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ANALYSIS OF AN ALGORITHM FOR COMPUTING ELECTROMAGNETIC BLOCH MODES USING NEDELEC SPACES

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ABSTRACT. The problem of approximating the band structure of electromagnetic Bloch modes in a three-dimensional periodic medium is studied. We analyze a mixed finite element approximation technique based on a variation of Nedelec edge elements. Convergence of approximate eigenvalues to those of the continuous system is proved.

Key words. photonic band gap, periodic structure, mixed finite elements, convergence analysis

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1. INTRODUCTION

In this paper, we provide an analysis for a method for computing the eigenvalues corresponding to Bloch modes for a wave propagation problem associated with photonic crystals. Photonic crystals are composite periodic structures constructed of dielectric materials. For certain material arrangements, these structures have been found to have interesting and useful spectral behavior with respect to electromagnetic waves, including the appearance of band gaps. Such structures are expected to have numerous applications in optics and microwaves. An introduction to photonic crystals, photonic band gap structures and some of their applications can be found in [14, 21]. Several numerical techniques have been developed for approximating Bloch modes in these structures. Methods for general 3D structures with arbitrary material distributions are described for example in [7] and the references therein; more specialized methods for 2D structures are described in [2, 12]. The most popular of the general 3D methods can be roughly classified as spectral methods, based on a truncated plane wave representation of the electromagnetic field. Discontinuities in the material parameters can lead to convergence difficulties with the plane wave method [23], and as far as we know, no general convergence theory has yet been established.

The approximation technique considered in this paper uses a variation of Nedelec edge finite element spaces. These spaces are generated by a basis of functions which result from the usual basis functions for the Nedelec spaces multiplied by phase functions. We will study the properties of these spaces and the convergence of the corresponding discrete eigenvalues. To the best of our knowledge, this paper provides the first convergence analysis of any method for computing electromagnetic Bloch modes in fully three-dimensional photonic crystals with arbitrary material configurations. A finite element method using the modified edge element spaces described here, utilizing an FFT-based preconditioner and a subspace iteration method for approximating eigenvalues, was presented in [11] along with numerical results. The issue of the iterative computation of

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these eigenvalues when the size of the system is too large for direct solvers was the focus of that work.

Our eigenvalue problem involves a divergence free condition and thus can be naturally posed in mixed form. Recently, the behavior of the discrete eigenvalues associated with mixed finite element problems has been studied [3, 4, 5, 6]. It was observed that even though the mixed discretization satisfies all of the usual conditions for convergence for the underlying problem, i.e., coercivity on the kernel and the discrete LBB condition, it can behave badly on the eigenvalue problem. For example, the numerical method can generate spurious eigenvalues and eigenvectors which have nothing to do with the eigenvalues and eigenvectors of the continuous system. An abstract framework for the convergence of the eigenvalues resulting from mixed finite element methods was given in [5]. Our goal will be to apply this theory to the modified Nedelec spaces mentioned above.

In the next section we formulate the basic eigenproblem to be studied in mixed form. In Section 3, we then use Fourier series to establish a Helmholtz decomposition of vector fields in spaces of periodic functions, and apply these results to obtain regularity of solutions. In Section 4 we construct modified Nedelec edge elements and nodal finite element spaces which allow us to establish discrete coercivity and LBB conditions for our mixed approximation in Section 5. Finally, in Section 6 we prove the convergence of approximate eigenvalues to the continuous ones, by establishing estimates necessary to apply the results of Boffi, Brezzi and Gastaldi [5].

2. Eigenproblem formulation

The underlying problem is classical electromagnetic wave propagation governed by Maxwell's equations in $\mathbb{R}^3$,

$$\begin{aligned} \nabla \times E - i\omega\mu H &= 0, \\ \nabla \times H + i\omega\epsilon E &= 0. \end{aligned} \tag{2.1}$$

Here E and H are the electric and magnetic field vectors, respectively. The magnetic permeability μ is assumed constant, and the dielectric coefficient ϵ is real, bounded and uniformly bounded away from zero. Setting $\rho = (\mu\epsilon)^{-1}$, it follows from (2.1) that

$$\begin{aligned} \nabla \times \rho \nabla \times H =& \omega^2 H, \quad \text{on } \mathbb{R}^3, \\ \nabla \cdot H =& \ 0, \qquad \text{on } \mathbb{R}^3. \end{aligned} \tag{2.2}$$

The medium is assumed to have unit periodicity on a cubic lattice. Thus denoting $Z = \{0, \pm 1, \pm 2, \dots\}$, and defining the lattice $\Lambda = Z^3$, we have

$$\rho(x+n) = \rho(x), \quad \text{for all } x \in \mathbb{R}^3, \text{ and for all } n \in \Lambda.$$

We define the periodic domain $\Omega = \mathbb{R}^3/\Lambda$ and the first Brillouin zone $K = [-\pi, \pi]^3$.

We wish to compute Bloch modes [20], that is, eigenfunctions H satisfying (2.2) for a particular frequency ω, and such that $H(x) = e^{i\alpha\cdot x}\mathbf{u}(x)$, where $\mathbf{u}$ is periodic in x, and $\alpha \in K$. It follows from (2.2) that

$$\begin{aligned} \nabla_\alpha \times \rho \nabla_\alpha \times \mathbf{u} =& \lambda\, \mathbf{u}, \quad \text{in } \Omega, \\ \nabla_\alpha \cdot \mathbf{u} =& 0, \qquad \text{in } \Omega, \end{aligned} \tag{2.3}$$

where $\nabla_\alpha = (\nabla + i\alpha)$ and $\lambda = \omega^2$. Computing the Bloch modes requires consideration of the transformed system (2.3), for each $\alpha \in K$.

Remark 2.1. *An alternative to transforming to system (2.3) is to use the original system (2.2). The problem with this is that unknown field H satisfies somewhat strange boundary conditions. The transformed system involves more complex operators but simpler boundary conditions and is easier to analyze.*

Let $\mathbf{L}^2(\Omega) = L^2(\Omega)^3$. Periodic versions of the standard vector Sobolev spaces will be useful:

$$\begin{aligned} H^1_p(\Omega) &= \{g \in L^2(\Omega) : \nabla g \in \mathbf{L}^2(\Omega)\}, \\ \mathbf{H}_p(curl) &= \{\mathbf{u} \in \mathbf{L}^2(\Omega) : \nabla \times \mathbf{u} \in \mathbf{L}^2(\Omega)\}, \\ \mathbf{H}_p(div) &= \{\mathbf{u} \in \mathbf{L}^2(\Omega) : \nabla \cdot \mathbf{u} \in L^2(\Omega)\}. \end{aligned}$$

For the above definitions, functions defined on Ω are implicitly periodic and the derivative operators respect the periodicity of the domain Ω, i.e., Ω has no boundary.

For $\mathbf{u}, \mathbf{v} \in \mathbf{H}_p(curl)$, and $q \in H^1_p(\Omega)$, we introduce the sesquilinear forms

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega \rho(\nabla_\alpha \times \mathbf{u}) \cdot \overline{(\nabla_\alpha \times \mathbf{v})}\, dx, \tag{2.4}$$

$$b(q, \mathbf{u}) = \int_\Omega \nabla_\alpha q \cdot \overline{\mathbf{u}}\, dx, \tag{2.5}$$

$$(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathbf{u} \cdot \overline{\mathbf{v}}\, dx. \tag{2.6}$$

Defining $V^{0,\alpha} = \{\mathbf{u} \in \mathbf{H}_p(curl) : b(q, \mathbf{u}) = 0 \quad \text{for all } q \in H^1_p(\Omega)\}$, the weak formulation of (2.3) is then to find $\lambda \in \mathbb{R}$ and $\mathbf{u} \in V^{0,\alpha}$ satisfying

$$a(\mathbf{u}, \mathbf{v}) = \lambda\, (\mathbf{u}, \mathbf{v}) \quad \text{for all } \mathbf{v} \in V^{0,\alpha}. \tag{2.7}$$

We consider (2.7) in mixed form: Find $\lambda \in \mathbb{R}$ and $(\mathbf{u}, w) \in \mathbf{H}_p(curl) \times H^1_p(\Omega)$ such that

$$a(\mathbf{u}, \mathbf{v}) + b(w, \mathbf{v}) = \lambda\, (\mathbf{u}, \mathbf{v}), \quad \text{for all } \mathbf{v} \in \mathbf{H}_p(curl), \tag{2.8}$$

$$\overline{b(q, \mathbf{u})} = 0, \quad \text{for all } q \in H^1_p(\Omega). \tag{2.9}$$

3. Decomposition and regularity results

In this section, we provide a number of facts concerning vector decompositions and solutions to the mixed problem: Find $(\mathbf{u}, w) \in \mathbf{H}_p(curl) \times H^1_p(\Omega)$ satisfying

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) + b(w, \mathbf{v}) = (\mathbf{f}, \mathbf{v}), &\quad \text{for all } \mathbf{v} \in \mathbf{H}_p(curl), \\ b(q, \mathbf{u}) = 0, &\quad \text{for all } q \in H^1_p(\Omega). \end{aligned} \tag{3.1}$$

These results can be understood in terms of Fourier analysis. Specifically, let $\mathcal{J}$ denote the set

$$\mathcal{J} = \{2\pi(i_1, i_2, i_3) \ : \ \text{for integer } i_1, i_2, i_3\}.$$

Functions in $\mathbf{L}^2(\Omega)$ can be expanded

$$\mathbf{u} = \sum_{I \in \mathcal{J}} e^{iI \cdot x} C_I$$

where C_I is a three dimensional vector with complex entries. Fix $\alpha \in K$ with $\alpha \neq (0, 0, 0)$ and define for $I \in \mathcal{J}$, $\gamma^I = \alpha + I$. Note that γ^I never vanishes and satisfies

$$C_0(1 + |I|^2) \leq |\gamma^I|^2 \leq C_1(1 + |I|^2) \tag{3.2}$$

with constants C_0,C_1 independent of $I \in \mathcal{J}$. For positive s, the Sobolev spaces of periodic functions can be characterized in terms of this expansion, i.e.,

$$\mathbf{H}_p^s(\Omega) = \left\{ \mathbf{u} \in \mathbf{L}^2(\Omega) \ : \ \mathbf{u} = \sum_{I\in\mathcal{J}} e^{iI\cdot x} C_I \text{ and } \sum_{I\in\mathcal{J}} (1+|I|^2)^s |C_I|^2 < \infty \right\}.$$

For techniques for verifying such results see the proof of Lemma 3.1 in [22]. By (3.2), $(1+|I|^2)^s$ can be replaced by $|\gamma^I|^{2s}$ in the above characterization. The Sobolev norm $\|\mathbf{u}\|_s$ is equivalent to the norm given by

$$\left(\sum_{I\in\mathcal{J}} |\gamma^I|^{2s} |C_I|^2 \right)^{1/2}.$$

The scalar valued Sobolev space $H_p^s(\Omega)$ is characterized analogously. We use the notation $\|\cdot\|_s$ to denote Sobolev norms on Ω in both the vector and scalar case. In the case of $s=0$, we drop the subscript and write $\|\cdot\|$.

For any $I \in \mathcal{J}$, we can decompose $C^3 = D_I \oplus G_I$ where G_I is the space spanned by the vector γ^I and D_I is its orthogonal complement. Note that D_I is the image of the hermitian matrix

$$N_I = i \begin{pmatrix} 0 & -\gamma_3^I & \gamma_2^I \\ \gamma_3^I & 0 & -\gamma_1^I \\ -\gamma_2^I & \gamma_1^I & 0 \end{pmatrix}$$

and $N_I \gamma^I = 0$. The eigenvalues of N_I are 0 and $\pm|\gamma^I|$ and hence for any $C_I \in C^3$,

$$|N_I C_I|^2 + |\gamma^I \cdot C_I|^2 = |\gamma_I|^2 |C_I|^2. \tag{3.3}$$

Clearly, N_I is invertible map on D_I and for a vector $C_I = d_I + g_I$ with $d_I \in D_I$ and $g_I \in G_I$, we define the pseudo-inverse $N_I^+ C_I = N_I^{-1} d_I \in D_I$. Since $g_I \in G_I$, we have that $g_I = \theta(C_I)\gamma^I$ where $\theta(C_I) = (C_I \cdot \gamma^I)/|\gamma^I|^2$.

For $\mathbf{u} = \sum_{I\in\mathcal{J}} e^{iI\cdot x} C_I \in \mathbf{H}_p^1(\Omega)$ and $u = \sum_{I\in\mathcal{J}} c_I e^{iI\cdot x} \in H_p^1(\Omega)$, we have

$$\nabla_\alpha \times \mathbf{u} = \sum_{I\in\mathcal{J}} e^{iI\cdot x} N_I C_I,$$
$$\nabla_\alpha \cdot \mathbf{u} = i \sum_{I\in\mathcal{J}} (\gamma^I \cdot C_I) e^{iI\cdot x},$$
$$\nabla_\alpha u = i \sum_{I\in\mathcal{J}} c_I e^{iI\cdot x} \gamma^I.$$

Let $\mathbf{u} = \sum_{I\in\mathcal{J}} e^{iI\cdot x} C_I$ be in $\mathbf{L}^2(\Omega)$. Then

$$\mathbf{u} = \sum_{I\in\mathcal{J}} e^{iI\cdot x} [N_I(N_I^+ C_I) + \theta(C_I)\gamma^I],$$

i.e.,

$$\mathbf{u} = \nabla_\alpha \times \mathbf{w} + \nabla_\alpha \phi$$

where

$$\mathbf{w} = \sum_{I\in\mathcal{J}} e^{iI\cdot x} N_I^+ C_I \text{ and } \phi = -i \sum_{I\in\mathcal{J}} \theta(C_I) e^{iI\cdot x}.$$

The next theorem easily follows from the above discussion.

Theorem 3.1. *Let α be in K with $\alpha \neq (0,0,0)$. Given $\mathbf{u} \in \mathbf{L}^2(\Omega)$ there exists unique functions $\mathbf{w} \in \mathbf{H}_p^1(\Omega)$ and $\phi \in H_p^1(\Omega)$ satisfying*

$$\mathbf{u} = \nabla_\alpha \times \mathbf{w} + \nabla_\alpha \phi \quad \text{and} \quad \nabla_\alpha \cdot \mathbf{w} = 0.$$

Furthermore, we have

$$\|\mathbf{w}\|_1 + \|\phi\|_1 \leq C\|\mathbf{u}\|,$$
$$\|\mathbf{w}\|_{1+s} \leq C\|\nabla_\alpha \times \mathbf{w}\|_s,$$
$$\|\phi\|_{1+s} \leq C\|\nabla_\alpha \phi\|_s.$$

Here s is any non-negative number.

Remark 3.1. *It easily follows that $\phi = 0$ if and only if $\nabla_\alpha \cdot \mathbf{u} = 0$ and $\mathbf{w} = 0$ if and only if $\nabla_\alpha \times \mathbf{u} = 0$.*

An easy consequence of the above discussion is that the sequence

$$0 \longrightarrow H_p^1(\Omega) \xrightarrow{\nabla_\alpha} \mathbf{H}_p(curl) \xrightarrow{\nabla_\alpha \times} \mathbf{H}_p(div) \xrightarrow{\nabla_\alpha \cdot} L^2(\Omega) \longrightarrow 0$$

is exact. We also have the following regularity estimate.

Theorem 3.2. *Let $\mathbf{u}$ and w be the solution of (3.1). Then*

$$\|\mathbf{u}\|_1 + \|w\|_1 \leq C\|\mathbf{f}\|.$$

If, in addition, ρ is Lipschitz continuous then

$$\|\mathbf{u}\|_2 + \|w\|_1 \leq C\|\mathbf{f}\|.$$

Finally, in the case when $\rho = 1$,

$$\|\mathbf{u}\|_{2+s} \leq \|\nabla_\alpha \times \mathbf{w}\|_s.$$

Here $s \geq 0$ and $\mathbf{f} = \nabla_\alpha \times \mathbf{w} + \nabla_\alpha \phi$ is the Helmholtz decomposition of $\mathbf{f}$ as in the previous theorem.

Proof. The first and third inequality of the theorem follow easily from the eigenfunction expansions discussed earlier. The second inequality follows with a simple modification of classical techniques for proving regularity for boundary value problems (see, e.g., the proof of Theorem 3.1.1 of [17]). □

4. α MODIFIED SPACES.

In this paper, we consider the lowest order Nedelec element on cubes in our discussion for simplicity. All of our results extend to higher order elements and elements based on tetrahedra. We start by partitioning the domain Ω into $N \times N \times N$ smaller cubes ($\Omega = \cup_j \tau_j$), each of side length $h = 1/N$. We shall consider a family of approximation spaces for $H^1(\Omega)$, $\mathbf{H}(curl)$, $\mathbf{H}(div)$ and $L^2(\Omega)$ defined with respect to this mesh. For simplicity, we consider the lowest order Raviart-Thomas-Nedelec spaces $\widetilde{W}_h$, $\widetilde{\mathbf{V}}_h$, $\widetilde{\mathbf{R}}_h$ and $\widetilde{S}_h$ given by

$$\widetilde{W}_h = \{\phi \in H^1(\Omega) \ : \phi|_{\tau_j} \in Q_{1,1,1}\},$$
$$\widetilde{\mathbf{V}}_h = \{\phi \in \mathbf{H}(curl) \ : \phi|_{\tau_j} \in Q_{0,1,1} \times Q_{1,0,1} \times Q_{1,1,0}\},$$
$$\widetilde{\mathbf{R}}_h = \{\phi \in \mathbf{H}(div) \ : \phi|_{\tau_j} \in Q_{1,0,0} \times Q_{0,1,0} \times Q_{0,0,1}\},$$
$$\widetilde{S}_h = \{\phi \in L^2(\Omega) \ : \phi|_{\tau_j} \in Q_{0,0,0}\}.$$

We do not impose periodicity constraints in the definitions of the above spaces. Here $Q_{i,j,k}$ denotes the polynomials of the form

$$p(x,y,z) = \sum_{l=0}^{i} \sum_{m=0}^{j} \sum_{n=0}^{k} c_{l,m,n} x^l y^m z^n.$$

Accompanying these spaces there are natural interpolation operators $\widetilde{I}_h$, $\widetilde{\pi}_h$, $\widetilde{r}_h$ and $\widetilde{Q}_h$ which make the following diagram commute (see, e.g., [9, 18, 19]):

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & H^1(\Omega)/R & \xrightarrow{\nabla} & \mathbf{H}(curl) & \xrightarrow{\nabla\times} & \mathbf{H}(div) & \xrightarrow{\nabla\cdot} & L^2(\Omega) & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \widetilde{I}_h} & & \Big\downarrow{\scriptstyle \widetilde{\pi}_h} & & \Big\downarrow{\scriptstyle \widetilde{r}_h} & & \Big\downarrow{\scriptstyle \widetilde{Q}_h} & & \\ 0 & \longrightarrow & \widetilde{W}_h/R & \xrightarrow{\nabla} & \widetilde{\mathbf{V}}_h & \xrightarrow{\nabla\times} & \widetilde{\mathbf{R}}_h & \xrightarrow{\nabla\cdot} & \widetilde{S}_h & \longrightarrow & 0 \end{array} \tag{4.1}$$

All of the squares above commute when applied to sufficiently smooth vector fields and the sequences from left to right are exact.

These operators are defined in terms of nodal bases. For example, the degrees of freedom for $\widetilde{W}_h$ are the nodal values and the interpolation operator is just nodal interpolation. In contrast, a function in $\widetilde{\mathbf{V}}_h$ is uniquely determined by the values of its tangential components on the edges of the elements.

We consider the space $\widetilde{\mathbf{V}}_h$ in more detail. One has a nodal basis $\{\widetilde{\Psi}_j\}$ where $\widetilde{\Psi}_j$ has a unit component tangential to the j'th edge with a vanishing tangential component along all other edges of the mesh. The support of $\widetilde{\Psi}_j$ is contained in the cubes which contain the j'th edge. Each edge function is associated with a degree of freedom: Given a sufficiently smooth function θ, we define

$$v_j(\theta) = \frac{1}{|e_j|} \int_{e_j} \theta \cdot \eta_j \, dx. \tag{4.2}$$

Here e_j is the edge associated with $\widetilde{\Psi}_j$ and η_j is a unit tangent vector along e_j. The corresponding interpolation operator is then defined by

$$\widetilde{\pi}_h \theta = \sum_j v_j(\theta) \widetilde{\Psi}_j$$

which is well defined provided that the integrals appearing in (4.2) make sense. For example, they make sense for functions $\theta \in \mathbf{W}^{1,s}(\Omega)$ with $s > 2$. Moreover, the integrals make sense for $\theta \in \mathbf{H}^{1+\gamma}(\Omega)$ with $\gamma > 0$, since by the Sobolev imbedding theorem, θ is in $\mathbf{W}^{1,s}(\Omega)$ for $s = 2/(1-\gamma)$.

We define α-modified spaces from those above in two steps. First we restrict to the periodic case. Specifically we set

$$\begin{aligned} W_h &= \{\phi \in H^1_p(\Omega) \cap \widetilde{W}_h\}, \\ \mathbf{V}_h &= \{\phi \in \mathbf{H}_p(curl) \cap \widetilde{\mathbf{V}}_h\}, \\ \mathbf{R}_h &= \{\phi \in \mathbf{H}_p(div) \cap \widetilde{\mathbf{R}}_h\}, \\ S_h &= \widetilde{S}_h. \end{aligned}$$

The only difference between, for example, the spaces W_h and $\widetilde{W}_h$ is that $\widetilde{W}_h$ has distinct degrees of freedom corresponding to the nodes on the periodic boundary. The nodal basis functions for W_h will be denoted by $\{\phi_j\}$. The nodal basis functions for $\mathbf{V}_h$ are denoted by $\{\Psi_j\}$. The commutativity of the diagram is a local property (element by element) and so is satisfied by the periodic spaces as well.

Let us consider $\alpha \in K$ with $\alpha \neq (0,0,0)$. We need to develop spaces $\mathbf{V}_h^\alpha$ and W_h^α which satisfy the condition

$$\|W\|_{H^1(\Omega)} \leq C \sup_{\substack{\mathbf{X}\in\mathbf{V}_h^\alpha \\ \mathbf{X}\neq 0}} \frac{|b(W,\mathbf{X})|}{\|\mathbf{X}\|_{\mathbf{H}(curl)}} = C \sup_{\substack{\mathbf{X}\in\mathbf{V}_h^\alpha \\ \mathbf{X}\neq 0}} \frac{|(\nabla_\alpha W,\mathbf{X})|}{\|\mathbf{X}\|_{\mathbf{H}(curl)}}, \tag{4.3}$$

for all $W \in W_h^\alpha$ with constant C independent of W and h. The analogous condition is proved in the case of $\alpha = (0,0,0)$ by using the fact that for any $W \in W_h$, $\nabla W \in \mathbf{V}_h$. Thus, we construct spaces $(\mathbf{V}_h^\alpha, W_h^\alpha)$ which satisfy $\nabla_\alpha W \in \mathbf{V}_h^\alpha$ for all $W \in W_h^\alpha$. Observe that the operator ∇_α arose in our equations (2.3) from the introduction of a phase factor. This motivates the introduction of a phase factor into the definition of the mixed finite element approximation spaces. Specifically, we define

$$\begin{aligned} \mathbf{V}_h^\alpha &= \operatorname*{span}_j\{e^{-i\alpha\cdot(x-x_j)}\Psi_j\}, \\ W_h^\alpha &= \operatorname*{span}_j\{e^{-i\alpha\cdot(x-y_j)}\phi_j\}. \end{aligned} \tag{4.4}$$

Here x_j is the center of the j'th edge and y_j is the node corresponding to the nodal function ϕ_j. In the above definition, x is taken to vary smoothly over the support of the basis functions for degrees of freedom on the periodic boundary. Spaces $\mathbf{R}_h^\alpha$ and S_h^α are defined analogously.

The degrees of freedom for the spaces W_h^α are again the nodal values. The corresponding interpolation operator I_h^α is just nodal interpolation. As in the case of $\alpha = (0,0,0)$, the functions in W_h^α can be characterized as

$$\begin{aligned} W_h^\alpha = \{\phi \in H_p^1(\Omega) \ &: \ \text{for each mesh element } \tau, \\ \phi|_\tau &= e^{-i\alpha\cdot x}\widetilde{\phi} \text{ for some } \widetilde{\phi} \in Q_{1,1,1}\}. \end{aligned} \tag{4.5}$$

The space $\mathbf{V}_h^\alpha$ inherits degrees of freedom from $\mathbf{V}_h$, specifically,

$$v_j^\alpha(\theta) = \frac{1}{|e_j|}\int_{e_j} e^{i\alpha\cdot(x-x_j)}\ \theta\cdot\eta_j\ dx$$

and the resulting interpolation operator is

$$\pi_h^\alpha\theta = \sum_j v_j^\alpha(\theta)e^{-i\alpha\cdot(x-x_j)}\Psi_j.$$

The degrees of freedom for $\mathbf{R}_h$ and S_h also involve integration. The corresponding degrees of freedom for $\mathbf{R}_h^\alpha$ and S_h^α are defined from those of $\mathbf{R}_h$ and S_h by the analogous phase shifted integrals. The interpolation operators r_h^α and Q_h^α immediately follow from the degrees of freedom. For the latter case, Q_h^α ends up being the $L^2(\Omega)$ projection onto S_h^α.

Remark 4.1. *We note that all of the above spaces consist of piecewise polynomial spaces (often discontinuous) multiplied by phase functions. For $0 < \gamma < 1/2$, the inverse inequality*

$$\|v\|_\gamma \leq C(\gamma)h^{-\gamma}\|v\|$$

holds for discontinuous piecewise polynomial functions of bounded order on a mesh of size h (cf., e.g., [8]). It follows that this inequality holds for functions in any one of the spaces W_h^α, $\mathbf{V}_h^\alpha$, $\mathbf{R}_h^\alpha$ or S_h^α.

Our first result concerning these spaces is that they satisfy the corresponding commutative diagram.

Theorem 4.1. *Let $\alpha \neq (0,0,0)$ be in K. The spaces W_h^α, $\mathbf{V}_h^\alpha$, $\mathbf{R}_h^\alpha$ and S_h^α satisfy the commutative diagram (when applied to sufficiently smooth vector fields so that the interpolation operators make sense)*

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & H_p^1(\Omega) & \xrightarrow{\nabla_\alpha} & \mathbf{H}_p(curl) & \xrightarrow{\nabla_\alpha\times} & \mathbf{H}_p(div) & \xrightarrow{\nabla_\alpha\cdot} & L^2(\Omega) & \longrightarrow & 0 \\ & & I_h^\alpha \big\downarrow & & \pi_h^\alpha \big\downarrow & & r_h^\alpha \big\downarrow & & Q_h^\alpha \big\downarrow & & \\ 0 & \longrightarrow & W_h^\alpha & \xrightarrow{\nabla_\alpha} & \mathbf{V}_h^\alpha & \xrightarrow{\nabla_\alpha\times} & \mathbf{R}_h^\alpha & \xrightarrow{\nabla_\alpha\cdot} & S_h^\alpha & \longrightarrow & 0 \end{array} \tag{4.6}$$

In addition, the spaces and operators above form exact sequences horizontally.

Proof. The following identities follow trivially,

$$\begin{aligned} \nabla_\alpha \times (e^{-i\alpha\cdot x}\mathbf{u}) &= e^{-i\alpha\cdot x}(\nabla \times \mathbf{u}) \\ \nabla_\alpha \cdot (e^{-i\alpha\cdot x}\mathbf{u}) &= e^{-i\alpha\cdot x}(\nabla \cdot \mathbf{u}) \\ \nabla_\alpha (e^{-i\alpha\cdot x} f) &= e^{-i\alpha\cdot x}\nabla f. \end{aligned} \tag{4.7}$$

Note that if W is in W_h^α then $\nabla_\alpha W$ is in $\mathbf{V}_h^\alpha$. Indeed, this is well known for $\alpha = (0,0,0)$ so for each basis function ϕ_j of W_h,

$$\nabla \phi_j = \sum_k c_{jk}\Psi_k$$

where the sum over k corresponds to the edges which have the node of ϕ_j as an endpoint. Thus,

$$\nabla_\alpha(e^{-i\alpha\cdot(x-y_j)}\phi_j) = e^{-i\alpha\cdot(x-y_j)}\sum_k c_{jk}\Psi_k = \sum_k (c_{jk}e^{i\alpha\cdot(y_j-y_k)})(e^{-i\alpha\cdot(x-y_k)}\Psi_k) \in \mathbf{V}_h^\alpha.$$

Similar arguments show that $\nabla_\alpha\times$ maps $\mathbf{V}_h^\alpha$ into $\mathbf{R}_h^\alpha$ and $\nabla_\alpha\cdot$ maps $\mathbf{R}_h^\alpha$ into S_h^α.

We next observe that the properties of the operators I_h^α, π_h^α, r_h^α and Q_h^α can be deduced from those of $\widetilde{I}_h$, $\widetilde{\pi}_h$, $\widetilde{r}_h$ and $\widetilde{Q}_h$. We illustrate this in the case of π_h^α. For $\mathbf{u} \in \mathbf{W}_p^{1,s}(\Omega)$ (with $s > 2$) define $\mathbf{g}(x) = e^{i\alpha\cdot x}\mathbf{u}(x)$. Note that $\mathbf{g}$ is not periodic. For any element τ, we have

$$\begin{aligned} e^{-i\alpha\cdot x}\widetilde{\pi}_h(\mathbf{g})(x) &= \sum_j \frac{e^{-i\alpha\cdot x}}{|e_j|}\Psi_j(x)\int_{e_j} e^{i\alpha\cdot t}\,\mathbf{u}\cdot\eta_j\,dt \\ &= \sum_j e^{-i\alpha\cdot(x-x_j)}\Psi_j(x)v_j^\alpha(\mathbf{u}) = \pi_h^\alpha(\mathbf{u})(x) \quad \text{for all } x \in \tau. \end{aligned} \tag{4.8}$$

The equalities (on τ)

$$\begin{aligned} e^{-i\alpha\cdot x}\ \widetilde{I}_h(e^{i\alpha\cdot x}u) &= I_h^\alpha(u), \\ e^{-i\alpha\cdot x}\ \widetilde{r}_h(e^{i\alpha\cdot x}\mathbf{u}) &= r_h^\alpha(\mathbf{u}), \\ e^{-i\alpha\cdot x}\ \widetilde{Q}_h(e^{i\alpha\cdot x}u) &= Q_h^\alpha(u) \end{aligned} \tag{4.9}$$

follow in an analogous way. In the above equalities, we implicitly assume that u and $\mathbf{u}$ are smooth enough so that corresponding interpolation operators can be applied. The commutativity of the diagram (4.6) follows immediately from the above identities and the fact that the commutativity of (4.1) holds element wise.

The exactness of the upper sequence in (4.6) has already been observed in the previous section. That $\nabla_\alpha \cdot$ maps $\mathbf{R}_h^\alpha$ onto S_h^α follows from commutativity, the exactness of the upper sequence and the density of smooth periodic functions in $\mathbf{H}_p(div)$. The injectivity of ∇_α restricted to W_h^α is immediate.

We next show that the kernel of $\nabla_\alpha \times : \mathbf{V}_h^\alpha \mapsto \mathbf{R}_h^\alpha$ is $\nabla_\alpha(W_h^\alpha)$. Clearly $\nabla_\alpha(W_h^\alpha) \subseteq \ker(\nabla_\alpha \times)$. If $\mathbf{u} \in \ker(\nabla_\alpha \times)$ is $(\cdot,\cdot)$ orthogonal to $\nabla_\alpha(W_h^\alpha)$ then by Remark 3.1, $\mathbf{u} = \nabla_\alpha \phi$ for some $\phi \in H_p^1(\Omega)$. It follows by Lemma 5.1 (given below) that there is an element $P \in W_h^\alpha$ satisfying $\mathbf{u} = \nabla_\alpha P$. Orthogonality implies that $\mathbf{u}$ is zero, i.e., the kernel of $\nabla_\alpha \times : \mathbf{V}_h^\alpha \mapsto \mathbf{R}_h^\alpha$ is $\nabla_\alpha(W_h^\alpha)$.

The argument showing that the kernel of $\nabla_\alpha \cdot : \mathbf{R}_h^\alpha \mapsto S_h^\alpha$ is $\nabla_\alpha \times (\mathbf{V}_h^\alpha)$ is similar. Clearly, $\nabla_\alpha \times (\mathbf{V}_h^\alpha)$ is contained in $\ker(\nabla_\alpha \cdot)$. If $\mathbf{u}$ is in $\ker(\nabla_\alpha \cdot)$ and is orthogonal to $\nabla_\alpha \times (\mathbf{V}_h^\alpha)$ then by Remark 3.1, $\mathbf{u} = \nabla_\alpha \times \eta$ for some $\eta \in \mathbf{H}_p^1(\Omega)$. Note that multiplication by $e^{i\alpha\cdot x}$ maps a function in $\mathbf{R}_h^\alpha$ to a function in $\widetilde{\mathbf{R}}_h$. By Remark 4.1, $\mathbf{u} \in \mathbf{H}_p^\gamma(\Omega)$ for $0 < \gamma < 1/2$. Consequently, π_h^α is well defined on η since by Theorem 3.1,

$$\|\eta\|_{1+\gamma} \le C\|\mathbf{u}\|_\gamma \le Ch^{-\gamma}\|\mathbf{u}\|$$

and commutivity implies that

$$\mathbf{u} = r_h^\alpha(\nabla_\alpha \times \eta) = \nabla_\alpha \times (\pi_h^\alpha \eta).$$

Orthogonality then implies that $\mathbf{u} = 0$. This completes the proof of the theorem. □

The α-modified spaces satisfy the same basic approximation properties as the original spaces as given in the following lemma.

Lemma 4.1. *There is a constant C not depending on h satisfying*

$$\|\mathbf{u} - \pi_h^\alpha \mathbf{u}\| + \|\nabla_\alpha \times (\mathbf{u} - \pi_h^\alpha \mathbf{u})\| \le ch\|\mathbf{u}\|_2, \quad \text{for all } \mathbf{u} \in \mathbf{H}_p^2(\Omega) \tag{4.10}$$

and

$$\|\mathbf{u} - \pi_h^\alpha \mathbf{u}\| + h\|\nabla_\alpha \times (\mathbf{u} - \pi_h^\alpha \mathbf{u})\| \le ch\|\mathbf{u}\|_{1,s}, \quad \text{for all } \mathbf{u} \in \mathbf{W}_p^{1,s}(\Omega). \tag{4.11}$$

An analogous inequality for W_h^α is

$$\|w - I_h^\alpha w\| + h\|w - I_h^\alpha w\|_1 \le Ch^2\|w\|_2, \quad \text{for all } w \in H_p^2(\Omega).$$

Proof. The results follow easily from (4.7), (4.8), (4.9) and the (well known) analogous properties for $\widetilde{\mathbf{V}}_h$ and $\widetilde{W}_h$. For example, by (4.7) and (4.8),

$$\begin{aligned} \|\mathbf{u} - \pi_h^\alpha \mathbf{u}\| + \|\nabla_\alpha \times (\mathbf{u} - \pi_h^\alpha \mathbf{u})\| &= \|\mathbf{g} - \widetilde{\pi}_h \mathbf{g}\| + \|\nabla \times (\mathbf{g} - \widetilde{\pi}_h \mathbf{g})\| \\ &\le Ch\|\mathbf{g}\|_2 \le Ch\|\mathbf{f}\|_2. \end{aligned}$$

□

We shall need additional approximation properties for W_h^α. We consider the approximation operator $\mathcal{I}_h^\alpha$ defined by

$$\mathcal{I}_h^\alpha w = \sum_j u_j^\alpha(w) e^{-i\alpha\cdot(x-y_j)} \phi_j$$

where

$$u_j^\alpha(w) = \frac{6}{\pi h^3} \int_{B_j} e^{i\alpha\cdot(x-y_j)} w(x)\, dx.$$

Here B_j is the ball of radius $h/2$ centered at y_j. A straightforward application of the Bramble-Hilbert argument gives

$$\|w - \mathcal{I}_h^\alpha w\| + h\|\mathcal{I}_h^\alpha w\|_1 \le Ch\|w\|_1, \quad \text{for all } w \in H_p^1(\Omega), \tag{4.12}$$

holds for $\alpha = (0,0,0)$. More precisely, let τ be an element and $w \in H^1(\widetilde{\tau})$ where $\widetilde{\tau}$ denotes the union of the elements which are (periodic) neighbors of τ. To prove (4.12) in the case of $\alpha = (0,0,0)$, one uses the Bramble-Hilbert argument to show that

$$\|w - \mathcal{I}_h w\|_{L^2(\tau)} + h\|\mathcal{I}_h w\|_{H^1(\tau)} \le Ch\|w\|_{H^1(\widetilde{\tau})}. \tag{4.13}$$

Here $\mathcal{I}_h$ denotes the approximation operator $\mathcal{I}_h^\alpha$ with $\alpha = (0,0,0)$. For $\alpha \ne 0$, we set $g = e^{i\alpha\cdot x} w$ and (4.12) follows from (4.13) and the identity (on $\widetilde{\tau}$)

$$e^{-i\alpha\cdot x}\mathcal{I}_h(g) = \mathcal{I}_h^\alpha(w).$$

Here x is taken to vary smoothly on $\widetilde{\tau}$. Using the above inequalities we can prove the following lemma.

Lemma 4.2. *Let $s \in [0,1]$. Then*

$$\|(I - Q_h^\alpha)w\|_s \le C_1 h^{1-s}\|w\|_1, \quad \textit{for all } w \in H_p^1(\Omega). \tag{4.14}$$

The constant C_1 above depends on s but is independent of h.

Proof. The lemma in the case of $s = 0$ follows immediately from (4.12). The subspace W_h^α satisfies the inverse inequality

$$\|W\|_1 \le Ch^{-1}\|W\|, \quad \text{for all } W \in W_h^\alpha.$$

Thus, (4.12) and the case of $s = 0$ give

$$\|(\mathcal{I}_h^\alpha - Q_h^\alpha)w\|_1 \le Ch^{-1}\|(\mathcal{I}_h^\alpha - Q_h^\alpha)w\| \le C\|w\|_1, \quad \text{for all } w \in H^1(\Omega).$$

That (4.14) holds for $s = 1$ follows from the triangle inequality and (4.12). The lemma follows by interpolation. ☐

5. Coercivity on the kernel.

In this section, we prove coercivity on the kernel for the pairs of spaces just constructed. For $\alpha \ne (0,0,0)$ and $\alpha \in K$, we have

$$\|W\|_{H^1(\Omega)} \le C\frac{\|\nabla_\alpha W\|^2}{\|\nabla_\alpha W\|} \le C \sup_{\substack{\mathbf{X}\in\mathbf{V}_h^\alpha \\ \mathbf{X}\ne 0}} \frac{|b(W,\mathbf{X})|}{\|\mathbf{X}\|_{\mathbf{H}(curl)}}.$$

The last inequality followed from taking $\mathbf{X} = \nabla_\alpha W$. The following theorem shows that coercivity on the kernel also holds.

Theorem 5.1. *Let α be in K with $\alpha \ne (0,0,0)$. There exists a constant c_0 not depending on h satisfying*

$$c_0\|\mathbf{U}\| \le \|\nabla_\alpha \times \mathbf{U}\| \quad \textit{for all } \mathbf{U} \in \mathbf{V}_h^{0,\alpha}.$$

Here $\mathbf{V}_h^{0,\alpha} = \{\mathbf{U} \in \mathbf{V}_h^\alpha \ : \ b(W,\mathbf{U}) = 0 \quad \textit{for all } W \in W_h^\alpha\}$.

The proof will follow the argument given in [13] for the case of $\alpha = (0,0,0)$. We start with the following lemma.

Lemma 5.1. *Let* $\mathbf{u} = \nabla_\alpha p$ *for some* $p \in H^1_p(\Omega)$. *Assume further that* $\pi^\alpha_h \mathbf{u}$ *is well defined. Then there exists* $P_h \in W^\alpha_h$ *satisfying* $\pi^\alpha_h \mathbf{u} = \nabla_\alpha P_h$.

Proof. Let $\mathbf{u}$ be as above and let $\mathbf{g} = e^{i\alpha\cdot x}\mathbf{u}$. Then

$$0 = \nabla_\alpha \times \mathbf{u} = e^{-i\alpha\cdot x}\nabla \times (e^{i\alpha\cdot x}\mathbf{u}). \tag{5.1}$$

It follows from e.g., [13], that

$$\nabla \times [\widetilde{\pi}_h(\mathbf{g})] = 0. \tag{5.2}$$

Applying (4.7) and (4.8) gives

$$0 = \nabla \times (e^{i\alpha\cdot x}\pi^\alpha_h \mathbf{u}) = e^{i\alpha\cdot x}\nabla_\alpha \times (\pi^\alpha_h \mathbf{u}). \tag{5.3}$$

By Remark 3.1, there exists $q \in H^1_p(\Omega)$ satisfying

$$\pi^\alpha_h \mathbf{u} = \nabla_\alpha q.$$

Now on any element τ, by (4.8),

$$\pi^\alpha_h \mathbf{u} = e^{-i\alpha\cdot x}\widetilde{\pi}_h(\mathbf{g})$$

where

$$\mathbf{g}|_\tau = e^{i\alpha\cdot x}\mathbf{u}|_\tau \in Q_{0,1,1} \times Q_{1,0,1} \times Q_{1,1,0}$$

and $\nabla \times \mathbf{g} = 0$. By Lemma 5.3 of [13],

$$\widetilde{\pi}_h \mathbf{g}|_\tau = \nabla \widetilde{q}|_\tau$$

for some $\widetilde{q} \in Q_{1,1,1}$, i.e.,

$$\pi^\alpha_h \mathbf{u} = e^{-i\alpha\cdot x}\nabla \widetilde{q} = \nabla_\alpha (e^{-i\alpha\cdot x}\widetilde{q}).$$

This means that $\nabla_\alpha(q - e^{-i\alpha\cdot x}\widetilde{q}) = 0$ from which it easily follows that $q|_\tau$ is of the form $e^{-i\alpha\cdot x}\bar{q}$ for some $\bar{q} \in Q_{1,1,1}$. The lemma follows from (4.5). ☐

Proof of Theorem 5.1. Let $\mathbf{U}$ be in $\mathbf{V}^{0,\alpha}_h$. Note that $e^{i\alpha\cdot x}\nabla_\alpha \times \mathbf{U}$ is piecewise polynomial with respect to the mesh defining $\mathbf{V}^\alpha_h$. By Remark 4.1, $\nabla_\alpha \times \mathbf{U}$ is in $\mathbf{H}^\gamma(\Omega)$ for any γ with $0 < \gamma < 1/2$ and satisfies the inverse inequality

$$\|\nabla_\alpha \times \mathbf{U}\|_\gamma \le C h^{-\gamma} \|\nabla_\alpha \times \mathbf{U}\|. \tag{5.4}$$

By Remark 3.1 and Theorem 3.2, there exists $\mathbf{w} \in H^{1+\gamma}_p(\Omega)$ such that

$$\begin{aligned} \nabla_\alpha \times \mathbf{w} &= \nabla_\alpha \times \mathbf{U}, \quad \text{in } \Omega,\\ \nabla_\alpha \cdot \mathbf{w} &= 0, \quad \text{in } \Omega\\ \|\mathbf{w}\|_{1+\gamma} &\le C\|\nabla_\alpha \times \mathbf{U}\|_\gamma \le C h^{-\gamma}\|\nabla_\alpha \times \mathbf{U}\|. \end{aligned} \tag{5.5}$$

Since $\nabla_\alpha \times (\mathbf{w} - \mathbf{U}) = 0$, by Remark 3.1, $\mathbf{w} - \mathbf{U} = \nabla_\alpha p$ for some $p \in H^1_p(\Omega)$. Note that π^α_h is well defined on $\mathbf{w} - \mathbf{U}$ since $\mathbf{U} \in \mathbf{V}^\alpha_h$ and $\mathbf{w} \in \mathbf{H}^{1+\gamma}_p(\Omega)$. By Lemma 5.1, there exists $P_h \in W^\alpha_h$ satisfying

$$\nabla_\alpha P_h = \pi^\alpha_h(\mathbf{w} - \mathbf{U}) = \pi^\alpha_h \mathbf{w} - \mathbf{U}.$$

Since $\nabla_\alpha P_h$ and $\mathbf{U}$ are orthogonal in $(\cdot,\cdot)$,

$$\begin{aligned} \|\mathbf{U}\| &\leq \|\pi_h^\alpha \mathbf{w}\| \leq \|\mathbf{w}\| + \|\mathbf{w} - \pi_h^\alpha \mathbf{w}\| \\ &\leq C\|\nabla_\alpha \times \mathbf{U}\| + \|\mathbf{w} - \pi_h^\alpha \mathbf{w}\| \end{aligned} \tag{5.6}$$

where we used Theorem 3.1 for the last inequality above. Applying (4.11) with $s = 2/(1-\gamma)$ and the Sobolev imbedding theorem gives

$$\|\mathbf{w} - \pi_h^\alpha \mathbf{w}\| \leq Ch\|\mathbf{w}\|_{1,s} \leq Ch\|\mathbf{w}\|_{1+\gamma}. \tag{5.7}$$

Combining (5.5)–(5.7) completes the proof of the theorem. □

Remark 5.1. *In the above proof, we cannot use commutativity of the diagram to avoid Lemma 5.1. To apply commutativity, we would need to have more smoothness on p so that $I_h^\alpha p$ was well defined.*

6. Convergence of the eigenvalue problem

In this section, we derive the estimates required to apply the results of [5]. We start with the discrete eigenvalue problem. The approximation to the eigenvalues are given by the discrete eigenvalues $\{\lambda^h\}$ associated with the pairs $\{\mathbf{U}, W\} \in \mathbf{V}_h^\alpha \times W_h^\alpha$ satisfying

$$\begin{aligned} a(\mathbf{U},\Theta) + b(W,\Theta) &= \lambda^h\,(\mathbf{U},\Theta), \quad \text{for all } \Theta \in \mathbf{V}_h^\alpha, \\ \overline{b(Q,\mathbf{U})} &= 0, \quad \text{for all } Q \in W_h^\alpha. \end{aligned} \tag{6.1}$$

It follows from Theorem 4.1 that the eigenvalues of (6.1) coincide with the positive eigenvalues associated with eigenvectors $\mathbf{U} \in \mathbf{V}_h^\alpha$ satisfying

$$a(\mathbf{U},\Theta) = \lambda^h(\mathbf{U},\Theta) \quad \text{for all } \Theta \in \mathbf{V}_h^\alpha. \tag{6.2}$$

The convergence of the approximate eigenvalue problem (6.2) was studied in the case of $\alpha = (0,0,0)$ by Boffi, Fernandes, Gastaldi and Perugia [6]. There they showed that this problem is equivalent to the discrete eigenvalue problem: Find λ^h and $(\mathbf{U}_h, \mathbf{P}_h) \in \mathbf{V}_h^\alpha \times \Sigma_h^\alpha$ satisfying

$$\begin{aligned} (\mathbf{U}_h, \tau_h) - (\rho^{1/2}\nabla_\alpha \times \tau_h, \mathbf{P}_h) &= 0, \quad \text{for all } \tau_h \in \mathbf{V}_h^\alpha, \\ (\rho^{1/2}\nabla_\alpha \times \mathbf{U}_h, \mathbf{Q}) &= \lambda^h(\mathbf{P}_h, \mathbf{Q}), \quad \text{for all } \mathbf{Q} \in \Sigma_h^\alpha. \end{aligned} \tag{6.3}$$

Here $\Sigma_h^\alpha = \rho^{1/2}\nabla_\alpha \times (\mathbf{V}_h^\alpha)$. They also proved convergence of the discrete eigenvalues by analyzing the discrete mixed system (6.3).

Our proof is based on formulation (6.1). We let $H = \sqrt{h}$ and consider the corresponding approximation subspace $\mathbf{V}_H^\alpha$. The main technical estimate is given by the following lemma. Its proof will be sketched at the end of this section.

Lemma 6.1. *Let $\mathbf{U}$ be $\mathbf{V}_h^{0,\alpha}$ and $0 < \gamma < 1/2$. Then there exists a function $\mathbf{U}_H \in \mathbf{V}_H^\alpha$ and a constant C not depending on H or h satisfying*

$$\|\mathbf{U} - \mathbf{U}_H\| \leq CHh^{-\gamma}\|\nabla_\alpha \times \mathbf{U}\|.$$

Remark 6.1. *This lemma also holds for $\gamma = 0$ although its proof is somewhat more complicated. The proof for $\gamma = 0$ follows that of the $\alpha = (0,0,0)$ case given for the first part of Lemma 5.2 of* [1].

To apply the results in [5] we need to verify two conditions. They both involve solutions of (3.1) Let w satisfy (3.1). Then for $\mathbf{U} \in \mathbf{V}_h^{0,\alpha}$,

$$b(w, \mathbf{U}) = b(w - Q_h^\alpha w, \mathbf{U}).$$

Applying Lemma 4.2 and Lemma 6.1 gives

$$\begin{aligned} |b(w, \mathbf{U})| &\leq |b(w - Q_h^\alpha w, \mathbf{U} - \mathbf{U}_H)| + |b(w - Q_h^\alpha w, \mathbf{U}_H)| \\ &\leq CHh^{-\gamma}\|w\|_1\|\nabla_\alpha \times \mathbf{U}\| + \|\nabla(w - Q_h^\alpha w)\|_{-\gamma}\|\mathbf{U}_H\|_\gamma \\ &\leq CHh^{-\gamma}\|w\|_1\|\nabla_\alpha \times \mathbf{U}\| + \|w - Q_h^\alpha w\|_{1-\gamma}\|\mathbf{U}_H\|_\gamma. \end{aligned}$$

Here $0 < \gamma < 1/2$. Applying Lemma 4.2 and Remark 4.1 gives

$$|b(w, \mathbf{U})| \leq C(Hh^{-\gamma} + (h/H)^\gamma)\|w\|_1\|\mathbf{U}\|_{\mathbf{H}(curl)}.$$

Taking $\gamma = 1/4$ and applying Theorem 3.2 gives

$$|b(w, \mathbf{U})| \leq CH^{1/4}\|\mathbf{f}\|\,\|\mathbf{U}\|_{\mathbf{H}(curl)}.$$

This shows that our application satisfies Definition 1 of [5].

Let $\mathbf{u}$ satisfy (3.1). We need to assume some additional regularity for solutions of the mixed problem. Specifically, we assume that there is some $\gamma > 0$ such that

$$\|\mathbf{u}\|_{1+\gamma} \leq C\|\mathbf{f}\|. \tag{6.4}$$

By Theorem 3.2, this inequality holds for $\gamma = 1$ when ρ is Lipschitz continuous. The case of jump coefficients with other boundary conditions has been studied in [10] for $\alpha = (0, 0, 0)$. Combining the following lemma with (6.4) shows that our application satisfies Definition 2 of [5]. We give the proof of the lemma at the end of this section.

Lemma 6.2. *Let $\mathbf{u}$ be the solution of (3.1) and assume that $\mathbf{u}$ is in $\mathbf{H}_p^{1+\gamma}(\Omega)$ for some $\gamma \in (0, 1/2)$. Then there is a constant C not depending on h such that*

$$\|\mathbf{u} - \pi_h^\alpha \mathbf{u}\|_{\mathbf{H}(curl)} \leq Ch^\gamma\|\mathbf{u}\|_{1+\gamma}.$$

Let $\mathbf{T} : \mathbf{L}^2(\Omega) \to \mathbf{L}^2(\Omega)$ be the solution operator corresponding to the mixed problem, i.e., $\mathbf{Tf} = \mathbf{u}$ solves (3.1). Similarly, let $\mathbf{T}_h : \mathbf{L}^2(\Omega) \to \mathbf{V}_h^{0,\alpha}$ be the discrete solution operator, i.e., $\mathbf{T}_h\mathbf{f} = \mathbf{U}$ where $(\mathbf{U}, W) \in \mathbf{V}_h^\alpha \times W_h$ solves

$$\begin{aligned} a(\mathbf{U}, \Phi) + b(W, \Phi) &= (\mathbf{f}, \Phi), \quad \text{for all } \Phi \in \mathbf{V}_h^\alpha, \\ b(Q, \mathbf{U}) &= 0, \quad \text{for all } Q \in W_h^\alpha. \end{aligned} \tag{6.5}$$

Since Definitions 1 and 2 of [5] hold, Theorem 1 of [5] shows that $\mathbf{T}_h$ converges uniformly to $\mathbf{T}$. The convergence of the eigenvectors and eigenvalues follows by classical perturbation techniques [15].

Sketch of the proof of Lemma 6.1. Let

$$\mathbf{Z}^\alpha = \{\mathbf{z} \in \mathbf{H}_p(div) \ : \ \nabla_\alpha \cdot \mathbf{z} = 0\}$$

and $\mathbf{Z}_h^\alpha = \mathbf{Z}^\alpha \cap \mathbf{R}_h^\alpha$. We consider the mixed problem: Find $(\mathbf{v}, \mathbf{z}) \in \mathbf{H}_p(curl) \times \mathbf{Z}^\alpha$ satisfying

$$\begin{aligned} (\mathbf{v}, \mathbf{w}) - (\mathbf{z}, \nabla_\alpha \times \mathbf{w}) &= 0, \quad \text{for all } \mathbf{w} \in \mathbf{H}_p(curl), \\ (\nabla_\alpha \times \mathbf{v}, \mathbf{q}) &= (\mathbf{f}, \mathbf{q}), \quad \text{for all } \mathbf{q} \in \mathbf{Z}^\alpha. \end{aligned} \tag{6.6}$$

The corresponding mixed approximation is to find $(\mathbf{V}, \mathbf{Z}) \in \mathbf{V}_h^\alpha \times \mathbf{Z}_h^\alpha$ satisfying

$$\begin{aligned} (\mathbf{V}, \mathbf{W}) - (\mathbf{Z}, \nabla_\alpha \times \mathbf{W}) &= 0, \quad \text{for all } \mathbf{W} \in \mathbf{V}_h^\alpha, \\ (\nabla_\alpha \times \mathbf{V}, \mathbf{Q}) &= (\mathbf{f}, \mathbf{Q}), \quad \text{for all } \mathbf{Q} \in \mathbf{Z}_h^\alpha. \end{aligned} \tag{6.7}$$

We note that for $\mathbf{f} \in \mathbf{Z}_h^\alpha$,

$$\|\mathbf{v} - \mathbf{V}\| \le \|\mathbf{v} - \pi_h^\alpha \mathbf{v}\|. \tag{6.8}$$

Let $\mathbf{V} \in \mathbf{V}_h^{0,\alpha}$ and set $\mathbf{f} = \nabla_\alpha \times \mathbf{V}$. Then there exists $\mathbf{Z} \in \mathbf{Z}_h^\alpha$ such that $(\mathbf{V}, \mathbf{Z})$ is the solution of (6.7) with $\mathbf{f} = \nabla_\alpha \times \mathbf{V}$. This means that (6.8) holds for $\mathbf{v}$ solving (6.6) with $\mathbf{f} = \nabla_\alpha \times \mathbf{V}$. Combining the above estimates with (4.11) for $s = 2/(1-\gamma)$ and a Sobolev inequality shows that

$$\|\mathbf{v} - \mathbf{V}\| \le Ch\|\mathbf{v}\|_{1+\gamma}.$$

Now (6.6) is an alternative form of (3.1) with $\rho = 1$, i.e., $\mathbf{z} = \mathbf{u}$ where $\mathbf{z}$ solves (6.6) and $\mathbf{u}$ solves (3.1) with $\rho = 1$. It follows from Theorem 3.2 that $\mathbf{v} = \nabla_\alpha \times \mathbf{u}$ satisfies

$$\|\mathbf{v}\|_{1+\gamma} \le C\|\mathbf{u}\|_{2+\gamma} \le C\|\nabla_\alpha \times \mathbf{V}\|_\gamma \le Ch^{-\gamma}\|\nabla_\alpha \times \mathbf{V}\|.$$

Thus,

$$\|\mathbf{v} - \mathbf{V}\| \le Ch^{1-\gamma}\|\nabla_\alpha \times \mathbf{V}\|.$$

Similarly,

$$\|\mathbf{v} - \pi_H^\alpha \mathbf{v}\| \le CHh^{-\gamma}\|\nabla_\alpha \times \mathbf{V}\|.$$

The lemma follows by taking $\mathbf{V}_H = \pi_H^\alpha \mathbf{v}$ and applying the triangle inequality. □

Proof of Lemma 6.2. As usual, the approximation properties in the case of $\alpha \ne (0,0,0)$ immediately follow from the $\alpha = (0,0,0)$ case. As already mentioned, $\nabla \times (\widetilde{\pi}_h \mathbf{u}) = \widetilde{r}_h(\nabla \times \mathbf{u})$ for sufficiently smooth $\mathbf{u}$. Thus, for $\mathbf{u} \in \mathbf{H}^{1+\gamma}(\Omega)$,

$$\|\nabla \times (\mathbf{u} - \widetilde{\pi}_h \mathbf{u})\| = \|(I - \widetilde{r}_h)(\nabla \times \mathbf{u})\|.$$

The following approximation property can be proved by a Bramble-Hilbert argument (see, e.g., [16]): For $\mathbf{w} \in \mathbf{H}^\gamma(\Omega) \cap \mathbf{H}(div)$,

$$\|\mathbf{w} - \widetilde{r}_h \mathbf{w}\| \le Ch^\gamma(\|\nabla \cdot \mathbf{w}\| + \|\mathbf{w}\|_\gamma).$$

Combining the above estimates with (4.11) completes the proof of the lemma. □

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