Math 3210 Exam I Sample Answers

September 29, 2006

1. See me if you need help with this one. Truth tables are hard to typeset!

2. (a) There exists $x \in \mathbb{R}$ such that for every rational number q, $q + x$ and qx are rational.

(b) ($\epsilon > 0$ and $x > 14$) and $\epsilon x \leq 0$.

3. The base case $1 = a_1 = \dfrac{1}{3^0}$ is clearly satisfied. For the induction step, assume $a_n = \dfrac{1}{3^{n-1}}$ for some n. Then $a_{n+1} = \dfrac{1}{3} \cdot \dfrac{1}{3^{n-1}} = \dfrac{1}{3^{(n+1)-1}}$, proving that the formula also holds for $n + 1$. $\square$

4. $\bigcup_{s\in[1,2)}[-s, s] = (-2, 2)$.

The complement is the set $A = \{x \in \mathbb{R} : x \leq -2 \text{ or } x \geq 2\}$.

Proof. First we show that every x not in A is in at least one of the intervals $[-s, s]$. If x is not in A, then $-2 < x < 2$. Choosing $s = \max\{|x|, 1\}$, we then have $1 \leq s < 2$ and $x \in [-s, s]$. Conversely, if $x \in A$, then x is in none of the intervals $[-s, s]$ where $s \in [1, 2)$ and hence x is not in the union. $\square$

5. If $\sup_B g = \infty$ then the conclusion holds automatically, so assume $\sup_B g$ is finite. By definition, $g(x) \leq \sup_B g$ for all $x \in B$. Since $A \subset B$, this inequality is also true for all $x \in A$. Then $f(x) \leq g(x) \leq \sup_B g$ for all $x \in A$. Hence $\sup_B g$ is an upper bound for f. Since $\sup_A f$ is the least upper bound for f over A, we must have $\sup_A f \leq \sup_B g$. $\square$

6. First assume $a = 0$. Given $\epsilon > 0$, since $a_n \to a$, there must exist an N such that $|a_n| < \epsilon$ whenever $n > N$. Then since $|(-1)^n a_n| = |a_n| < \epsilon$, we also have $(-1)^n a_n \to 0$.

To prove the converse (equivalent to the inverse), assume $a \neq 0$, and $a_n \to a$. Let $\epsilon = |a|$ and choose N such that $|a_n - a| < \epsilon$ whenever $n > N$. Then for all odd $n > N$,

$$|(-1)^n a_n - a| = |a_n + a| = |a_n - a + 2a| \geq |2a| - |a_n - a| \geq |a| > 0.$$

Thus there exists no N such that $|(-1)^n a_n - a| < |a|$ for all $n > N$. It follows that $(-1)^n a_n$ does not converge to a. $\square$