Abstract commensurators of lattices in Lie groups

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Partial automorphisms

Let $\Gamma$ be an infinite group.

**Definition**

A *partial automorphism* of $\Gamma$ is an isomorphism $\phi : H \to K$ between finite index subgroups $H, K \leq \Gamma$.

**Example:** Let $\Gamma = \mathbb{Z}$. Then $2n \mapsto 3n$ defines an isomorphism $\phi_1 : 2\mathbb{Z} \to 3\mathbb{Z}$. 
Definition and Examples

**Equivalence of partial automorphisms**

Define $\phi_1 \sim \phi_2$ if there is some finite index subgroup $\Lambda \leq \Gamma$ so that

$$\phi_1|_{\Lambda} = \phi_2|_{\Lambda}.$$ 

**Example:** Let $\phi_1 : 2\mathbb{Z} \to 3\mathbb{Z}$ map $2n \mapsto 3n$ and $\phi_2 : 4\mathbb{Z} \to 6\mathbb{Z}$ map $4n \mapsto 6n$. Then $\phi_1 \sim \phi_2$ because $\phi_1|_{4\mathbb{Z}} = \phi_2|_{4\mathbb{Z}}$.

**Remark:** Can’t compose

$$\phi_1 : 2\mathbb{Z} \to 3\mathbb{Z} \quad \text{and} \quad \phi_2^{-1} : 6\mathbb{Z} \to 4\mathbb{Z}$$

$$\begin{align*}
2n & \mapsto 3n \\
6n & \mapsto 4n
\end{align*}$$

But $\phi_1 \sim \phi_2$, so can say $[\phi_1] \circ [\phi_2^{-1}] = [\text{Id}]$.

**Key Point:** Can compose any two partial automorphisms up to equivalence.
Abstract commensurator

**Definition**

The *abstract commensurator* of $\Gamma$, denoted $\text{Comm}(\Gamma)$, is the group of equivalence classes of partial automorphisms of $\Gamma$.

Write $\Gamma_1 \bowtie \Gamma_2$ if $\Gamma_1$ and $\Gamma_2$ contain isomorphic finite index subgroups.

**Remark:** If $\Gamma_1 \bowtie \Gamma_2$, then $\text{Comm}(\Gamma_1) = \text{Comm}(\Gamma_2)$.

**Example**

$\text{Comm}(\mathbb{Z}) = \mathbb{Q}^*$

**Proof:** $[\phi] \in \text{Comm}(\mathbb{Z})$ satisfying $\phi(m) = n$ corresponds to $\frac{n}{m} \in \mathbb{Q}^*$.  

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Examples

- Exercise: \( \text{Comm}(\mathbb{Z}^n) = \text{GL}_n(\mathbb{Q}) \)

- (Ivanov) Mapping class groups:
  \[ \text{Comm}(\text{Mod}^{\pm}(\Sigma_g)) = \text{Mod}^{\pm}(\Sigma_g) \text{ for } g \geq 2 \]

- (Farb–Handel) Outer automorphisms of free groups:
  \[ \text{Comm}(\text{Out}(F_n)) = \text{Out}(F_n) \text{ for } n \geq 4 \]

- (Leininger–Margalit) Braid groups:
  \[ \text{Comm}(B_n) = \text{Mod}^{\pm}(\Sigma_{0,n+1}) \rtimes (\mathbb{Q}^\times \rtimes \mathbb{Q}^\infty) \text{ for } n \geq 4 \]
Motivating question

Question
Suppose \( \Gamma \) is a lattice in a Lie group \( G \). What is \( \text{Comm}(\Gamma) \)?

Example: \( G = \mathbb{R}^n \) and \( \Gamma = \mathbb{Z}^n \).

\[
\text{Aut}(\mathbb{Z}^n) = \text{GL}_n(\mathbb{Z}) \\
\text{Comm}(\mathbb{Z}^n) = \text{GL}_n(\mathbb{Q}) \\
\text{Aut}(\mathbb{R}^n) = \text{GL}_n(\mathbb{R})
\]
Outline of approach

Question
Suppose $\Gamma$ is a lattice in a Lie group $G$. What is $\text{Comm}(\Gamma)$?

Levi Decomposition: If $G$ an (algebraically) simply-connected Lie group,

$$G = G^{\text{sol}} \rtimes G^{\text{ss}},$$

where $G^{\text{sol}}$ is the maximal connected solvable normal subgroup, and $G^{\text{ss}}$ is semisimple. Decompose question:

1. Semisimple case
2. Solvable case
3. Combine semisimple and solvable cases

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Step 1: Lattices in semisimple groups

Example (Margulis, Borel)

Let $G = \text{PGL}_n(\mathbb{R})$ and $\Gamma = \text{PGL}_n(\mathbb{Z})$ for $n \geq 3$. Then

$$\text{Comm}(\text{PGL}_n(\mathbb{Z})) \cong \text{PGL}_n(\mathbb{Q}).$$

Theorem (Mostow, Prasad, Margulis, Borel)

Let $G$ be a ‘nice’ semisimple Lie group without center, without compact factors, not $\text{PSL}_2(\mathbb{R})$, and $\Gamma \leq G$ an irreducible lattice. Then

$$\Gamma \cong G(\mathbb{Z}) \iff \text{Comm}(\Gamma) = G(\mathbb{Q})$$

$\Gamma$ non-arithmetic $\iff$ Comm$(\Gamma) \cong \Gamma$.

Main tool: Strong rigidity: Commensurations of $\Gamma$ extend to automorphisms of $G$. 
Step 2: Lattices in solvable groups

**Theorem (S.)**

Suppose $\Gamma$ is a lattice in a simply-connected solvable Lie group $G$. Then there is some linear algebraic group $A$ such that $\text{Comm}(\Gamma) = A(\mathbb{Q})$ and $\text{Aut}(\Gamma) \cong A(\mathbb{Z})$.

**Remark:** (Malcev) If $G$ nilpotent then $A(\mathbb{R}) = \text{Aut}(G)$. 
Good example: lattice in SOL

Let $\Gamma := \mathbb{Z}^2 \rtimes_M \mathbb{Z}$, where $M = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

- Fact: Every $[\phi] \in \text{Comm}(\Gamma)$ satisfies $\phi(H \cap \mathbb{Z}^2) = K \cap \mathbb{Z}^2$.
- Restriction induces a map

  $$\text{Comm}(\Gamma) \to \text{Comm}(\mathbb{Z}^2) = \text{GL}_2(\mathbb{Q}).$$

Proposition

*There is (virtually) an exact sequence*

$$1 \to \mathbb{Q}^2 \to \text{Comm}(\Gamma) \to C_{\text{GL}_2(\mathbb{Q})}(M) \to 1.$$
Illustrating difficulty

**First attempt at proof:** As in semisimple case, try to use rigidity of $\Gamma$ in $G$ to extend elements of $\text{Comm}(\Gamma)$ to automorphisms of $G$.

**Key Difficulty:** Rigidity fails!

**Example:** Let $G = \mathbb{R}^2 \rtimes \widetilde{\text{SO}}(2)$ and $p : \widetilde{\text{SO}}(2) \to \text{SO}(2)$. Let $\Gamma = \mathbb{Z}^2 \rtimes p^{-1}(0)$. Then

- $\Gamma = \mathbb{Z}^3$
- Automorphism swapping a $\mathbb{Z}$ factor with $p^{-1}(0)$ does not extend to an automorphism of $G$.
- $\text{Comm}(\Gamma) = \text{GL}_3(\mathbb{Q})$ while $\text{Aut}(G) \neq \text{GL}_3(\mathbb{R})$
Overcoming the difficulty

**Goal:** Find linear algebraic group $A$ such that $\text{Comm}(\Gamma) = A(\mathbb{Q})$.

**Steps:**

1. (Virtually) embed $\Gamma$ in an algebraic group $H$ with $\text{Comm}(\Gamma) \rightarrow \text{Aut}(H)$ (the *algebraic hull*, using ideas of Mostow, Raghunathan)

2. Find an algebraic subgroup $A \leq \text{Aut}(H)$ such that $\text{Comm}(\Gamma) = A(\mathbb{Q})$ (techniques of Baues–Grunewald)

Thank you!