## SELECT HW SOLUTIONS: MATH 3160, SPRING 2014

For solutions to problems not appearing here, feel free to consult the student solutions manual.

Section 29, Problem 4: The easiest way to see that  $f(z) = e^{z^2}$  is entire is to recognize it as a composition of two entire functions (i.e.,  $e^z$  and  $z^2$ ). Moreover, the chain rule allows us to compute that  $f'(z) = e^{z^2} \cdot 2z$ . Alternatively, we may write  $f(z) = e^{x^2-y^2}(\cos(2xy) + i\sin(2xy))$ . If u and v denote the real and imaginary parts of f(z), then the student may verify that

$$u_x = v_y = 2e^{x^2 - y^2} (x \cdot \cos(2xy) - y\sin(2xy))$$

and that

$$-u_y = v_x = 2e^{x^2 - y^2} \left( x \cdot \sin(2xy) + y \cdot \cos(2xy) \right)$$

In particular, we see that the partials are all continuous everywhere, and also satisfy the CR equations everywhere. Therefore, we may write

$$f'(z) = u_x + iv_x = 2e^{x^2 - y^2} \cdot (x \cdot \cos(2xy) - y\sin(2xy) - i \cdot (x \cdot \sin(2xy) + y \cdot \cos(2xy)))$$
  
= 2(x + iy) \cdot e^{x^2 - y^2} (\cos(2xy) + i \sin(2xy)) = 2z \cdot e^{z^2}.

- Section 26, Problem 1: Since these are all similar, we'll only do (a): If v is a harmonic conjugate of u = 2x 2xy, we see that  $v_y = u_x = 2 2y$ , and integrating with respect to y shows that  $v = 2y y^2 + \phi(x)$ , where here,  $\phi(x)$  denotes an arbitrary function of x. The remaining CR equation shows that  $v_x = \phi'(x) = -u_y = 2x$ , and integrating with respect to x shows that  $\phi(x) = x^2 + C$ , where C can be any constant. Finally, subbing into our earlier expression for v shows that  $v = 2y y^2 + x^2 + C$ .
- Section 25, Problem 2: We went over a bunch of examples like this in class, so I'll only give the solution to (b): Here, u = 2xy and  $v = x^2 y^2$ . The CR equations state that

$$u_x = 2y = v_y = -2y$$
 and  $u_y = 2x = -v_x = -2x$ ,

which we see are both satisfied only when x = y = 0. Since the partial derivatives are all continuous everywhere, we have that f'(0) exists, and is equal to  $u_x(0,0) + iv_x(0,0) = 0$ . However, the function is *not* analytic at the origin, since in every neighborhood of 0, the only point where f is differentiable is when z = 0.

- Section 25, Problem 3: The fact that the composition of two entire functions is entire follows from the chain rule: If f(z) and g(z) are both entire, and f(g(z)) is defined, then  $\frac{d}{dz}f(g(z)) = f'(g(z))g'(z)$ , and hence is differentiable everywhere (i.e., entire). Similarly, if  $c_1$  and  $c_2$  are two complex constants, then  $\frac{d}{dz}(c_1f + c_2g) = c_1f' + c_2g'$ , by the sum rule for derivatives, and hence any linear combination of two entire functions is entire.
- Section 23, Problem 5: Let  $f(z) = x^3 + i(1-y)^3$ , where here both x and y are real numbers (this will be important later on). In this setting,  $u = x^3$  and  $v = (1-y)^3$ . If u and v are to satisfy the CR equations, then we must have that

$$u_x = 3x^2$$
 equals  $v_y = -3(1-y)^2$  and  $u_y = 0$  equals  $-v_x = -0 = 0$ .

Thus, only the first of these equations is relevant. Moreover, this equation shows that  $x^2 = -(1-y)^2$ , which shows that  $x = \pm i \cdot (1-y)$ . Since both x and y are real, we see the only way this happens is when x = 0 and y = 1. Thus, the only point where the CR equations are satisfied is when x = 0 and y = 1 (i.e., z = i). Since all of the partial derivates

are continuous everywhere, and since we just showed that the CR equations are only when z = i, it follows that we may write  $f'(z) = u_x + iv_x$  only when z = i.

Section 18, Problem 10: Recall that we went over a bunch of similar examples in class. (a) To see that  $\lim_{z\to\infty} \frac{4z^2}{(z-1)^2} = 4$ , we must show that  $\lim_{z\to0} \frac{4(1/z)^2}{((1/z)-1)^2} = 4$ . However,

$$\lim_{z \to 0} \frac{4(1/z)^2}{((1/z) - 1)^2} = \lim_{z \to 0} \frac{4}{z^2((1/z) - 1)^2} = \lim_{z \to 0} \frac{4}{(1-z)^2} = 4.$$

(b) By definition,  $\lim_{z\to 1} \frac{1}{(z-1)^3} = \infty$ , since  $\lim_{z\to 1} \frac{(z-1)^3}{1} = 0$ . (c) To see that  $\lim_{z\to\infty} \frac{z^2+1}{z-1} = \infty$ , we must show that  $\lim_{z\to 0} \frac{(1/z)-1}{(1/z)^2+1} = 0$ . However,

$$\lim_{z \to 0} \frac{(1/z) - 1}{(1/z)^2 + 1} = \lim_{z \to 0} \frac{z - z^2}{1 + z^2} = 0.$$

**Section 14, Problem 7:** Under the function  $f(z) = e^z = e^x e^{iy}$ , the region

$$D = \{x + iy : x \ge 0 \text{ and } 0 \le y \le \pi\}$$

gets mapped to the region

$$R = \{ z = x + iy : y \ge 0 \text{ and } |z| \ge 1 \}$$

Geometrically, this image consists of all points in the upper half-plane not contained strictly inside the unit circle centered at the origin. To see why this is true, note that the vertical slice of the region D in which x = a and  $0 \le y \le \pi$  gets mapped to the top half of the circle of radius  $e^a$  centered at the origin.

Section 13, Problem 4: Give  $z = re^{i\theta}$ , then

$$f(z) = z + \frac{1}{z} = re^{i\theta} + \frac{1}{r} \cdot e^{-i\theta} = (r\cos(\theta) + i \cdot r\sin(\theta)) + \left(\frac{1}{r}\cos(-\theta) + i \cdot \frac{1}{r}\sin(-\theta)\right)$$
$$= (r\cos(\theta) + i \cdot r\sin(\theta)) + \left(\frac{1}{r}\cos(\theta) - i \cdot \frac{1}{r}\sin(\theta)\right)$$
$$= \left(r + \frac{1}{r}\right)\cos(\theta) + i\left(r - \frac{1}{r}\right)\sin(\theta).$$

Section 10, Problem 3(a): We seek solutions to  $z^3 = -1$ , and we follow the process outlined in lecture: Note that, since  $-1 = e^{i\pi}$ , the obvious solution is  $z_{\text{obvious}} = e^{i\pi/3}$ . Moreover, every other solution differs from this one by a cubed root of unity. Since the cubed roots of unity are  $1, e^{i2\pi/3}, e^{i4\pi/3}$  it follows that the three solutions are

$$z_{\text{obvious}} \cdot 1 = e^{i\pi/3} = \cos(\pi/3) + i\sin(\pi/3) = \frac{1}{2} + i \cdot \frac{\sqrt{3}}{2}$$
$$z_{\text{obvious}} \cdot e^{i2\pi/3} = e^{i\pi} = -1$$
$$z_{\text{obvious}} \cdot e^{i\pi/3} = e^{i5\pi/3} = \cos(5\pi/3) + i\sin(5\pi/3) = \frac{1}{2} - i\frac{\sqrt{3}}{2}$$

HW 0, Problem 1 (d): First off, note that

$$\frac{1+2i}{3-4i} = \frac{(1+2i)(3+4i)}{(3-4i)(3+4i)} = \frac{-5+10i}{25} = \frac{-1+2i}{5},$$

and similarly,  $\frac{2-i}{5i} = \frac{-1-2i}{5}$ . Thus, after adding these, we obtain -2. **HW 0, Problem 2:** All but the last two are straight-forward.

(d) Using properties of conjugation,  $\overline{\overline{z}-i} = \overline{\overline{z}} - \overline{i} = z + i = x + i \cdot (y+1)$ . Consequently,  $\operatorname{Im}(\overline{\overline{z}+i}) = \operatorname{Im}(x+i \cdot (y+1)) = y+1,$ 

and hence the set in question consists of all  $z = x + i \cdot y$  with y = 1.

(e) Once again, we'll need to use basic properties of conjugation and complex norm (i.e., modulus). Note that

$$\left|2\overline{z}+i\right| = 2 \cdot \left|\overline{z}+\frac{i}{2}\right| = 2\left|\overline{z}+\frac{i}{2}\right| = 2\left|\overline{z}+\frac{\overline{i}}{2}\right| = 2\left|z-\frac{i}{2}\right|.$$

Consequently, the equation  $|2\overline{z} + i| = 4$  becomes  $|z - \frac{i}{2}| = 2$ , hence the set in question is nothing more than the circle of radius 2 centered at  $\frac{i}{2}$ .

**HW 0, Problem 4 (c):** As  $\sqrt{3} - i = 2e^{-i\pi/3}$ , we have that  $(\sqrt[3]{3} - i)^8 = 2^8 e^{-8\pi/3}$ , a complex number whose principal argument is  $-8\pi/3 + 2\pi = -2\pi/3$ .

**HW 0, Problem 5:** This is just like we did in lecture! By foiling out (twice), we see that  $(12)^3$ 

$$\left(e^{i\theta}\right)^{\circ} = \left(\cos(\theta) + i\sin(\theta)\right)^{3} = \cos^{3}(\theta) - 3\sin^{2}(\theta)\cos(\theta) + i\cdot\left(3\cos^{2}(\theta)\sin(\theta) - \sin^{3}(\theta)\right)$$

On the other hand,

$$\left(e^{i\theta}\right)^3 = e^{i3\theta} = \cos(3\theta) + i\sin(3\theta).$$

Now, since two complex numbers are the same if and only if their real and imaginary parts are the same, comparing the real and imaginary parts of these formulas gives the two identities we seek.