ON THE SCATTERING BY A BIPERIODIC STRUCTURE

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ABSTRACT. Consider scattering of electromagnetic waves by a nonmagnetic biperiodic structure. The structure separates the whole space into three regions: above and below the structure the medium is assumed to be homogeneous. Inside the structure, the medium is assumed to be defined by a bounded measurable dielectric coefficient. Given the structure and a time-harmonic electromagnetic plane wave incident on the structure, the scattering (diffraction) problem is to predict the field distributions away from the structure. In this note, the problem is reduced to a bounded domain and solved by a variational method. The main result establishes existence and uniqueness of the weak solutions in $W^{1,2}$.

1. INTRODUCTION

Consider scattering of electromagnetic waves by a biperiodic structure. The structure separates the whole space into three regions: Above and below the structure the medium is assumed to be homogeneous. However, inside the structure, the medium can be very general. In fact, the dielectric coefficient only needs to be bounded measurable. The medium is assumed to be nonmagnetic with a constant magnetic permeability throughout. Given the structure and a time-harmonic electromagnetic plane wave incident on the structure, the scattering (diffraction) problem is to predict the field distributions away from the structure. Scattering of electromagnetic waves in a biperiodic structure has recently received considerable attention. We refer to Dobson and Friedman [6], Abboud [1], Dobson [5], and Bao [4] for other results on existence, uniqueness, and numerical approximations of solutions. The present note gives a new method of proof for existence and uniqueness, using a very simple penalty term in the problem formulation. This alternative problem formulation may also be advantageous for the numerical approximation of solutions, since it allows the use of nodal finite element spaces. The regularity result of the weak solutions obtained in this work is optimal within the Sobolev scales.

Scattering theory in periodic structures has many applications in micro-optics, where doubly periodic structures are often called *crossed diffraction gratings*. A description of the biperiodic scattering problem and other problems which arise in micro-optics can be found in Friedman [7]. A good introduction to the problem of electromagnetic diffraction through periodic structures, along with some numerical methods, can be found in Petit [8].

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2. Maxwell's equations

In this section we outline the basic setup for the scattering problem. The electromagnetic fields are governed by the time harmonic Maxwell equations (time dependence $e^{-i\omega t}$):

$$\nabla \times E - i\omega\mu H = 0 , \tag{2.1}$$

$$\nabla \times H + i\omega\epsilon E = 0 , \tag{2.2}$$

where E and H denote the electric and magnetic fields in $\mathbf{R}^3$, respectively. The magnetic permeability μ is assumed to be one everywhere. There are two constants Λ_1 and Λ_2, such that the dielectric coefficient ϵ satisfies, for any n_1, $n_2 \in Z = \{0, \pm 1, \pm 2, \dots\}$, and for almost all $x = (x_1, x_2, x_3) \in \mathbf{R}^3$,

$$\epsilon(x_1 + n_1\Lambda_1, x_2 + n_2\Lambda_2, x_3) = \epsilon(x_1, x_2, x_3) . \tag{2.3}$$

Further, it is assumed that, for some fixed positive constant b and sufficiently small $\delta > 0$,

$$\epsilon(x_1, x_2, x_3) = \epsilon_1 , \text{ for } x_3 > b - \delta , \tag{2.4}$$

$$\epsilon(x_1, x_2, x_3) = \epsilon_2 , \text{ for } x_3 < -b + \delta , \tag{2.5}$$

where $\epsilon(x) \in L^\infty$, $Re(\epsilon(x)) \geq \epsilon_0$, $Im(\epsilon(x)) \geq 0$, ϵ_0, ϵ_1 and ϵ_2 are constants, ϵ_0, ϵ_1 are real and positive, and $Re\ \epsilon_2 > 0$, $Im\ \epsilon_2 \geq 0$. The case $Im\ \epsilon > 0$ accounts for materials which absorb energy.

Let $\Omega_0 = \{x \in \mathbf{R}^3 : -b < x_3 < b\}$, $\Omega_1 = \{x \in \mathbf{R}^3 : x_3 > b\}$, $\Omega_2 = \{x \in \mathbf{R}^3 : x_3 < -b\}$.

Consider a plane wave in Ω_1

$$E_I = s e^{iq\cdot x} \ , \ H_I = p e^{iq\cdot x} \ , \tag{2.6}$$

incident on Ω_0. Here $q = (\alpha_1, \alpha_2, -\beta) = \omega\sqrt{\epsilon_1}(\cos\theta_1 \cos\theta_2, \cos\theta_1 \sin\theta_2, -\sin\theta_1)$ is the incident wave vector whose direction is specified by θ_1 and θ_2, with $0 < \theta_1 < \pi$ and $0 < \theta_2 \leq 2\pi$. The vectors s and p satisfy

$$s = \frac{1}{\omega\epsilon_1}(p \times q) \ , \quad q \cdot q = \omega^2\epsilon_1 \ , \quad p \cdot q = 0 \ . \tag{2.7}$$

The incident wave vector is not constrained to lie in a plane orthogonal to one of the linear grating structures, thus corresponding to the general *conical diffraction* problem [8].

We are interested in quasiperiodic solutions, *i.e.*, solutions E and H such that the fields E_α, H_α defined by, for $\alpha = (\alpha_1, \alpha_2, 0)$,

$$E_\alpha = e^{-i\alpha\cdot x} E(x), \tag{2.8}$$

$$H_\alpha = e^{-i\alpha\cdot x} H(x), \tag{2.9}$$

are periodic, with period Λ_1 in the x_1 direction, and period Λ_2 in the x_2 direction.

Denote

$$\nabla_\alpha = \nabla + i\alpha = \nabla + i(\alpha_1, \alpha_2, 0) .$$

It is easy to see from (2.1) and (2.2) that E_α and H_α satisfy

$$\nabla_\alpha \times E_\alpha - i\omega H_\alpha = 0 , \tag{2.10}$$

$$\nabla_\alpha \times H_\alpha + i\omega\epsilon E_\alpha = 0 . \tag{2.11}$$

It follows that for differentiable E_α and H_α or in a weak sense, the system (2.10) and (2.11) is equivalent to:
Problem A:

$$\nabla_\alpha \times (\frac{1}{\epsilon}\nabla_\alpha \times H_\alpha) - \omega^2 H_\alpha = 0 \,, \tag{2.12}$$

$$\nabla_\alpha \times H_\alpha + i\omega\epsilon E_\alpha = 0 \,. \tag{2.13}$$

Due to a consideration for coercivity, it turns out to be natural to solve the following problem:
Problem B:

$$\nabla_\alpha \times (\frac{1}{\epsilon}\nabla_\alpha \times H_\alpha) - \nabla_\alpha(\frac{1}{\epsilon_C}\nabla_\alpha \cdot H_\alpha) - \omega^2 H_\alpha = 0 \,, \tag{2.14}$$

$$\nabla_\alpha \times H_\alpha + i\omega\epsilon E_\alpha = 0 \,. \tag{2.15}$$

instead of Problem A. Here ϵ_C is a fixed positive constant which will be specified later. Obviously, if (E_α, H_α) is a solution to Problem A, it must be a solution to Problem B in the sense of distributions. Conversely, if H_α satisfies (2.14) along with the condition $\nabla_\alpha \cdot H_\alpha = 0$, then H_α satisfies (2.12).

3. Scattering problem in a truncated domain

In order to solve the system of differential equations, we need boundary conditions in the x_3 direction. These conditions may be derived by the radiation condition, the periodicity of the structure, and the Green functions.

Since H_α is Λ periodic, we can expand H_α in a Fourier series:

$$H_\alpha(x) = H_{I,\alpha}(x) + \sum_{n \in Z^2} U_\alpha^{(n)}(x_3) e^{i\alpha_n \cdot x}, \tag{3.1}$$

where $H_{I,\alpha} = H_I e^{-i\alpha\cdot x}$ and

$$U_\alpha^{(n)}(x_3) = \frac{1}{\Lambda_1\Lambda_2}\int_0^{\Lambda_1}\int_0^{\Lambda_2}(H_\alpha(x) - H_{I,\alpha}(x))e^{-i\alpha_n\cdot x}dx_1dx_2$$

and

$$\alpha_n = (2\pi n_1/\Lambda_1, 2\pi n_2/\Lambda_2, 0).$$

Denote

$$,_1 = \{x \in \mathbf{R}^3 : x_3 = b\} \text{ and } ,_2 = \{x_3 = -b\}.$$

Define for $j = 1, 2$ the coefficients

$$\beta_j^{(n)}(\alpha) = e^{i\gamma_j^n/2}|\omega^2\epsilon_j - |\alpha_n + \alpha|^2|^{1/2}, \quad n \in Z^2, \tag{3.2}$$

where

$$\gamma_j^n = arg(\omega^2\epsilon_j - |\alpha_n + \alpha|^2), \;\; 0 \le \gamma_j^n < 2\pi. \tag{3.3}$$

We assume that $\omega^2\epsilon_j \neq |\alpha_n + \alpha|^2$ for all $n \in Z$, $j = 1, 2$. This condition excludes "resonance". Note that $\beta_j^{(n)}$ is real at most for finitely many n, and for the rest of n, β_j^n has a positive imaginary part. In particular, for real ϵ_2, we have the following equivalent form of (3.2)

$$\beta_j^{(n)}(\alpha) = \begin{cases} \sqrt{\omega^2\epsilon_j - |\alpha_n + \alpha|^2}, & \omega^2\epsilon_j > |\alpha_n + \alpha|^2 \,, \\ i\sqrt{|\alpha_n + \alpha|^2 - \omega^2\epsilon_j}, & \omega^2\epsilon_j < |\alpha_n + \alpha|^2 \,. \end{cases} \tag{3.4}$$

Observe that inside Ω_1 and Ω_2 the dielectric coefficients ϵ are constants, Maxwell's equations then become

$$(\Delta_\alpha + \omega^2 \epsilon_j) H_\alpha = 0 \ , \tag{3.5}$$

where $\Delta_\alpha = \Delta + 2i\alpha \cdot \nabla - |\alpha|^2$, with the additional constraint $\nabla_\alpha \cdot H_\alpha = 0$.

Since the medium in Ω_j is homogeneous ($\epsilon = \epsilon_j$), the method of separation of variables implies that H_α can be expressed as a sum of plane waves:

$$H_\alpha|_{\Omega_j} = H_{I,\alpha}(x) + \sum_{n \in Z} A_j^{(n)} e^{\pm i\beta_j^{(n)} x_3 + i\alpha_n \cdot x}, \quad j = 1, 2, \tag{3.6}$$

where the $A_j^{(n)}$ are constant (complex) vectors, and $H_{I,\alpha}(x) = 0$ in Ω_2.

We next impose a radiation condition on the scattering problem. Due to the (infinite) periodic structure, the usual Sommerfeld or Silver-Müller radiation condition is no longer valid. Instead, the following radiation condition based on diffraction theory is employed: Since β_j^n is real for at most finitely many n, there are only a finite number of propagating plane waves in the sum (3.4), the remaining waves are exponentially decayed (or unbounded) as $|x_3| \to \infty$. We will insist that H_α is composed of bounded outgoing plane waves in Ω_1 and Ω_2, plus the incident (incoming) wave in Ω_1.

From (3.1) and (3.4) we deduce

$$H_\alpha^{(n)}(x_3) = \begin{cases} U_\alpha^{(n)}(b) e^{i\beta_1^{(n)}(x_3 - b)}, & \text{in } \Omega_1, \\ U_\alpha^{(n)}(-b) e^{-i\beta_2^{(n)}(x_3 + b)}, & \text{in } \Omega_2. \end{cases} \tag{3.7}$$

We can then calculate the derivative of $H_\alpha^n(x_3)$ with respect to ν, the unit normal, on $\partial\Omega_0$:

$$\frac{\partial H_\alpha^{(n)}}{\partial \nu} = \begin{cases} i\beta_1^{(n)} U_\alpha^{(n)}(b) \ , & \text{on } \Gamma_1, \\ i\beta_2^{(n)} U_\alpha^{(n)}(-b) \ , & \text{on } \Gamma_2. \end{cases} \tag{3.8}$$

Thus from (3.6) and (3.8),

$$\begin{aligned} \left.\frac{\partial H_\alpha}{\partial \nu}\right|_{\Gamma_1} &= \sum_{n \in Z} i\beta_1^{(n)} U_\alpha^{(n)}(b) e^{i\alpha_n \cdot x} - i\beta_1 p e^{-i\beta_1 b} \\ &= \sum_{n \in Z} i\beta_1^{(n)} H_\alpha^{(n)}(b) e^{i\alpha_n \cdot x} - 2i\beta_1 p e^{-i\beta_1 b}, \end{aligned} \tag{3.9}$$

$$\left.\frac{\partial H_\alpha}{\partial \nu}\right|_{\Gamma_2} = \sum_{n \in Z} i\beta_2^n H_\alpha^{(n)}(-b) e^{i\alpha_n \cdot x} \ , \tag{3.10}$$

where the unit vector $\nu = (0, 0, 1)$ on Γ_1 and $(0, 0, -1)$ on Γ_2.

Define the lattice

$$\Lambda = \Lambda_1 Z \times \Lambda_2 Z \times \{0\} \subset \mathbf{R}^3 \ . \tag{3.11}$$

Since the fields H_α are Λ-periodic, we can move the problem from $\mathbf{R}^3$ to the quotient space $\mathbf{R}^3/\Lambda$. For the remainder of the paper, we shall identify Ω_0 with the cylinder Ω_0/Λ, and similarly for the boundaries $\Gamma_j \equiv \Gamma_j/\Lambda$. Thus from now on, all functions defined on Ω_0 and Γ_j are implicitly Λ-periodic. Let $\mathcal{H}^m(\Omega_0)$ be the mth order L^2-based Sobolev spaces of complex valued functions defined on Ω_0. Note that any $f \in \mathcal{H}^m(\Omega_0)$, when extended by periodicity to $\mathbf{R}^2 \times (-b, b)$, is automatically in $\mathcal{H}^m_{loc}(\mathbf{R}^2 \times (-b, b))$.

For functions $f \in \mathcal{H}^{\frac{1}{2}}(\,,_j)^3$, define the operator T_j^α by

$$(T_j^\alpha f)(x_1, x_2) = \sum_{n\in\Lambda} i\beta_j^{(n)} f^{(n)} e^{i\alpha_n\cdot x}, \tag{3.12}$$

where $f^{(n)} = \frac{1}{\Lambda_1\Lambda_2}\int_0^{\Lambda_1}\int_0^{\Lambda_2} f(x)e^{-i\alpha_n\cdot x}$, and equality is taken in the sense of distributions.

It is necessary in our study to understand the continuity properties of the above "Dirichlet-Neumann" maps. Fortunately, this is trivial by observing that T_j^α is a standard pseudodifferential operator (in fact, a convolution operator) of order one from the definition of $\beta_j^n(\alpha)$. Thus the standard theory on pseudodifferential operators applies to establish the following lemma.

Lemma 3.1. *For $j = 1, 2$, the operator $T_j^\alpha : \mathcal{H}^{\frac{1}{2}}(\,,_j)^3 \to \mathcal{H}^{-\frac{1}{2}}(\,,_j)^3$ is continuous.*

By matching the two expansions (3.1) and (3.6), we get

$$A_1^{(n)} = U_\alpha^{(n)}(b)e^{-i\beta_1^{(n)}b} \quad \text{on } ,_1\,, \tag{3.13}$$

$$A_2^{(n)} = U_\alpha^{(n)}(-b)e^{-i\beta_2^{(n)}b} \quad \text{on } ,_2\,. \tag{3.14}$$

Further, since

$$\nabla_\alpha \cdot H_\alpha = 0\,, \;\; \nabla_\alpha \cdot H_{I,\alpha} = 0\,,$$

we have from (3.6) that

$$\{(\alpha_n + \alpha) + (0, 0, \beta_1^{(n)})\} \cdot U_\alpha^{(n)}(b) = 0 \quad \text{on } ,_1\,, \tag{3.15}$$

$$\{(\alpha_n + \alpha) - (0, 0, \beta_2^{(n)})\} \cdot U_\alpha^{(n)}(-b) = 0 \quad \text{on } ,_2\,, \tag{3.16}$$

Lemma 3.2. *There exist boundary pseudo-differential operators B_j $(j = 1, 2)$ of order one, such that*

$$\nu \times (\nabla_\alpha \times (H_\alpha - H_{I,\alpha})) = B_1(P(H_\alpha - H_{I,\alpha})) \quad on\ ,_1, \tag{3.17}$$

$$\nu \times (\nabla_\alpha \times H_\alpha) = B_2(P(H_\alpha)) \quad on\ ,_2, \tag{3.18}$$

where the operator B_j is defined by

$$B_j f = -i\sum_{n\in\Lambda} \frac{1}{\beta_j^{(n)}}\{(\beta_j^{(n)})^2(f_1^{(n)}, f_2^{(n)}, 0) + ((\alpha + \alpha_n)\cdot f^{(n)})(\alpha + \alpha_n)\}e^{i\alpha_n\cdot x},$$

where P is the projection onto the plane orthogonal to ν, i.e.,

$$Pf = -\nu \times (\nu \times f),$$

and

$$f^{(n)} = \Lambda_1^{-1}\Lambda_2^{-1}\int_0^{\Lambda_1}\int_0^{\Lambda_2} f(x)e^{-i\alpha_n\cdot x}dx_1dx_2\,.$$

The proof may be given by using the expansion (3.6) together with (3.13–3.16), and some simple calculations.

Remark 3.3. The significance of this result is that the Dirichlet to Neumann operator B carries the information on radiation condition in an explicit form. Here it is crucial to assume that $\beta^{(n)}$ is nonzero. The present form of the result is equivalent to the one in Abboud [1]. A slightly different form was derived by Dobson [5].

Thus, the scattering problem can be formulated as follows:

$$\nabla_\alpha \times (\frac{1}{\epsilon}\nabla_\alpha \times H_\alpha) - \nabla_\alpha(\frac{1}{\epsilon_C}\nabla_\alpha \cdot H_\alpha) - \omega^2 H_\alpha = 0 \text{ , in } \Omega_0 \text{ ,} \tag{3.19}$$

$$\nu \times (\nabla_\alpha \times (H_\alpha - H_{I,\alpha})) - B_1(P(H_\alpha - H_{I,\alpha})) = 0 \text{ , on } ,_{1} \text{ ,} \tag{3.20}$$

$$\nu \times (\nabla_\alpha \times H_\alpha) - B_2(P(H_\alpha)) = 0 \text{ , on } ,_{2} \text{ ,} \tag{3.21}$$

$$(T_1^\alpha - \frac{\partial}{\partial \nu})H_{\alpha,3} - 2i\beta_1 p_3 e^{-i\beta_1 b} = 0 \text{ , on } ,_{1} \text{ ,} \tag{3.22}$$

$$(T_2^\alpha - \frac{\partial}{\partial \nu})H_{\alpha,3} = 0 \text{ , on } ,_{2} \text{ ,} \tag{3.23}$$

where ϵ_C is a constant (penalty) which will be determined later.

Remark 3.4. Similarly, one can derive the transparent boundary conditions for Problem A. In fact, for x_3 sufficiently large, H_α satisfies

$$(\Delta_\alpha + \omega^2 \epsilon_j \mu) H_\alpha = 0$$

which is identical to Equation (3.5). Thus, formally, the resulting boundary conditions for Problem A are the same as those for Problem B.

4. Well-posedness of the model

In this section we study the well-posedness of the scattering problem (3.19–3.23). Denote $r = 1/\epsilon$ and $r_c = 1/\epsilon_C$. The real constant ϵ_C is chosen such that $\inf_{x\in\Omega_0} Re\ r(x) \geq \frac{3}{4} r_c$. For simplicity, we shall drop α from H_α, T_j^α. Multiplying both sides of (3.19) by $\overline{F}$ and integrating over Ω_0 yield

$$\int_{\Omega_0} \nabla_\alpha \times (r\nabla_\alpha \times H) \cdot \overline{F} - \int_{\Omega_0} \nabla_\alpha (r_c \nabla_\alpha \cdot H) \cdot \overline{F} - \omega^2 \int_{\Omega_0} H \cdot \overline{F} = 0 \ .$$

The left hand side of the above equation may be simplified by simple vector identities to

$$\int_{\Omega_0} r(\nabla_\alpha \times H) \cdot (\overline{\nabla_\alpha \times F}) + \int_{\Omega_0} r_c (\nabla_\alpha \cdot H)(\overline{\nabla_\alpha \cdot F}) - \omega^2 \int_{\Omega_0} H \cdot \overline{F}$$
$$+ \int_{\partial\Omega_0} \nu \times (r\nabla_\alpha \times H) \cdot \overline{F} - \int_{\partial\Omega_0} r_c (\nabla_\alpha \cdot H)(\overline{\nu \cdot F}) \ .$$

Applying the boundary conditions, we have

$$\int_{\Omega_0} r(\nabla_\alpha \times H) \cdot (\overline{\nabla_\alpha \times F}) + \int_{\Omega_0} r_c (\nabla_\alpha \cdot H)(\overline{\nabla_\alpha \cdot F}) - \omega^2 \int_{\Omega_0} H \cdot \overline{F}$$
$$+ \int_{\Gamma_1} r_1 B_1(P(H)) \cdot \overline{F} - \int_{\Gamma_1} r_c (\nabla_{\alpha t} \cdot H)(\overline{\nu \cdot F}) - \int_{\Gamma_1} r_c T_1(H_3) \overline{F_3}$$
$$+ \int_{\Gamma_2} r_2 B_2(P(H)) \cdot \overline{F} - \int_{\Gamma_2} r_c (\nabla_{\alpha t} \cdot H)(\overline{\nu \cdot F}) - \int_{\Gamma_2} r_c T_2(H_3) \overline{F_3}$$
$$= \int_{\Gamma_1} (\nu \times \nabla_\alpha \times H_I - B_1 P(H_I)) \cdot \overline{F} + \int_{\Gamma_1} 2i\beta_1 r_c e^{-i\beta_1 b} \overline{F_3} \ ,$$

where $\nabla_{\alpha t} = (\partial_{x_1} + i\alpha_1, \partial_{x_2} + i\alpha_2, 0)$.

Therefore the weak form of the scattering problem takes the form

$$a(H, F) = R(F) \ ,$$

where

$$\begin{aligned} a(H,F) &= a_1(H,F) - \omega^2 a_2(H,F) \\ &= \int_{\Omega_0} r(\nabla_\alpha \times H)\cdot(\overline{\nabla_\alpha \times F}) + \int_{\Omega_0} r_c(\nabla_\alpha \cdot H)(\overline{\nabla_\alpha \cdot F}) \\ &\quad + \int_{\Gamma_1} r_1 B_1(P(H))\cdot \overline{F} - \int_{\Gamma_1} r_c T_1(H_3)\overline{F_3} \\ &\quad + \int_{\Gamma_2} r_2 B_2(P(H))\cdot \overline{F} - \int_{\Gamma_2} r_c T_2(H_3)\overline{F_3} \\ &\quad - \int_{\partial\Omega_0} r_c(\nabla_{\alpha t}\cdot H)(\overline{\nu\cdot F}) - \omega^2 \int_{\Omega_0} H\cdot\overline{F} \end{aligned}$$

and

$$R(F) = \int_{\Gamma_1} (\nu\times\nabla_\alpha\times H_I - B_1 P(H_I))\cdot\overline{F} + \int_{\Gamma_1} 2i\beta_1 r_c p_3 e^{-i\beta_1 b}\overline{F_3}\ .$$

Next we establish coercivity for

$$\begin{aligned} a_1(H,H) &= \int_{\Omega_0} r|\nabla_\alpha\times H|^2 + r_c|\nabla_\alpha\cdot H|^2 + \int_{\Gamma_1} r_1 B_1(P(H))\cdot\overline{H} - \int_{\Gamma_1} r_c T_1(H_3)\overline{H_3} \\ &\quad - \int_{\partial\Omega_0} r_c(\nabla_{\alpha t}\cdot H)(\overline{\nu\cdot H}) + \int_{\Gamma_2} r_2 B_2(P(H))\cdot\overline{H} - \int_{\Gamma_2} r_c T_2(H_3)\overline{H_3}\ . \end{aligned}$$

We begin with a useful identity.

Lemma 4.1. *For a vector function $u \in C^1(\overline{\Omega_0})^3$ which is Λ-periodic in x_1, x_2, the following identity holds*

$$\int_{\Omega_0} |\nabla_\alpha\times u|^2 + |\nabla_\alpha\cdot u|^2 = \int_{\Omega_0} |\nabla_\alpha u|^2 + 2Re\left\{\int_{\partial\Omega_0} \nabla_{\alpha t}\cdot u\ \overline{\nu_3 u_3}\right\}\ ,$$

where the operators ∇_α, $\nabla_{\alpha t}$ are defined as before and ν_3 is the third component of the outward normal vector ν.

Proof. The proof follows from the vector identity

$$\int_{\Omega_0} |\nabla u|^2 = \int_{\Omega_0} (|\nabla\cdot u|^2 + |\nabla\times u|^2) + \sum_{i,j=1}^{3} \int_{\partial\Omega_0} (\nu_i u_i \partial_{x_j} u_j - \nu_j u_i \partial_{x_i} u_j)$$

and the periodicity of u. □

An application of Lemma 4.1 yields that

$$\begin{aligned} Re\ a_1(H,H) &= \int_{\Omega_0} [(Re(r) - \frac{r_c}{2})|\nabla_\alpha\times H|^2 + \frac{r_c}{2}|\nabla_\alpha\cdot H|^2 \\ &\quad + \frac{r_c}{2}|\nabla_\alpha H|^2] + Re\left\{\int_{\Gamma_1} r_1 B_1(P(H))\cdot\overline{H} + \int_{\Gamma_2} r_2 B_2(P(H))\cdot\overline{H} \right. \\ &\quad \left. - \int_{\Gamma_1} r_c T_1(H_3)\overline{H_3} - \int_{\Gamma_2} r_c T_2(H_3)\overline{H_3}\right\}\ . \end{aligned}$$

Denote for $j = 1,2$ the (possibly empty) sets

$$\Lambda_j^+ = \{n :\ Im\ \beta_j^{(n)} = 0\} \text{ and } \Lambda_j^- = \text{ the complement } of \Lambda_j^+\ .$$

Then

$$
\begin{aligned}
-Re\left\{\int_{\Gamma_1} r_c T_1(H_3)\overline{H_3} \right. &+ \left. \int_{\Gamma_2} r_c T_2(H_3)\overline{H_3}\right\} \\
&= -Re\sum_{n\in\Lambda} i\beta_1^{(n)} r_c |H_3^{(n)}(b)|^2 - Re\sum_{n\in\Lambda} i\beta_2^{(n)} r_c |H_3^{(n)}(-b)|^2 \\
&= r_c \sum_{n\in\Lambda_1^-} |\beta_1^{(n)}|\, |H_3^{(n)}(b)|^2 + r_c \sum_{n\in\Lambda} Im(\beta_2^{(n)})\, |H_3^{(n)}(-b)|^2 \\
&\geq 0.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
& Re\left\{\int_{\Gamma_1} r_1 B_1(P(H))\cdot\overline{H}\right\} \\
= \; & Im\left\{\sum_{n\in\Lambda} \frac{1}{\epsilon_1\beta_1^{(n)}}[(\beta_1^{(n)})^2(|H_1^{(n)}(b)|^2 + |H_2^{(n)}(b)|^2) + |(\alpha+\alpha_n)\cdot H^{(n)}(b)|^2]\right\} \\
= \; & Im\left\{\sum_{n\in\Lambda} \frac{1}{\epsilon_1\beta_1^{(n)}}[(\omega^2\epsilon_1 - |\alpha+\alpha_n|^2)(|H_1^{(n)}(b)|^2 + |H_2^{(n)}(b)|^2) \right. \\
& \left. + |(\alpha+\alpha_n)\cdot H^{(n)}(b)|^2]\right\} \\
= \; & Im\left\{\sum_{n\in\Lambda} \frac{\omega^2}{\beta_1^{(n)}}(|H_1^{(n)}(b)|^2 + |H_2^{(n)}(b)|^2) \right. \\
& \left. + \frac{1}{\epsilon_1\beta_1^{(n)}}[|(\alpha+\alpha_n)\cdot H^{(n)}(b)|^2] - |\alpha+\alpha_n|^2(|H_1^{(n)}(b)|^2 + |H_2^{(n)}(b)|^2)]\right\}
\end{aligned} \tag{4.1}
$$

Since the expression in square brackets in (4.1) is nonpositive, we then have

$$
Re\left\{\int_{\Gamma_1} r_1 B_1(P(H))\cdot\overline{H}\right\} \geq -\sum_{n\in\Lambda_1^-} \frac{\omega^2}{|\beta_1^{(n)}|}(|H_1^{(n)}(b)|^2 + |H_2^{(n)}(b)|^2). \tag{4.2}
$$

Similarly, as one can check that $Im(\epsilon_2\beta_2^{(n)}) > 0$, we find

$$
Re\left\{\int_{\Gamma_2} r_2 B_2(P(H))\cdot\overline{H}\right\} \geq -\sum_{n\in\Lambda_2^-} \frac{\omega^2 Im(\beta_2^{(n)})}{|\beta_2^{(n)}|^2}(|H_1^{(n)}(-b)|^2 + |H_2^{(n)}(-b)|^2). \tag{4.3}
$$

Note that $\beta_2^{(n)}$ is asymptotic to the positive imaginary axis as $|n| \to \infty$. Combining (4.2) and (4.3), and using the properties of $\beta_j^{(n)}$ along with the discrete Fourier representation of the $\mathcal{H}^{-1/2}$ norm, it follows

$$
Re\left\{\int_{\Gamma_1} r_1 B_1(P(H))\cdot\overline{H} + \int_{\Gamma_2} r_2 B_2(P(H))\cdot\overline{H}\right\} \geq -C\omega^2||\nu\times H||^2_{\mathcal{H}^{-1/2}(\partial\Omega_0)^3}.
$$

We next state a trace regularity result.

Lemma 4.2 (Abboud [2]). *For any $\eta > 0$, there is a constant $C(\eta)$ such that the following estimate*

$$
||n\times u||_{\mathcal{H}^{-1/2}(\partial\Omega_0)^3} \leq \eta||\nabla\times u||_{L^2(\Omega_0)^3} + C(\eta)||u||_{L^2(\Omega_0)^3}
$$

holds.

Therefore, we have shown with the help of Lemma 4.2 that

$$Re\ a(H,H) \ \geq C_1||\nabla H||_{L^2}^2 - C_2\omega^2||H||_{L^2}^2 \ .$$

An application of the Fredholm alternative gives

Theorem 4.3. *For all but possibly a discrete set of ω, the scattering problem (3.19–3.23) attains a unique weak solution $H \in \mathcal{H}^1(\Omega_0)^3$.*

Theorem 4.3 establishes existence and uniqueness of solutions to Problem B. Our next result is concerned with the equivalence of Problem A and Problem B.

Theorem 4.4. *Suppose that $H \in \mathcal{H}^1(\Omega_0)^3$ is a solution of the scattering problem. Then for all but a discrete set of frequencies ω,*

$$\nabla_\alpha \cdot H = 0 \ .$$

In order to prove Theorem 4.4, we set $W = \nabla_\alpha \cdot H$ and examine uniqueness of solutions for the equation

$$(\Delta_\alpha + \omega^2\epsilon_C)W = 0 \ .$$

One may derive the boundary conditions similar to (3.9), (3.10) except that they are homogeneous since $\nabla_\alpha \cdot H_I = 0$. A detailed proof of the theorem may be found in Bao [4].

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