MATH 3160 UPDATE: SPRING 2014

The most recent lecture will always appear first; please email me if you notice any errors/typos.

Lecture 26: Tuesday, April 15
Today, we recalled the definition of reside, as well as the (very important!) Residue Thereom, which we discussed last time. After going over a few examples to jog our memory, we turned our attention to finding a (sometimes) more efficient way to compute resides. Recall that a singular point z_0 of $f(z)$ is a called a pole of order m if and only if we can write

$$f(z) = \frac{\phi(z)}{(z-z_0)^m},$$

where $\phi(z)$ is a function that is analytic (and non-vanishing) at z_0. Using such a representation of $f(z)$, we used the fact that $\phi(z)$ has a Laurent expansion in a neighborhood of z to obtain the following **Theorem:** If z_0 is a pole of order m, and if $f(z) = \frac{\phi(z)}{(z-z_0)^m}$, with ϕ as above, then

$$\text{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}.$$

A really nice case of this is when $m = 1$ (in this case, we called z_0 a simple pole), since the formula simplifies down to $\text{Res}_{z=z_0} f(z) = \phi(z_0)$. We spent the rest of the class going over lots of examples. **Note:** Students should expect this material to appear (in some way) on the final exam (which takes place next Friday, April 25).

Lecture 25: Thursday, April 10
We started class by defining the notion of *residue*. **Definition:** If a function $f(z)$ is analytic in some deleted neighborhood of a point z_0, the reside of f at z_0, denoted $\text{Res}_{z=z_0} f$, is the coefficient of $\frac{1}{z-z_0}$ in the Laurent series expansion of $f(z)$ centered at z_0. We computed a bunch of examples of residues (especially residues at non-zero points, which required us to find Laurent expansions *not* centeret the origin), and then stated and proved the main theorem of the day (and one of the main theorems of the course). **Theorem (Cauchy's Residue Theorem)**: If C is a closed contour, and if $f(z)$ is analytic at all points on C, and at all points inside of C, except for the points $z_0, \cdots, z_n$, then

$$\int_C f(z)dz = \sum_{k=0}^{n} 2\pi i \cdot \text{Res}_{z=z_k} f(z).$$

Using this theorem, we computed many examples of contour integrals. Moreover, though many of these examples could have been dealt with using the (generalized) Cauchy Integral Formula, we also went over examples in which none of our previous techniques would have worked. We'll start next time by computing more examples of integrals using the Residue Theorem.

Lecture 24: Tuesday, April 8
Today, we went over the following generalization of an earlier theorem: **Theorem:** If $f(z)$ is analytic on a simple closed contour C, and fails to be analytic only at a finite number of points $z_0, \cdots, z_n$ contained strictly inside of C, then $\int_C f(z)dz = \sum_{k=0}^{n} \int_{C_k} f(z)dz$, where here, $C_0, \cdots, C_n$ are small contours lying entirely inside of C, with each contour C_k being centered at z_k. We gave a sketch of the proof of this (it involved connecting $C_0, \cdots, C_n$ (with the reverse orientation) with

the original contour C in a clever way, and then applying Cauchy-Goursat. After going over some examples of how to use this theorem, we ended class by discussing (the results of) Midterm 2.

Lecture 23: Thursday, April 3

Today, we took Midterm 2.

Lecture 22: Tuesday, April 1

We gave a detailed $\epsilon - \delta$ proof of Laurent's theorem. Changing track, we then defined the notion of *complex manifolds*. After discussing/reviewing the necessary background from topology (e.g, the notion of homotopy equivalence), we talked a bit about Teichmuller spaces, and eventually, moduli spaces. We finished the class by discussing the mapping class group; we'll pick up here next time. April Fools! We actually spend some time reviewing the statements of Taylor/Laurent series, and spent the rest of the class computing examples of Laurent series. Don't forget that Midterm 2 is in two days!

Lecture 21: Thursday, March 27

Today, we recalled Taylor's theorem (which was covered by Jack on Tuesday). **Theorem (Taylor series)**: Suppose f is analytic at z_0, and that the nearest point to z_0 at which f is *not* analytic is a distance R from z_0 (note, it is possible to have $R = \infty$). Then, for every point z inside the circle of radius R centered at z_0 (i.e, $|z - z_0| < R$), then $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$. The series you get when $z_0 = 0$ is called the Maclaurin series of f. After thinking a bit more about series of these types, we then asked whether power series expansions exist for functions that are *not* analytic at a point z_0. In this case, we saw that such power series do exist, but we have to allow to possibility of having negative powers of z appear in our expansion. **Theorem (Laurent series)** Suppose f is not analytic at a point z_0, but is analytic in the annular region $A < |z - z_0| < B$ centered at z_0 (here, we may have that $A = 0$, and that $B = \infty$). Then, there exists a power series expansion

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=0}^{\infty} \frac{b_n}{(z - z_0)^n},$$

where the coefficients a_n and b_n are explicitly determined in terms of some contour integrals (we de-emphasized this point of view in class). Using the important formula $\sum_{n=0}^{\infty} w^n = \frac{1}{1-w}$ if $|w| < 1$, we computed a bunch of examples of Laurent series (we'll get more practice computing these in HW).

Lecture 20: Tuesday, March 25

Today's lecture was covered by Jack Jeffries. Jack went over Sections 56, 57, and 59 (skipping Section 58).

Lecture 19: Thursday, March 20

Today, we had a more theoretical focus. We began by receiving the (generalized) Cauchy integral formula, and then continued our discussion of its consequences. We started out by deriving Cauchy's inequality, which describes how big $f^{(n)}(z_0)$ can be at an analytic point z_0 (see Theorem 3 from Section 52). Using this, we were able to state and prove **Liouville's Theorem:** If f is bounded and entire, then f must be constant. Using this new result, we were able to state and prove the **Fundamental Theorem of Algebra:** If P is a non-constant polynomial with complex coefficients, then P must have a root (which is also a complex numbers). We discussed how this is the same thing as saying every polynomial with complex coefficients can be factored completely into a product of linear factors (each with some multiplicity). *This is probably the most important property of the complex numbers that we will discuss in Math 3160.*

Lecture 18: Tuesday, March 18

We started the lecture by reviewing (and sketching the key ideas behind the proofs) of the most important facts regarding contour integration that we discussed before the break (the Cauchy-Goursat theorem, the theorem on invariance of contour, and the very important Cauchy Integral formula). We then went over some example of how to use the Cauchy integral formula, and then started exploring generalizations. Indeed, without too much work, we were able to prove the following generalization of the Cauchy Integral formula (which, by abuse of notation, we will *also* call the Cauchy Integral formula): **Theorem:** If f is analytic on and inside a simple closed contour C, and z_0 is a point contained inside of C, then

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz.$$

After working through a bunch of practical examples, we also were able to use this to prove the following (amazing) theorem: **Theorem:** If f is analytic at z_0, then *all* higher derivatives $f^{(n)}$ are also analytic at z_0. Next time, we'll give other interesting consequences of the Cauchy Integral formula.

Lecture 17: Thursday, 3/6

We began the class by recalling the Cauchy-Goursat theorem, and went over some more examples. Next, we examined what can be said in the situation that we are integrating over a domain with holes (i.e., over a domain that is *not* simply connected). By dividing up contours in a clever way, we were able to deduce the following **Theorem**: If C_1 and C_2 are two simple closed contours (both oriented in the same direction) such that f is analytic in the region contained between C_1 and C_2, then $\int_{C_1} f dz = \int_{C_2} f dz$. This (amazing) result make computing integrals much easier, in that (in many cases) we can replace a given (random) contour with one that is really nice. This set us up to being our discussion of the (famous) Cauchy Integral formula, which is another great way to evaluate (special) contour integrals.

Lecture 16: Tuesday, 3/4

We recalled the main theorem from last time (i.e., the result relating the three points appearing below), and went over an example of how to use it to compute $\int_C \sqrt{z} dz$, where C denotes any contour starting at 10, and ending at -10, and lying entirely above the real axis. After this, we defined what it means for a region to be *simply connected*. Intuitively, this condition can be restated as follows: A region is simply connected if and only if it contains no "holes." After going over some (non)examples, we presented (a version of) the **Cauchy Goursat Theorem:** If a closed contour C is contained a region D such that 1) f is analytic at all points of D, and 2) D is simply connected, then $\int_C f dz = 0$. Note: For those reading the text, this appears as the main theorem in Section 48. We then went over an example of how to (not) use the Cauchy-Goursat theorem. Using the third point of the theorem described last class (you can see it in the previous entry), we saw that the Cauchy-Goursat theorem allows us to note the following string of implications:

- Suppose f is analytic in some region D, and that D happens to be simply connected (i.e., contains no holes)
- Cauchy-Goursat then says that $\int_C f dz = 0$ for every closed contour C contained in D.
- This is just the last point of the Theorem described during last class, which tells us that f has an antiderivative F throughout D.
- Summarizing, we see the following (rough) idea: If f happens to be differentiable at every point of D, then f is a derivative at every point of D!
- This observation appears as the Corollary in Section 49.

We'll pick up here next time.

Lecture 15: Thursday, 2/27

We started by recalling an earlier computation showing that $\int_C f dz$ sometimes depends on the contour C itself, and not just its starting/end points. We then recalled the definition of what it means for a function $f(z)$ to have an antiderivative $F(z)$ on a domain D: For every point z in the domain, we have that $F'(z) = f(z)$. We then stated a *very* important theorem (which may be thought of as a complex version of the Fundamental Theorem of Calculus). **Theorem:** For a function $f(z)$ defined on a domain D, the following are equivalent:

(1) $f(z)$ has an antiderivative on the domain D.

(2) If C is a contour contained in D, then $\int_C f dz$ depends only on the starting/ending points of C. Moreover, if $F(z)$ is an antiderivative of f on the domain D, then

$$\int_C f(z) = F(z_2) - F(z_1),$$

where z_1 is the starting point of C, z_2 the ending point of C, and C is contained entirely in D.

(3) $\int_C f dz = 0$ for every *closed* (i.e., same starting/ending point) contour C contained in D

We went over a bunch of examples on how to use this theorem. The most interesting of this was the computation that $\int_C \frac{1}{z} dz = 2\pi i$, where C is the circle of radius 10 centered at the origin. The key point here was that we had to break the contour up into multiple pieces, and on each piece, find an antiderivative of $\frac{1}{z}$ by looking at an appropriate branch of $\log z$. We'll pick up here next time.

Lecture 14: Tuesday, 2/25

Today, we gave more examples of contour integrals. We started the class by recalling a computation done in the previous class: For a certain function $f(z)$, and for a given contour C with starting point 0 and ending point $1+i$, we computed the integral $\int_C f dz$. On the other hand, for the *same* function $f(z)$, and for a different contour C' with the same start/end points as C, we saw that the value of $\int_{C'} f dz$ and $\int_C f dz$ were different. Thus, we observed the follwing **Observation:** Even though the function being integrated and the endpoints of the path don't change, the value of the contour integral may change for different contours! On the other hand, we showed directly that, if $f(z) = z^n$ for some $n \geq 1$, then $\int_C f dz$ depended only on the endpoints of the contour C, but not on the actual contour itself! This subtle difference will become a key point shortly.

We then showed that the arc length of a contour C given by $z = z(t), a \leq t \leq b$, is given by $\int_a^b |z'(t)| dt$, and using this, we proved the following **Theorem:** If $|f(z)| \leq M$ for every point z on the contour C, then $\left|\int_C f dz\right| \leq M \cdot L$, where L is the arc length of the contour C. This theorem is very useful, in that it allows us to understand integrals without actually computing them! We highlight two points that appear often.

How to find M: If $f(z) = \frac{g(z)}{h(z)}$ is a fraction, then to find an M that works, do the following:

- Find a number N such that $|g(z)| \leq N$ for every point z on the contour.
- Find a number D such that $|h(z)| \geq D$ for every point z on the contour.

Then, it follows that $|f(z)| = \frac{|g(z)|}{|h(z)|} \leq \frac{N}{D}$, so you can take $M = \frac{N}{D}$. When finding the values of N and D described above, the following inequalities will be key:

- $|z \pm w| \leq |z| + |w|$
- $|z \pm w| \geq ||z| - |w||$ (often, the stuff on the inside will be positive, so the outer absolute value symbol won't be necessary).

Lecture 13: Thursday, 2/20

Today, we recalled the definitions of *arcs* and *contours*, and also recalled what it means to take a function $w(t) = u(t) + iv(t)$ (with t being a real variable), and compute the integral $\int_a^b w(t)dt$ to obtain a complex function. After this, we then defined what it means to integrate a complex valued function over a contour: If C is the contour given by $z = z(t)$ with $a \leq t \leq b$, then

$$\int_C f dz = \int_a^b f(z(t))z'(t)dt.$$

We stated some basic properties of contour integrals (i.e., the behavior under addition of functions, multiplication by a constant, and "breakdown" of contour into the sum of other contours), and then computed a couple of examples. We'll pick up here next time. **Note:** We will be skipping Section 42. I will also be accepting HW until tomorrow at noon, in my mailbox.

Lecture 12: Tuesday, 2/18

Lecture 12 was covered by Jack Jeffries. Jack covered Sections 37-39

Lecture 11: Thursday, 2/13

Today, the class took Midterm 1.

Lecture 10: Tuesday, 2/11

We called the definition of $\sin(z)$ and $\cos(z)$. These functions are entire, and satisfy many of the same familair properties. We spent some time verifying some of these properties (e.g., the fact that $\frac{d}{dz}\sin(z) = \cos(z)$, $\frac{d}{dz}\cos(z) = -\sin(z)$, the identity $\sin^2(z) + \cos^2(z) = 1$, and others). We also pointed out one important difference: Even though the norm of the real-valued functions sine and cosine is always less than or equal to 1, we saw an example in which $|\cos(z)|$ went to infinity as z moved away from the origin on the positive imaginary axis. We then briefly defined the complex hyperbolic functions, and computed their derivates. We concluded the course by defining inverse trig functions, and gave a detailed description of the value of $\cos^{-1}(z)$ (note, your book omits this case, but covers the $\sin^{-1}(z)$ case instead). These inverse trig functions are multivalued functions, and even have multiple-valued inputs! NOTE: There will be no assigned HW for this section; instead, students are asked to study for Midterm 1.

Lecture 9: Thursday, 2/6

We recalled that the principal value $\text{Log}(z)$ is not continuous anywhere on the negative real axis. The remedy this, we restricted the domain further to avoid all points on the negative real axis; the resulting function is called the *principal branch* of $\text{Log}(z)$. Using the CR criteria, we were able to show that the prinicipal branch of $\text{Log}(z)$, is analytic, with derivative $\frac{1}{z}$. We then discussed the basic properties of the prinicipal value of $\text{Log}(z)$, and observed some odd behavior (e.g., the fact that, sometimes, $\text{Log}(z_1 z_2) \neq \text{Log}(z_1 z_2)$). We then discussed what it means (over the real numbers) to take non-integer exponents. Motivated by this, we defined what it means to raise complex numbers to complex exponents. If z and c are complex, then z^c is a multi-valued function, and to remedy this, we defined the principal value and principal branch of z^c, the latter being a differentiable function with derivative equal to z^{c-1}. We concluded by observing some odd behavior that can occur when raising to complex exponents (e.g., the fact that, sometimes, $(z_1 z_2)^c \neq z_1^c z_2^c$, even when considering only principal values)! At the conclusion of the course, we defined the complex trig functions, and will pick up here next time.

Lecture 8: Tuesday, 2/4

We recalled the definition of a harmonic function, and defined what it means for jarmonic real-valued functions $u(x, y)$ and $v(x, y)$ to be *harmonic conjugates*. We then related harmonic conjugates to analytic functions as follows: **Theorem** The function $f = u + iv$ is analytic if and only if u and v are harmonic conjugates. We then showed how the CR equations + integration allow one to, given any harmonic function, construct a harmonic conjugate, and hence an analytic function! After this, we discussed the complex exponential, and considered how to solve equations of the form $z = e^w$. This lead us to define the complex logarithm $\log z$ (which is a multi-valued function), and its principal branch $\operatorname{Log} z$ (which is actually a function). We concluded the class by showing that $\operatorname{Log} z$ is *discontinuous* at every point on the negative real axis. Next time, we'll investigate the differentiability of this function.

Lecture 7: Thursday, 1/30

We began by recalling how to compute $f'(z)$ using the CR equations (provided that the real/imaginary parts of f are continuous). After defining what it means for a function f to be *analytic* at a point z_0, we then used the methods of the previous class to construct various (non) examples of analytic functions. We then defined what it means for a function to be *entire*, and then went over examples of entire functions. After this, we discussed a handful of theoretical results. Indeed, we showed that a function with $f'(z) = 0$ everywhere on an open set must be constant on that open set. Using this, we were able to show the following **Theorem:** If $f = u + iv$ and $\overline{f} = u - iv$ are both analytic, then both f and its conjugate $\overline{f}$ must be constant! Recall, we've seen this type of behavior often in examples. **Example:** $f(z) = z$ is analytic everywhere, but $\overline{f(z)} = \overline{z}$ is *not* analytic anywhere! **Example:** $f(z) = e^z = e^x \cos(x) + i \cdot e^x \sin(x)$ is analytic everywhere, but its conjugate $\overline{f(z)} = \overline{e^z} = e^x \cos(x) - i \cdot e^x \sin(x)$ *not* analytic anywhere! In the final theoretical result of the lecture, we showed that an analytic function $f(z)$ such that the norm $|f(z)|$ is constant everywhere on an open set must be itself constant. We concluded the lecture by defining the notion of a *harmonic* function. We briefly discussed how harmonic functions are related to analytic functions, and will pick up here next time.

Lecture 6: Tuesday, 1/28

We began by recalling the definition of $f'(z)$, and, using the limit definition, computed some examples (including some in which $f'(z)$ did not exist). We then proved that differentiability implied continuity, and went over some of the basic rules for (complex) differentiation: These were exactly the same as the rules over $\mathbb{R}$, and include the sum/product/quotient/chain rules. We then discussed how to fill in the blanks $f'(z) = \boxed{?} + i \cdot \boxed{?}$ in terms of $f = u + iv$, in the case that the derivative actually exists. By computing the limit appearing in $f'(z)$ in two ways (along the purely real and imaginary axes), we were able to derive the *Cauchy-Riemann* equations

$$u_x = v_y \text{ and } u_y = -v_x.$$

Moreover, when $f'(z)$ exists, we have that

$$f'(z) = u_x + iv_x.$$

We also showed that when the CR equations for $f = u + iv$ are satisfied at a point z_0, and if all of the partial derivatives of u and v are continuous at z_0, then we can guarantee that $f'(z_0)$ exists, and is given by $f'(z_0) = u_x(x_0, y_0) + iv(x_0, y_0)$. We then went over a bunch of examples of how to use this. **Note:** This is the most important theorem we've discussed so far, and it is important that students know how to use it. If you are having trouble with this material, please let me know.

Lecture 5: Thursday, 1/23

We continued discussing limits of complex valued functions. In particular, we took real and complex parts of the statement $\lim_{z \to z_0} f(z) = w_0$ to show that this limit converges if and only if the real part of $f(z)$ goes to the real part of w_0 as $z \to z_0$ *and* the imaginary part of $f(z)$ goes to the imaginary part of w_0 as $z \to z_0$. Using this, we observed that the basic properties of limits (i.e., formulas for limits of sums/products/quotients) that we know from $\mathbb{R}$ functions also hold in the $\mathbb{C}$ setting. Using this, we showed that $\lim_{z \to z_0} P(z) = P(z_0)$ whenever $P(z)$ is a polynomial. We then briefly discussed what ∞ means in the complex numbers, and restated limits involving ∞ in terms of limits involving regular (i.e., non-infinite) complex numbers. After computing a bunch of examples, we moved on.and defined the notion of *continous* and *differentiable* complex functions. We'll pick up here next time.

Lecture 4: Tuesday, 1/21

Today, we continued our discussion of complex-valued functions of a complex variable. Any such function $f(z)$ can be written either as an expression involving z, or equivalently, in the forms $f(z) = u(x,y) + iv(x,y) = \tilde{u}(r,\theta) + i\tilde{v}(r,\theta)$, where $z = x + iy = re^{i\theta}$, and the functions $u, v, \tilde{u}, \tilde{v}$ are all real-valued. We went over examples showing the advantages of this form, and showed via the example $f(z) = \text{Arg}\left(\frac{1}{z}\right)$ that it is not always obvious what the largest domain a function may have. We then discussed briefly the notion of a multi-valued function, and gave two examples. After this, we discussed mappings by functions $f(z) = z^2$ and $f(z) = e^z$, spending a lot of time on the latter. We concluded the day by defining what it means for $\lim_{z\to z_0} f(z) = w_0$. In particular, we stressed that the "new" issue here (namely, the fact that when, computing a limit, one may approach z_0 in whatever way they wany) makes computing such limits more subtle than what students may be familiar with over the real numbers.

Lecture 3: Thursday, 1/16

We continued our discussion of n^{th} roots of unity. In particular, we discussed a geometric interpretation of roots of unity: the n^{th} roots of unity are the vertices of a regular n-gon inscribed inside the unit circle, with the first vertex being 1. Motivated by this, we defined the notion of a *principal* n^{th} *root of unity*, which we denoted by ω_n. The key point is that all of the roots of unity can be described by raising ω_n to appropriate powers. After this, we saw how to use n^{th} roots of unity to solve any any equation of the form

$$z^n = z_0 = r_0 e^{i\theta_0}.$$

Indeed, there is always an obvious solution $z_{obvious} = \sqrt[n]{r_0} \cdot e^{i\theta_0/n}$, and every other solution is of the form $z = z_{obvious} \cdot (\text{some } n^{\text{th}} \text{ root of unity})$. Afterwards, we went over a bunch of examples, and expressed roots of a given complex number as vertices of certain polygons inscribed inside of circles of a certain radius. We concluded lecture by introducing the notion of a complex-valued function of a complex variable, and giving some elementary examples. We'll pick up here next time.

Lecture 2: Tuesday, 1/14

We began by recalling some of the topics discussed in the previous lectures. In particular, we derived the identity

$$e^{i\theta} = \cos(\theta) + i\sin(\theta),$$

using nothing more than the basic power series expansions for $e^x, \sin(x)$, and $\cos(x)$. We also recalled the *exponential form* $z = re^{i\theta}$ of a complex number, as well as the notions of (principal) arguments. As an application, we saw that exponential forms allowed us to (often) quickly compute powers of a given complex number. We also used the fact that

$$\cos(n\theta) + i\sin(n\theta) = \left(e^{i\theta}\right)^n = e^{in\theta} = \cos(n\theta) + i\sin(n\theta)$$

to derive the well-known double angle formulas for *sin* and *cos* (recall, we just set $n = 2$ in the above identity). We then proceeded to discuss solutions to the equation $z^n = 1$, and saw that the complex numbers

$$e^{i\cdot\frac{2\pi k}{n}} \text{ with } k = 0, 1, \cdots, n-1$$

give the n-distinct solutions to this equation. We call these numbers n^{th} roots of unity, and will continue our discussion of them next lecture.

Lecture 1: Thursday, 1/9

Lectures 0 and 1 were covered by Jack Jeffries. Jack covered roughly the first 6 sections of the text.

Lecture 0: Tuesday, 1/7

Lectures 0 and 1 were covered by Jack Jeffries. Jack covered roughly the first 6 sections of the text.