the converse of the above corollary. Given a compact operator, does there exist a sequence $\{T_n\}$ of operators of finite rank such that $\|T_n - T\|_{\mathcal{L}(E,F)} \to 0$? In general, the answer is no, even for certain closed subspaces of ℓ^p, $(1 < p < \infty,\ p \neq 2)$. However, the answer is yes in many cases, for example if F is a Hilbert space. To see this, let $K = \overline{T(B_E)}$. Then given $\epsilon > 0$, we can cover K by

$$\bigcup_{i\in I} B(f_i, \epsilon)$$

for some finite set I. Let G be the vector space generated by the f_i, and let $T_\epsilon = P_G \circ T$. Then T_ϵ is of finite rank. We want to show that $\|T_\epsilon - T\|_{\mathcal{L}(E,F)} < 2\epsilon$. If $x \in B_E$, then there exists i_0 such that

$$\|Tx - f_{i_0}\| < \epsilon.$$

Therefore, since the projection operator is Lipschitz with constant 1, we have that

$$\|(P_G \circ T)x - P_G f_{i_0}\| = \|(P_G \circ T)x - f_{i_0}\| < \epsilon$$

Combining the two inequalities above, we find that

$$\|(P_G \circ T)x - Tx\| < 2\epsilon \ \forall\ x \in B_E,$$

and therefore,

$$\|T_\epsilon - T\|_{\mathcal{L}(E,F)} < 2\epsilon.$$

It is also easy to show this result if F has a Schauder basis.

We will also demonstrate a strong technique used in nonlinear analysis that permits the approximation of a continuous (linear or non-linear) function by non-linear functions of finite rank. Let X be a topological space, F be a Banach space, and $T : X \to F$ be a continuous function such that $T(X)$ is relatively compact in F. For every $\epsilon > 0$, there exists a continuous function $T_\epsilon : X \to F$ with finite rank such that

$$\|T_\epsilon(x) - T(x)\| < \epsilon \;\forall\; x \in X. \tag{21}$$

To see this, note that if $K = \overline{T(X)}$, then K is compact, and can be covered by a finite number of balls, say

$$K \subset \bigcup_{i\in I} B\left(f_i, \frac{\epsilon}{2}\right),$$

with I a finite set. Let

$$T_\epsilon(x) = \frac{\sum_{i\in I} q_i(x) f_i}{\sum_{i\in I} q_i(x)},$$

where $q_i(x) = \max\{\epsilon - \|Tx - f_i\|, 0\}$. Then T_ϵ defined in this way satisfies (21), since T_ϵ is a convex combination and $B(Tx, \epsilon)$ is convex. This method can be used to establish the Schauder fixed point theorem and the Brauer fixed point theorem. Recently, this technique has also been used with much success to demonstrate the existence of relatively invariant subspaces of certain linear operators.

Proposition 19 *Let E, F, and G be three Banach spaces. If $T \in \mathcal{L}(E,F)$ and $S \in \mathcal{K}(F,G)$ (respectively $T \in \mathcal{K}(E,F)$ and $S \in \mathcal{L}(F,G)$), then $S \circ T \in \mathcal{K}(E,G)$.*

Proof : Obvious.

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Theorem 51 (Schauder) *Let $T \in \mathcal{K}(E,F)$, where E and F are Banach spaces. Then $T^* \in \mathcal{K}(F', E')$.*

Proof : To prove the theorem, we want to show that If $\{v_n\} \subset B_{F'}$, then there exists a subsequence $\{T^* v_{n_k}\} \subset \{T^* v_n\}$ which converges in E'. Note that since T is compact, the set $K = \overline{T(B_E)}$ is compact. Let us define

$$M = \{\varphi_n : K \to \mathbb{R} : \varphi_n(x) = \langle v_n, x\rangle_{F',F}\}.$$

We will think about K as a compact metric space. Then $M \subset C(K)$. We claim that M is relatively compact in $C(K)$. To see this, we must recall

Theorem 52 (Arzela-Ascoli) *Let $M \subset C(K)$, where (K, d) is a compact metric space. Assume that*

(i) M is uniformly bounded in $C(K)$,

(ii) M is equicontinuous ($\forall\, \epsilon > 0\; \exists\, \delta > 0$ st $d(x_1, x_2) < \delta \Rightarrow |f(x_1) - f(x_2)| < \epsilon \;\forall\; f \in M$).

Then M is relatively compact in $C(K)$.

First we will show that M is bounded in $C(K)$. We have that

$$\|\varphi_n\|_{C(K)} = \sup_{x\in K} |\varphi_n(x)| = \sup_{x\in K} |\langle v_n, x\rangle| \le \sup_{x\in K} \|v_n\|_{F'}\|x\|_F \le \|T\|_{\mathcal{L}(E,F)}$$

(Recall that $\|v_n\}_{F'} \le 1$ and that $K = \overline{T(B_E)})$. To see the equicontinuity, let $\epsilon > 0$. Then

$$|\varphi_n(x_1) - \varphi_n(x_2)| = |\langle v_n, x_1 - x_2\rangle| \le \|v_n\|_{F'}\|x_1 - x_2\|_F \le \|x_1 - x_2\|_F.$$

Thus we take $\delta = \epsilon$.

The Arzela-Ascoli theorem now implies that our claim holds, so there exists a subsequence $\{\varphi_{n_k}\}$ and a $\varphi \in C(K)$ such that $\varphi_{n_k} \to \varphi$ in $C(K)$. In particlular

$$\sup_{u\in K} |\varphi_{n_k}(u) - \varphi(u)| \to 0 \text{ as } k \to \infty.$$

Therefore, we have that

$$\sup_{u\in K=\overline{T(B_E)}} |\varphi_{n_k}(u) - \varphi_{n_j}| = \sup_{u\in B_E} |\varphi_{n_k}(Tu) - \varphi_{n_j}(Tu)|$$

$$= \sup_{u\in B_E} |\langle v_{n_k} - v_{n_j}, Tu\rangle| = \sup_{u\in B_E} |\langle T^*(v_{n_k} - v_{n_j}, u\rangle| \to 0 \text{ as } k, j \to \infty,$$

which implies that

$$\|T^* v_{n_k} - T^* v_{n_j}\|_{E'} \to 0 \text{ as } j.k \to \infty.$$

Thus, $\{T^* v_{n_k}\}$ is a Cauchy sequence and convergent in E'.

Lemma 12 (Riesz) *Suppose that $(E, \|\cdot\|)$ is a normed vector space, and let M be a closed subspace of E such that $M \ne E$. Then for every $\epsilon > 0$ there exists $u \in E$ with $\|u\| = 1$ such that dist$(u, M) \ge 1 - \epsilon$.*

Remark : This lemma still holds with $\epsilon = 0$ if either $\dim(E) < \infty$ or E is a reflexive Banach space.

Proof : Let $v \in E\backslash M$, and let $d = \text{dist}(v, M) > 0$. Pick $m_0 \in M$ such that $d \le \|v - m_0\| < d/(1-\epsilon)$, and take

$$u = \frac{v - m_0}{\|v - m_0\|}.$$

Then $\|u\| = 1$ and if $m \in M$, we have that

$$\|u - m\| = \left\|\frac{v - m_0}{\|v - m_0\|} - m\right\| = \frac{\|v - (m_0 + m\|v - m_0\|)\|}{\|v - m_0\|} \ge \frac{d}{\frac{d}{1-\epsilon}}.$$

We now pass to the infimum over $m \in M$ to find that $\text{dist}(u, M) \ge 1 - \epsilon$.

Theorem 53 (Riesz) *Let E be a normed vector space such that $B_E = \{x \in E : \|x\| \le 1\}$ is compact (in the topology of the norm). Then dim$(E) < \infty$.*

Proof : Assume that $\dim(E) = \infty$. Then one can construct a sequence of closed subspaces $\{E_n\}$ such that $E_n \supset E_{n+t}$ and $E_n \ne E_{n+1}$ for each $n \in \mathbb{N}$. By the previous lemma, for each n, there exists $u_n \in E_n$ with $\|u_n\| = 1$ such that $\text{dist}(u_n, E_{n-1}) \ge 1/2$. Then $\{u_n\}$ is bounded but not Cauchy and has no Cauchy subsequences, since if $m < n$ then

$$\|u_n - u_m\| \ge \text{dist}(u_n, E_m) \ge 1/2.$$

Theorem 54 (The Fredholm Alternative) *Let E be a Banach space and $T \in \mathcal{K}(E) = \mathcal{K}(E, E)$. Then*

(i) dim$(N(I - T))$ = dim$(N(I - T^*)) < \infty$,

(ii) $R(I - T) = (N(I - T^*))^\perp$ is closed,

(iii) $N(I - T) = \{0\} \;\Leftrightarrow\; R(I - T) = E$.

Here $I : E \to E$ is the identity and T^* is the adjoint of T.

Remark : The theorem tells us something about the equation

$$u - Tu = f, \;\; f \in E. \tag{22}$$

Either (22) has a unique solution, or the homogeneous equation

$$u - Tu = 0 \tag{23}$$

has finitely many (say n) linearly independent solutions, in which case (22) admits solutions if and only if f satisfies n orthogonality conditions. This is called the Fredholm alternative. Indeed, either $R(I - T) = E$ or $R(I - T) \neq E$. In the first case, for every $f \in E$, there exists $u \in E$ such that $u - Tu = f$, and part (iii) gives uniqueness of u. If $R(I - T) \neq E$, by (i) $\dim(N(I - T)) < \infty$, so (**??**) admits $n \in \mathbb{N}$ linearly independent solutions, and the orthogonality relations are given by (ii).

Remark : Part (iii) of the theorem implies that

$$[I - T \text{ is injective } \Leftrightarrow \; I - T \text{ is surjective.}]$$

Recall that if $\dim(E) < \infty$, a linear mapping is injective if and only if it is surjective. This is false in general if $\dim(E) = \infty$. For example, let $S : \ell^2 \to \ell^2$, where

$$\ell^2 = \left\{ \{u_n\} : \sum_{n=1}^{\infty} |u_n|^2 < \infty \right\}$$

and

$$S(u_1, u_2, \ldots) = (0, u_1, u_2, \ldots).$$

We notice that S is linear, injective, and an isometry, but not surjective.

Spectral Theory

Definition 20 *Let $T \in \mathcal{L}(E)$, where E is a Banach space. Then the set*

$$\rho(T) = \{\lambda \in \mathbb{R} : T - \lambda I \text{ is bijective}\}$$

is called the resolvent set for the operator T, and

$$\sigma(T) = \mathbb{R} \backslash \rho(T)$$

is the spectrum. The number $\lambda \in \mathbb{R}$ is an eigenvalue of T if $N(T - \lambda I) \neq \{0\}$. If λ is an eigenvalue, then $N(T - \lambda I)$ is the eigenspace associated with λ.

Remark : If λ is an eigenvalue, then $\lambda \in \sigma(T)$, since $N(T - \lambda I) \neq \{0\}$ implies that $T - \lambda I$ is not injective. The converse is false. Consider again the shift operator S defined before. We see that $S - 0I$ is not injective, but 0 is not an eigenvalue of S. In finite dimensions, the converse is true. If $\dim(E) < \infty$ and $\lambda \in \sigma(T)$, then $T - \lambda I$ is not bijective, and therefore not injective. Thus $N(T - \lambda I) \neq \{0\}$, and λ is an eigenvalue.

Proposition 20 *The set $\sigma(T)$ is compact and*

$$\sigma(T) \subset [-\|T\|_{\mathcal{L}(E)}, \|T\|_{\mathcal{L}(E)}]. \tag{24}$$

Proof : First, we show that (24) holds. Let $\lambda \in \mathbb{R}$ be such that $|\lambda| > \|T\|_{\mathcal{L}(E)}$. We want to show that $\lambda \notin \sigma(T)$. Let us look at the equation $Tu - \lambda u = f$, which is the same as $u = (1/\lambda)(Tu - f)$. We will denote the function $u \mapsto (1/\lambda)(Tu - f)$ by S. Note that S is continuous, and

$$\|Su_1 - Su_2\| = \left\| \frac{1}{\lambda}(Tu_1 - Tu_2) \right\| \leq \frac{\|T\|}{\lambda} \|u_1 - u_2\| \ \forall\ u_1, u_2 \in E.$$

Since $\|T\|/\lambda < 1$, we have that S is a strict contraction, to by the Banach fixed point theorem, S has a unique fixed point. Thus $Tu - \lambda u = f$ has a unique solution. Next, we must show that $\sigma(T)$ is closed, which is equivalent to showing that $\rho(T)$ is open. Let $\lambda_0 \in \rho(T)$, and consider the equation $Tu - \lambda u = f$ for λ close to λ_0. We may rewrite the equation as

$$Tu - \lambda_0 u = f + u(\lambda - \lambda_0) \ \Rightarrow \ u = (T - \lambda_0 I)^{-1}(f + u(\lambda - \lambda_0)).$$

The open mapping theorem implies that $(T - \lambda_0 I)^{-1}$ exists and is continuous since $(T - \lambda_0 I)$ is bijective. Now let S be defined by $Su = (T - \lambda_0 I)^{-1}(f + u(\lambda - \lambda_0))$. We calculate that

$$\|Su_1 - Su_2\| = \|(T - \lambda_0 I)^{-1}((\lambda - \lambda_0)(u_1 - u_2))\| \leq \|(T - \lambda_0 I)^{-1}\| |\lambda - \lambda_0| \|u_1 - u_2\|,$$

so S is a strict contraction if $\|(T - \lambda_0 I)^{-1}\| |\lambda - \lambda_0| < 1$, which means that for

$$\lambda \in \left(\lambda_0 - \frac{1}{\|T - \lambda_0 I)^{-1}\|}, \lambda_0 + \frac{1}{\|T - \lambda_0 I)^{-1}} \right)$$

we have a unique fixed point, so $\rho(T)$ is open.