RESEARCH STATEMENT

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1. Introduction

My research interest is in the field of commutative algebra. Commutative algebra is the study of commutative rings and modules over such rings. Commutative algebra has applications to many other mathematical fields such as algebraic geometry and algebraic number theory. To an algebraic geometer, commutative algebra studies the underlying ring structure of algebraic varieties and schemes. Methods from commutative algebra are used to classify many properties of varieties such as classifying singularities and the topological structure.

I am interested in commutative algebra for its intrinsic beauty and rich theory. To state in a simplified manner, my own interest concerns the following the questions. What do certain homological properties tell us about a complex or the ring it self? How do homological properties of rings, modules or complexes behave when passing through certain maps? When does a set of modules generate an whole class of modules. We say that a subcategory S is generated by a set of modules T if every module in S is a surjective image of a direct sum of modules in T. These questions are very broad and perhaps impossible to answer in full generality, but I focus on some special cases outlined below.

2. Homological properties over Contracting Endomorphisms

Kunz [Kun69] proved that a local ring $(R, \mathfrak{m}, k)$ of positive characteristic is regular if and only some (equivalently, every) power of the Frobenius endomorphism is flat as an R-module. Since then analogous characterizations of other properties of the ring, such as complete intersections (by Rodicio [Rod88]), Gorenstein (by Goto [Got77]) and Cohen-Macualay (by Takahashi and Yoshino [TY04]), have been obtained. Many of these results has been established for the larger family of *contracting* endomorphisms. Following [AIM06], an endomorphism $\varphi\colon R \to R$ is said to be *contracting* if $\varphi^i(\mathfrak{m}) \subseteq \mathfrak{m}^2$ for some $i > 0$. The Frobenius endomorphism is one example but there are many interesting examples even when R is of characteristic 0.

Avramov, Iyengar and Miller [AIM06] generalized Kunz's theorem to apply to any contracting endomorphism. For other results concerning contracting endomorphisms see, for example, Avramov, Hochster, Iyengar and Yao [AHIY12], Rahmati [Rah09] and Nasseh and Sather-Wagstaff [NSW15].

Recently, Falahola and Marley [?FaMa18] gave the following characterization of Gorenstein rings. Let $(R, \mathfrak{m}, k)$ be a local Cohen-Macaulay ring with a canonical module ω_R. Let φ be a contracting endomorphism of R, then R is Gorenstein if and only if $\mathrm{R}^{\varphi} \otimes \omega_R$ has finite injective dimension. They show that this cannot be generalized to dualizing complexes. They however ask if the derived analogous statement holds. I gave a positive answer, and in fact proved something stronger.

Theorem 2.1. [Dir19] *Let $\varphi\colon R \to R$ be a contracting endomorphism. The following conditions are equivalent.*

(i) *R is Gorenstein.*
(ii) *There exists an R-complex X with finitely generated nonzero homology and finite injective dimension for which the base change $\mathrm{R}^{\varphi} \otimes^{\boldsymbol{L}}_R X$ has finite injective dimension.*
(iii) *For every X with finitely generated nonzero homology and finite injective dimension the base change $\mathrm{R}^{\varphi} \otimes^{\boldsymbol{L}}_R X$ has finite injective dimension.*

There are further questions I am interested to explore in this direction.

Question 2.2. Is the finitely generated homology in Theorem 2.1 necessary? The methods I use in my proof only works for complexes with finitely generated homology but I don't know at this moment whether the same result holds for general complexes of finite injective dimension.

Question 2.3. Do analogous results hold for complexes of finite G-injective dimension?

Theorem 2.1 shows that there are two extreme situations. When X has finite projective dimension, the iterate base change along any endomorphism is always homologically bounded. On the other hand, if R is not Gorenstein and X has finite injective dimension then even the first base change of X along a contracting endomorphism is homologically unbounded. However, there are examples of complexes of infinite projective and injective dimension for which the first base change along a contracting endomorphism is homologically, bounded but eventually the iterate base change become unbounded in homology.

Question 2.4. Suppose X is a complex with bounded finitely generated homology and infinite projective and injective dimension, what can the threshold of when the iterate base becomes unbounded tell us about the homological properties of X?

3. Complexes of Finite Injective Dimension with Full Support

In the previous section, we saw that some properties that were known to hold for dualizing complexes do in fact hold for a much larger class of complexes of finite injective dimension. This, along with some methods I have used to prove Theorem 2.1, has motivated me to study these complexes further.

Sharp [Sha79] has shown that if a ring R admits a dualizing complex, then R satisfies the following conditions: 1) R is a catenary ring of finite Krull dimension. 2) The Gorenstein locus or R is open. 3) R has Gorenstein formal fibers. Sharp called these rings 'acceptable'. Sharp conjectured that all acceptable rings admit dualizing complexes. This was indeed proven later on Kawasaki [Kaw00] who proved that a ring is acceptable if and only if it is a homomorphic image of a Gorenstein ring. But Grothendiek [Har66] has already shown that such rings admit dualizing complexes. As a corollary of Kawasaki's theorem, we know that a ring admits a dualizing complex if and only if it is a homomorphic image of a Gorenstein ring.

Question 3.1. Can we extend Kawasaki's result to a larger class of complexes of finite injective dimension?

It is not hard to show that every local ring admits a complex of finite injective dimension with finitely generated homology which are supported only at the maximal ideal. Hence, if we are looking for restricting conditions on the ambient ring, we need to make some assumptions on the support of the complex. In my thesis, I have proved that if a ring R admits a complex of finite injective dimension with finitely generated homology with full support, then R satisfies the following conditions: 1) R has finite Krull dimension. 2) The Gorenstein locus or R is open. 3) R has Gorenstein formal fibers.

These are almost all of the conditions that sharp proved for rings which admit a dualizing complex, with the exception of the catenary condition. I am currently working to see whether this last condition is also necessarily satisfied.

It should be pointed out that if we can give an affirmative answer to Question 3.1 then it, together with a result by Letz [Let19], will give a complete description when an analogous statement to Hopkins and Neeman hold for the category of finite injective dimension.

4. Syzygies of the Residue Field

Let $(R, \mathfrak{m}, k)$ be a local ring. For every $n \geq 0$ we denote the n'th syzygy of k as $\operatorname{Syz}_n^R(k)$. The higher syzygies of k give information on the singularities of R and sometimes we can read off some geometrical properties, see [Eis05] for an extensive study on the geometry of syzygies. The syzygies of the residue ring have been studied extensively. I recall some well known theorems that provided me with motivation for my project.

Dutta [Dut89] proved that if R is singular then there are no surjections from a syzygy of k onto a nonzero free module. Martsinkovsky [Mar96] later extended Dutta's result and proved that if R is singular, then no direct sum of the syzygy modules of k surjects onto a non-zero R-module of finite projective dimension. Similar results have been obtained since. For example, Ghosh, Gupta and Puthenpurakal [GGP18] gave the following characterization of regular local rings: If a finite direct sum of syzygy modules of k surjects onto a semidualizing R-module, then R is regular.

These results can be thought of as obstructions to the existence of surjections from syzygies of k onto certain modules. In particular, these results tell us that modules with certain properties cannot be generated by $\{\mathrm{Syz}_i(k)\}$. I am interested in a converse question, can we classify the subcategory generated by $\{\mathrm{Syz}_i(k)\}$? To make this more concrete we need the notion of complexity and curvature of a module.

Let M be a finitely generated R-module. The size of a minimal free resolution of a finite R–module M is given by its Betti numbers $\beta^R_n(M) = \mathrm{Rank}_k \mathrm{Ext}^R_n(M, k)$. The *complexity* of M, denoted $\mathrm{cx}_R(M)$, is at most d if the growth of the Betti numbers of M is bounded by a polynomial of degree $d - 1$. The *curvature of* M, denoted $\mathrm{curv}_R(M)$, is the reciprocal of the radius of convergence of the Hilbert series of M. Every syzygy $\mathrm{Syz}^R_n(M)$ has the same complexity and curvature as M.

For every local ring, k always has maximal complexity and curvature. That is for any finitely generated module M we have

$$\mathrm{cx}_R(M) \leq \mathrm{cx}_R(k)$$
$$\mathrm{curv}_R(M) \leq \mathrm{curv}_R(k)$$

Modules with maximal complexity and curvature occur in obundance. Avramov [Avr96] proved that for any finitely generated module M, either $\mathfrak{m}M = 0$ or $\mathfrak{m}M$ has maximal complexity and curvature. Avramov, Hochster, Iyengar and Miller [AHIY12] show that if φ is a contracting endomorphism then for any finitely generated module M the base change $\mathrm{R}^\varphi \otimes_R M$ has maximal complexity and curvature as a module over the source ring.

Avramov [Avr96] proved the following, Let N is a finitely generated R module, and suppose there is a surjective map $\beta : \mathrm{Syz}^n(k) \to N$ for some n, then N has maximal complexity and curvature. Naively, one may ask if the converse is true, that is if M has maximal complexity and curvature does there exists a surjection $\beta : \mathrm{Syz}^n(k) \to M$ for some n? The first obvious obstruction comes from Martsinkovsky's theorem. If M has maximal complexity then so will $M \oplus R$, but by Martsinkovsky's theorem there cannot be any surjection from a direct sum of syzygy module of k onto $M \oplus R$. However, even if we restrict the question to indecomposable modules, it is not very hard to show that the answer is negative. Even for negative syzygies of k in an Artinian ring we cannot in general find surjections from positive syzygies of k. However we may ask a weaker question.

Question 4.1. Let M be an indecomposable module of maximal complexity, does there exists a surjection $\beta : \mathrm{Syz}^n(k) \twoheadrightarrow \mathrm{Syz}^R_m(M)$ for some n and some m?

4.1. **Partial Progress.** The key point in Avramov's proof is that if there are maps $\mathrm{Syz}^n(k) \to N \to N/\mathfrak{m}N$ whose composition is nonzero then these maps induce maps

$$\mathrm{Ext}^*_R(N/\mathfrak{m}N, k) \hookrightarrow \mathrm{Ext}^*_R(N, k) \to \mathrm{Ext}^{\geq n}_R(k, k)$$

whose composition is also nonzero. This yields that the syzygies of N grows at least as fast as the syzygies of k hence N has maximal complexity and curvature. I am trying to use ideas employed in Avramov's proof. In that direction I have been able to make some progress in the case when R is an Artinian complete intersection. When R is a complete intersection we have that $\mathrm{Ext}^*_R(k, k)$ is a graded Noetherian ring and $\mathrm{Ext}^*_R(M, k)$ is a module over $\mathrm{Ext}^*_R(k, k)$. I am able to show that M has maximal complexity if and only if there are maps

$$\mathrm{Ext}^*_R(k, k) \hookrightarrow \mathrm{Ext}^*_R(M, k) \to \mathrm{Ext}^{\geq n}_R(k, k)$$

Whose composition is nonzero. This leads me to a new question:

Question 4.2. Suppose we have maps $\operatorname{Ext}_R^*(k,k) \hookrightarrow \operatorname{Ext}_R^*(M,k) \to \operatorname{Ext}_R^{\geq n}(k,k)$ whose composition in nonzero, under what conditions are these maps induced by a map $\operatorname{Syz}_n^R(k) \to M$?

This seems to be a hard question. Currently, I have been able to show that the maps in Question 4.2 extend to nonzero maps in the stable category. I am trying to use ideas by Beligiannis and Krause [BK03], who gave a criterion for when maps between stable cohomology modules are realized as maps between the underlying modules.

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