

Schur multipliers of functions extended by zero

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Abstract

We show that the extension by zero of a function in $M_{d \geq 3}$ has the same norm as the original function. This was only known for $d = 2$ (see [1]).

Let $f \in M_d(H)$, where $d \geq 2$. Assume we have the embedding $H \subset G$ and extend f by 0 :

$$\hat{f} : G \rightarrow \mathbb{C} : g \mapsto \begin{cases} f(g) & g \in H \\ 0 & g \notin H \end{cases}$$

Proposition 0.1. *The functions f and \hat{f} have the same norm:*

$$|f|_{M_d(H)} = |\hat{f}|_{M_d(G)}$$

Proof. It is clear that $|f|_{M_d(H)} \leq |\hat{f}|_{M_d(G)}$ it is therefor sufficient to show the opposite inequality.

Assume that f factors through ξ_1 and ξ_2 :

$$f(t_1 t_2) = \xi_1(t_1) \dots \xi_d(t_d)$$

where $\xi_d : H_{d-1} \rightarrow H_d$, et $H_0 = H_d = \mathbb{C}$. We define the new functions $\hat{\xi}_d$:

$$\hat{\xi}_d(t) = \begin{cases} \xi_d(t) & t \in H \\ 0 & t \notin H \end{cases}$$

Fix a set of right H -coset representatives in G , i.e. $G = H \sqcup Hg_1 \dots \sqcup Hg_k$, and call it $\mathcal{C} = \{e, g_1, \dots, g_k\}$.

Consider now the following operator which are given in block-form :

$$\begin{aligned} \Xi_1(t) &= \left(\hat{\xi}_1(tg^{-1}) \right)_{g \in \mathcal{C}} \\ \Xi_{\lambda=2, \dots, d-1}(t) &= \left(\hat{\xi}_\lambda(gt\tilde{g}^{-1}) \right)_{g, \tilde{g} \in \mathcal{C}} \\ \Xi_d(t) &= \left(\hat{\xi}_1(gt) \right)_{g \in \mathcal{C}}^t \end{aligned}$$

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The domains and ranges are $\Xi_1 : \mathbb{C} \rightarrow H_1^{[G:H]}$, $\Xi_{\lambda=2,\dots,d-1} : H_{\lambda-1}^{[G:H]} \rightarrow H_{\lambda}^{[G:H]}$ and $\Xi_d : H_{d-1}^{[G:H]} \rightarrow \mathbb{C}$. These Ξ factor \hat{f} (see lemma 0.2) :

$$\hat{f}(t_1 \dots t_d) = \Xi_1(t_1) \dots \Xi_d(t_d)$$

We will show that $\|\Xi_k\| = \|\xi_k\|$. The cases $k = 1, d$ are trivial we will thus focus on the central Ξ 's.

Let g be an element of G , for a given $g_i \in \mathcal{C}$ there exists a unique $g_j \in \mathcal{C}$ such that $g \in g_i H g_j^{-1}$. Therefore, Ξ_i is a permutation matrix and its norm is the supremum of the norm of its entries. Since these entries are in ξ_i , the norms of the lower case and upper case operators are the same. \square

Lemma 0.2. *Let the function \hat{f} and the operators Ξ_d be as above. We have the following Schur factorization :*

$$\hat{f}(t_1 \dots t_d) = \Xi_1(t_1) \dots \Xi_d(t_d)$$

Proof. The element t_1 belongs to a unique $Hg_{(1)}$. Given this element $g_{(1)} \in \mathcal{C}$, t_2 belongs to a unique $g_{(1)}^{-1} H g_{(2)}$. Inductively, all $t_{i=2\dots d-1}$ belong to a unique $g_{(i-1)}^{-1} H g_{(i)}$. Therefore,

$$\begin{aligned} & \Xi_1(t_1) \cdots \Xi_2(t_2) \cdots \Xi_{d-1}(t_{d-1}) \\ &= \left(\hat{\xi}_1(t_1 g^{-1}) \right)_{g \in \mathcal{C}} \cdot \left(\hat{\xi}_2(g t_2 h^{-1}) \right)_{g, h \in \mathcal{C}} \cdot \Xi_2(t_3) \cdots \Xi_{d-1}(t_{d-1}) \\ &= \left(\sum_{g \in \mathcal{C}} \hat{\xi}_1(t_1 g^{-1}) \cdot \hat{\xi}_2(g t_2 h^{-1}) \right)_{h \in \mathcal{C}} \cdot \Xi_2(t_3) \cdots \Xi_{d-1}(t_{d-1}) \\ &= \left(\xi_1(t_1 g_{(1)}^{-1}) \cdot \hat{\xi}_2(g_{(1)} t_2 h^{-1}) \right)_{h \in \mathcal{C}} \cdot \Xi_2(t_3) \cdots \Xi_{d-1}(t_{d-1}) \\ & \vdots \\ &= \left(\xi_1(t_1 g_{(1)}^{-1}) \cdot \xi_2(g_{(1)} t_2 g_{(2)}^{-1}) \cdots \hat{\xi}_{d-1}(g_{(d-1)} t_d h^{-1}) \right)_{h \in \mathcal{C}} \end{aligned}$$

Therefore,

$$\begin{aligned} & \Xi_1(t_1) \cdots \Xi_2(t_2) \cdots \Xi_d(t_d) \\ &= \left(\xi_1(t_1 g_{(1)}^{-1}) \cdot \xi_2(g_{(1)} t_2 g_{(2)}^{-1}) \cdots \hat{\xi}_{d-1}(g_{(d-1)} t_d h^{-1}) \right)_{h \in \mathcal{C}} \left(\hat{\xi}_d(h t_d) \right)_{h \in \mathcal{C}}^t \\ &= \sum_{h \in \mathcal{C}} \xi_1(t_1 g_{(1)}^{-1}) \cdots \hat{\xi}_{d-1}(g_{(d-1)} t_d h^{-1}) \hat{\xi}_d(h t_d) \\ &= \xi_1(t_1 g_{(1)}^{-1}) \cdots \hat{\xi}_{d-1}(g_{(d-1)} t_d g_{(d)}^{-1}) \hat{\xi}_d(h t_d) \\ &= \begin{cases} \xi_1(t_1 g_{(1)}^{-1}) \cdots \xi_{d-1}(g_{(d-1)} t_d g_{(d)}^{-1}) \xi_d(g_{(d)} t_d) & \text{if } t_d \in g_{(d)}^{-1} H \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

The second case occurs exactly when $t_1 \dots t_d \notin H$. The first case corresponds to $t_1 \dots t_d \in H$. Now, we note that the last line is nothing but

$$\begin{aligned} & \xi_1(t_1 g_{(1)}^{-1}) \dots \xi_{d-1}(g_{(d-1)} t_d g_{(d)}^{-1}) \xi_d(g_{(d)} t_d) \\ = & f(t_1 g_{(1)}^{-1} \dots x_{i_{d-1}}(g_{(d-1)} t_d g_{(d)}^{-1}) \xi_d(g_{(d)} t_d) \\ = & f(t_1 t_2 \dots t_d) \end{aligned}$$

□

Example in M_3 :

Take the extension $H = \{0\} \subset \mathbb{Z}/2\mathbb{Z}$ and assume $f(0) = 3 \in \mathbb{C}$. The function f factors through

$$\begin{aligned} \xi_1(0) : & \quad \mathbb{C} \rightarrow H_1 = \mathbb{C} & : z \mapsto 3 \cdot z \\ \xi_2(0) : & \quad H_1 = \mathbb{C} \rightarrow H_2 = \mathbb{C} & : z \mapsto z \\ \xi_3(0) : & \quad H_2 = \mathbb{C} \rightarrow \mathbb{C} & : z \mapsto z. \end{aligned}$$

The function \hat{f} is simply defined by $\hat{f}(0) = 3$ and $\hat{f}(1) = 0$. As $\mathbb{Z}/2\mathbb{Z} = 1 + H \sqcup 0 + H$, the set \mathcal{C} contains two elements : 0 and 1. Therefore:

- $\Xi_1(t) := \begin{pmatrix} \hat{\xi}_1(t-0) & \hat{\xi}_1(t-1) \end{pmatrix} = \begin{pmatrix} \hat{\xi}_1(t) & \hat{\xi}_1(t+1) \end{pmatrix}$
- $\Xi_2(t) := \begin{pmatrix} \hat{\xi}_2(0+t-0) & \hat{\xi}_2(0+t-1) \\ \hat{\xi}_2(1+t-0) & \hat{\xi}_2(1+t-1) \end{pmatrix} = \begin{pmatrix} \hat{\xi}_2(t) & \hat{\xi}_2(t+1) \\ \hat{\xi}_2(t+1) & \hat{\xi}_2(t) \end{pmatrix}$
- $\Xi_3(t) := \begin{pmatrix} \hat{\xi}_3(0+t) & \hat{\xi}_3(1+t) \end{pmatrix}^t = \begin{pmatrix} \hat{\xi}_3(t) & \hat{\xi}_3(t+1) \end{pmatrix}^t$

for the decomposition $\hat{f}(0) = \hat{f}(1+0+1) = f(0)$, we have

$$\begin{aligned} & \Xi_1(1) \cdot \Xi_2(0) \cdot \Xi_3(1) \\ = & \begin{pmatrix} \hat{\xi}_1(1) & \hat{\xi}_1(1+1) \end{pmatrix} \begin{pmatrix} \hat{\xi}_2(0+0+0) & \hat{\xi}_2(0+0+1) \\ \hat{\xi}_2(0+0+1) & \hat{\xi}_2(1+0+1) \end{pmatrix} \begin{pmatrix} \hat{\xi}_3(1) \\ \hat{\xi}_3(1+1) \end{pmatrix} \\ = & \begin{pmatrix} 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ = & 3 \end{aligned}$$

As expected !

References

- [1] Pisier, Gilles. Are unitarizable groups amenable? Infinite groups: geometric, combinatorial and dynamical aspects, 323–362, Progr. Math., 248, Birkhäuser, Basel, 2005.

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