MATH 2250 UPDATE: FALL 2013

The most recent lecture will always appear first; please email me if you notice any errors/typos.

Lecture 27: Thursday, 12/10

Today, the pace of lecture was slowed down to make sure important concepts are clear before the midterm. We reviewed how to obtain two real valued solutions from a complex eigenvector/eigenvalue pair. After doing this, we solved a bunch of examples (including some IVPs) involving complex eigenvalues/eigenvectors. I expect that, after this, students should have a good idea of how to solve these types of problems. If this is not the case, please let me know.

After going through more examples, we next dealt with a limitation of the eigenvalue/eigenvector method to solve ODEs. In particular, we addressed the following situation: If an eigenspace associated to an eigenvector with multiplicity two contains only one linearly independent vector, how do we find another? Below, we summarize the conclusions of our discussion.

Algorithm for finding an extra solution: If λ is an eigenvector (with multiplicity two) associated to the matrix A whose eigenspace does not contain two linearly independent vectors, then we may produce two linearly independent solutions to $A\mathbf{x} = \mathbf{x}'$ in the following way.

- First, let $\mathbf{w}$ denote *any* solution to the equation
$$(A - \lambda\mathbb{1})(A - \lambda\mathbb{1})\mathbf{w} = 0.$$
- Once you've determined such a $\mathbf{w}$, set $\mathbf{v} = (A - \lambda\mathbb{1})\mathbf{w}$. Note that $\mathbf{v}$ will automatically be an eigenvector of A associated to λ.
- Two linearly independent solutions are
$$\mathbf{x}_1 = e^{\lambda t}\mathbf{v} \text{ and } \mathbf{x}_2 = te^{\lambda t}\mathbf{v} + e^{\lambda t}\mathbf{w}.$$

We'll discuss this more next time.

Lecture 26: Thursday, 12/5

In this lecture, we considered **homogeneous** systems of ODEs with **constant coefficients**. That is, we considered systems of the form
$$A\mathbf{x} = \mathbf{x}'$$
for matrices $A = (a_{ij})$ with constant (real-valued) coefficients. Given such a system, we once again try the naive approach and look for solutions of the form $\left(e^{\lambda t}v_1, \cdots, e^{\lambda t}v_n\right) = e^{\lambda t} \cdot \mathbf{v}$, where $\mathbf{v} = (v_1, \cdots, v_n)$ is a vector of (possibly complex) constants. We then directly verified that $e^{\lambda t} \cdot \mathbf{v}$ is a solution of the system $A\mathbf{x} = \mathbf{x}'$ if and only if $\mathbf{v}$ is an eigenvector of A with eigenvalue λ. We referred to a solution of this form as an **eigensolution** of the system. Using this, we outlined the **eigenvalue method** for solving a constant-coefficient system $A\mathbf{x} = \mathbf{x}'$, which we summarize below.

(1) Find the eigenvalues $\lambda_1, \cdots, \lambda_n$ of A. NOTE: This list may contain repeats (due to the fact that some eigenvalues may appear with higher multiplicity).
(2) **If you are lucky**, you will be able to find n linearly independent eigenvectors $\mathbf{v}_1, \cdots, \mathbf{v}_n$. NOTE: This happens if and only if A is diagonalizable.
(3) In this case, setting $\mathbf{x}_1 = e^{\lambda_1 t}\mathbf{v}_1, \cdots, \mathbf{x}_n = e^{\lambda_n t}\mathbf{v}_n$ produces a list of n linearly independent solutions. NOTE: This solutions are guaranteed to be independent once you know the vectors $\mathbf{v}_1, \cdots, \mathbf{v}_n$ are independent.

(4) At this point, it follows from the previous lecture that the general solution $\mathbf{x}$ to $A\mathbf{x} = \mathbf{x}'$ is of the form

$$\mathbf{x} = c_1\mathbf{x}_1 + \cdots c_n\mathbf{x}_n$$

for some constants $c_1, \cdots, c_n$.

We then computed general solutions to two system of homogeneous ODEs with constant coefficients using this method. In these two examples, the associated matrix A was 2×2 and had two real, distinct eigenvectors.

We next talked about that to do if an eigenvalue λ of the associated matrix A was complex. We recalled the concept of **conjugation** of complex numbers and vectors: If $\lambda = a + ib$, then its conjugate is $\overline{\lambda} = a - ib$. Similarly, if we have a vector $\mathbf{v} = \mathbf{u} + i\mathbf{w}$ for some vectors $\mathbf{u}$ and $\mathbf{w}$ with real entries, we set $\overline{\mathbf{v}} = \mathbf{u} - i\mathbf{w}$. We then recalled the following important **FACT:** If A is a square matrix with real entries, then $\mathbf{v}$ is an eigenvalue of A with eigenvector $\mathbf{v}$ if and only if $\overline{\lambda}$ is an eigenvalue of A with eigenvector $\overline{\mathbf{v}}$. Using these ideas, along with Euler's formula $e^{a+ibt} = e^a t\,(\cos(bt) + i\sin(bt))$, we ended the lecture by showing the following.

Theorem Suppose $\lambda = a + ib$ and $\overline{\lambda} = a - ib$ are conjugate eigenvalues of A, with associated eigenvectors $\mathbf{v} = \mathbf{u} + i\mathbf{w}$ and $\overline{\mathbf{v}} = \mathbf{u} - i\mathbf{w}$. Using these conjugate eigenvectors/values, we are able to build the following two solutions:

- $\mathbf{x}_1 = e^{at}\,(\cos(bt) \cdot \mathbf{u} - \sin(bt) \cdot \mathbf{w})$, and
- $\mathbf{x}_2 = e^{at}\,(\cos(bt) \cdot \mathbf{w} + \sin(bt) \cdot \mathbf{u})$.

We then went over an example of how to put this result into action. **NOTE:** Based on feedback, I am aware of the fact that this last point may not yet be totally clear to many students. Consequently, we will review this at the start of the next lecture. In the meantime, please review Section 7.3, and read ahead to Section 7.4. We finally concluded the lecture with the last Super Quiz of the semester.

Lecture 25: Thursday, 12/3

Today, we started Chapter 7. The purpose of this chapter is to study *systems* of ODEs (rather than just a single ODE). A system of ODEs is any number of functions relating the variable t and any number of functions of t and their derivatives.

Systems of ODEs occur often *in nature*. Indeed, we saw in class that a system consisting of two masses connected by a string produce a system of two 2^{nd} order linear ODEs in two unknown functions. We also saw an example of a mixture problem that produced a system of ODEs.

Systems of ODEs that are of special interest are those of *first-order*. To prove this point, we went over lots of examples in which we turned a higher order system into a first order system (at the expense of introducing new variables and equations). We also showed how special systems of ODEs (after some clever manipulation) can be solved by "eliminating" an unknown function to obtain a single ODE (no longer a system). Admittedly, this approach is limited and will not work to solve all systems of ODEs.

We ended the break by stating the folowing **Theorem:** Consider the system

$$\begin{aligned} x_1(t) =& a_{11}(t)x_1 + \cdots + a_{1n}(t)x_n \\ \vdots\vdots& \qquad\qquad\qquad\qquad\qquad\qquad \vdots \\ x_n(t) =& a_{n1}(t)x_1 + \cdots + a_{1n}(t)x_n \end{aligned}$$

If each $a_{ij}(t)$ and $f_i(t)$ is continuous in some interval containing a, then there is a unique solution to the system satisfying the n initial conditions $x_1(a) = b_1, \cdots, x_n(a) = b_n$ for any collection $b_1, \cdots, b_n$ of real numbers.

After the break, we started with the following definition. **Definition:** An $m \times n$ *matrix-valued function* is an $m \times n$ matrix whose entries are functions. That is, $A(t) = (a_{ij}(t))$ for some functions

$a_{ij}(t)$. A *vector-valued function of size* n is a vector $\mathbf{x}(t) = (x_1(t), \cdots, x_n(t))$ of size n whose entries are functions. The main motivation for for these definitions is that they simplify our notation. For example, the derivative of a matrix-valued function $A(t) = (a_{ij}(t))$ is taken component-wise. That is, $A'(t) = (a'_{ij}(t))$, and similarly if $\mathbf{x}(t) = (x_1(t), \cdots, x_n(t))$, then $\mathbf{x}'(t) = (x'_1(t), \cdots, x'_n(t))$.

Under this notation, the system

$$\begin{aligned} x_1(t) =& a_{11}(t)x_1 + \cdots + a_{1n}(t)x_n & +f_1(t) \\ \vdots\vdots & & \vdots \\ x_n(t) =& a_{n1}(t)x_1 + \cdots + a_{1n}(t)x_n & +f_n(t) \end{aligned}$$

becomes the simple equation

$$\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{f}(t),$$

where $\mathbf{x}(t) = (x_1(t), \cdots, x_n(t))$, $A = (a_{ij}(t))$,and $\mathbf{f}(t) = (f_1(t), \cdots, f_n(t))$. An equation of this form with $\mathbf{f}(t) = 0$ is called homogeneous.

Homogeneous systems are important, because of the following **Theorem:** Consider the homogeneous system $A\mathbf{x} = \mathbf{x}'$ for some $n \times n$ matrix of functions A and for some vector valued function $\mathbf{x}$ of size n. Then, the set of all solutions to this system forms a vector space (and in fact is a subspace of the vector space of all vector valued functions of size n). Furthermore, the solution space has dimension n. What this means is that once we find n linearly independent solutions $\mathbf{x}_1(t), \cdots, \mathbf{x}_n(t)$ to the system $A\mathbf{x} = \mathbf{x}'$, then every solution $\mathbf{x}(t)$ is of the form $\mathbf{x}(t) = \alpha_1 \cdot \mathbf{x}_1(t) + \cdots + \alpha_n \cdot \mathbf{x}_n(t)$.

We then talked about how to verify whether n solutions to a system $A\mathbf{x} = \mathbf{x}'$ is linearly independent. **Theorem**: Suppose that $\mathbf{x}_1, \cdots, \mathbf{x}_n$ are n solutions to the system $A\mathbf{x} = \mathbf{x}'$ (where A is some $n \times n$ matrix valued function). Then, there are two options: **Option 1**: If $\mathbf{x}_1, \cdots, \mathbf{x}_n$ are linearly independent, then the *Wronskian* $W(\mathbf{x}_1, \cdots, x_n) = \det(\mathbf{x}_1 \cdots \mathbf{x}_n)$ is non-zero at every point. **Option 2**: If $\mathbf{x}_1, \cdots, \mathbf{x}_n$ are linearly dependent, then the Wronskian $W(\mathbf{x}_1, \cdots, \mathbf{x}_n)$ is always zero!

Finally, we concluded the lecture by giving an example of a specific system of ODEs $A\mathbf{x} = \mathbf{x}'$ given by a 2×2 matrix A. We verified that 2 solutions were linearly independent (by computing the Wronskian) and then used this to solve an initial value problem.

Lecture 24: Tuesday, 11/26

Today, we covered topics from Chapter 6. Our focus during this lecture was on eigenvectors and eigenvalues. Recall that if A is an $n \times n$ matrix, then a real number λ is an eigenvalue of A if $A\mathbf{v} = \lambda \cdot \mathbf{v}$ for some non-zero vector $\mathbf{v}$ in $\mathbb{R}^n$. Note that λ may be zero, though the vector $\mathbf{v}$ may not! In this case, we call $\mathbf{v}$ an eigenvector associated to λ. We computed a couple of examples of eigenvalues and eigenvectors, and then made the following observations:

(1) The eigenvalues of A are the roots of the polynomial

$$\det(A - \lambda \cdot \mathbb{1})$$

in the variable λ. This determinant (which is a polynomial of degree n) is called the characteristic function (or polynomial) of the matrix A.

(2) If λ is an eigenvector of A, then the collection of all non-zero eigenvectors (plus the zero vector) form a subspace, called the *eigenspace* of λ. That is, the set $\{\mathbf{v} : A\mathbf{v} = \lambda\mathbf{v}\}$ (which by definition, is the eigenspace of λ) is a subspace of $\mathbb{R}^n$. In fact, we see that the eigenspace of λ is equal to the kernel of the matrix $A - \lambda\mathbb{1}$.

We continued computing a bunch of examples, up until the break. After the break, we discussed **diagonalization** of matrices. **Definition:** We say that an $n \times n$ matrix A is *diagonalizable* if and only if $A = PDP^{-1}$ for some diagonal matrix D and some invertible matrix P.

We then stated the following Theorem (and proved one implication): **Theorem:** An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors $\mathbf{v}_1, \cdots, \mathbf{v}_n$. Using this theorem, we computed examples of matrices that were diagonalizable, and we even diagonalized them using the following **Fact:** If the $n \times n$ matrix A has n linearly independent eigenvectors $\mathbf{v}_1, \cdots, \mathbf{v}_n$ with associated eigenvalues $\lambda_1, \cdots, \lambda_n$, then if $P = (\mathbf{v}_1 \cdots \mathbf{v}_n)$ and D is the diagonal matrix with diagonal entries $\lambda_1, \cdots, \lambda_n$, then $A = PDP^{-1}$.

We then showed one useful application of diagonal matrices: If $A = PDP^{-1}$, then $A^n = PD^nP^{-1}$ for every $n \geq 1$. The upshot here is that it is much easier to compute D^n, since it is already diagonal. As an application, we computed the 1000^{th} power of the diagonalizable matrix $\begin{pmatrix} 5 & -6 \\ 3 & -4 \end{pmatrix}$ in about 10 seconds!

Lecture 23: Thursday, 11/21

Today, we concluded our discussion of Laplace transforms by discussing how to solve problems involving piecewise continous functions. We recalled the definition of $u_a(t)$, the function that is zero until $t = a$, and then is equal to 1 for all $t \geq a$. Once again, we computed $\mathscr{L}\{u_a(t)\}$ by definition, and then gave examples of what graphs of functions $f(t-a) \cdot u_a(t)$ look like. Afterwards, we proved the following important result: **Theorem:** If $\mathscr{L}\{f(t)\} = F(s)$, then $\mathscr{L}\{u_a(t) \cdot f(t-a)\} = e^{-as}F(s)$. Restatee in terms of inverses, we have that $\mathscr{L}^{-1}\{e^{-as}F(s)\} = u_a(t) \cdot f(t-a)$.

Using this result, we computed lots of new examples of (inverse) Laplace transforms. As an application, we derived the formula for the motion of a pendulum using the principle of conservation of energy, and once we had this formula, we used it to construct an ODE for the motion of a pendulum under forcing.

We discussed the forcing case at length, and using Laplace transforms, were able to give an explicit solution describing the motion of a child moving along a swing being pushed in piecewise fashion.

Note: This is our last lecture on Laplace transforms (though they may appear later on in the course, they are no longer our main focus). We will be returning to pure linear algebra-type material during the next lecture, so be sure to read ahead.

Lecture 22: Tuesday, 11/19

We started by reviewing some of the material covered last time. In particular, we used the formula

$$\mathscr{L}\left\{\int_0^t f(w)dw\right\} = \frac{\mathscr{L}\{f(t)\}}{s},$$

derived at the end of last class, to compute $\mathscr{L}^{-1}\left\{\frac{1}{s(s-3)}\right\}$, and then once again to compute $\mathscr{L}^{-1}\left\{\frac{1}{s^2(s-3)}\right\}$. We then showed (since it was easy) the so-called **translation rule**, which states the following:

$$\text{If } F(s) = \mathscr{L}\{f(t)\}, \text{ then } \mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a).$$

We then used this to compute a bunch of examples, and took our break.

After the break, we proved the following important and useful formula:

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \cdot \frac{d^n}{ds^n}\left(\mathscr{L}\{f(t)\}\right).$$

To illustrate the usefulness of this, we used it to compute $\mathscr{L}\{t\sin(t)\}$, which according to the formula is just $-\frac{d}{ds}\left(\mathscr{L}\{\sin(kt)\}\right)$, which is easy since we actually know $\mathscr{L}\{\sin(kt)\}$! After computing other examples, we combined *everything* we know so far about Laplace transforms to solve the following ODE

$$tx'' + (t-2)x' + x = 0.$$

Note, this is quite impressive, since none of the techniques from earlier chapters can be used to solve this ODE! **Note:** The last problem on the lab is very similiar to this example; if you weren't in lecture today, it might be a good idea to get notes from a friend.

Lecture 21: Thursday, 11/14

Due to Midterm 2, we didn't spend a lot of time lecturing.

We began by recalling the formula relating the Laplace transform of the higher derivatives of a function with the Laplace transform of the function. We then did more examples where we used the method of "Transforming, solving, and then un-transforming" to solve some ODEs.

We also pointed out the following important fact: The formula was have relating $\mathscr{L}\left\{f^{(n)}\right\}$ with $\mathscr{L}\{f\}$ is also useful when computing the Laplace transforms of functions. For example, we used to to compute the transform of $t \cdot \sin(kt)$ and $t \cdot e^{at}$. The point is that anytime a function shows up in one of its own derivatives, then we can transform everything in sight to derive some interesting formulas.

We concluded by showing how to relate the transform of an integral with the transform of the function being integrated. In particular, we showed that

$$\mathscr{L}\left\{\int_0^t f(w)dw\right\} = \frac{\mathscr{L}\{f(t)\}}{s}.$$

After this, we took Midterm 2.

Lecture 20: Tuesday, 11/12

We continued with §10.1, and started by reviewing the definition of $\mathscr{L}\{f(t)\}$, and computing $\mathscr{L}\{t^a\}, \mathscr{L}\{\sin(kt)\}$, and $\mathscr{L}\{\cos(kt)\}$ directly using integration by parts. We continued by computing directly $\mathscr{L}\{u_a(t)\}$, where $u_a(t)$ is the function that is zero for values of $t < a$ and 1 for values of $t \geq a$. As an application of this, we computed the Laplace transform of the *infinite staircase* (see Problem 39 from §10.1). We then defined the notion of inverse Laplace transforms (i.e., $\mathscr{L}^{-1}$), and computed a few examples.

After the break, we began by asking how Laplace transforms behave with respect to differentiation. Earlier, we had seen that if $f(t) = t^n$, then $\mathscr{L}\{f(t)'\} = s\mathscr{L}\{f(t)\}$, and if $f(t) = \cos kt$, then $\mathscr{L}\{f'(t)\} = s\mathscr{L}\{f(t)\} - 1$. Motivated by this, showed the following: **Theorem**: Quite generally, $\mathscr{L}\{f'(t)\} = s\mathscr{L}\{f(t)\} - f(0)$. By applying this again, we saw that

$$\begin{aligned}\mathscr{L}\left\{f''(t)\right\} &= s\mathscr{L}\left\{f'(t)\right\} - f'(0) = s\left(s\mathscr{L}\{f(t)\} - f(0)\right) - f'(0) \\ &= s^2\mathscr{L}\{f(t)\} - sf(0) - f'(0).\end{aligned}$$

In fact, we saw that **Corollary**: If f and its derivates satisfy the conditions of the previous theorem, then

$$\mathscr{L}\left\{f^{(n)}(t)\right\} = s^n\mathscr{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f^{(1)}(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0).$$

We then showed how this result would help us to solve ODEs! Here is our method.

Step 1: Start with an ODE in the function $x(t)$.

Step 2: Apply $\mathscr{L}$ to the system. That is, transform your ODE to obtain an equation in the new variable s.

Step 3: Using your new system (in the variable s), solve the for $\mathscr{L}\{x(t)\}$. This step may require the use of partial fraction decompositions.

Step 4: To find $x(t)$, apply the inverse transform! (this last part is where the partial fractions come in)

We went over some examples of this procedure, and we'll see more examples of this during the next lecture.

Lecture 19: Thursday, 11/7

We started lecture by continuing our discussion of mass-spring-dashpot systems, and spent some time talking about (undamped) resonance, and went through an example that is similar to the first problem on the lab. After this, we took the Quiz, and after the Quiz, we discussed the situation of practical resonance, which occurs even in the presence of damping. We saw a video showing one of the most famous examples of this behavior; click here to see a video of the collapse of the Tacoma Narrows bridge, and click here for a discussion of this event.

We began §10.1, and introduced the definition of a **Laplace transform** $\mathscr{L}\{f(t)\}$ of a function $f(t)$ (which is defined for all $t \geq 0$ as follows: $\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)dt$. Note, since we are integrating with respect to t, there will be no t's appearing in the expression of $\mathscr{L}\{f(t)\}$. That is, $\mathscr{L}\{f(t)\}$ is a function of the variable s, and *not* of t. We computed many examples by hand, and these are summarized in a table on page 581. In making these calculations, we had to rely on the following important fact (whose proof is basically by definition): **Theorem**: If a and b are constants, then $\mathscr{L}\{af(t)+bg(t)\} = a\mathscr{L}\{f(t)\} + b\mathscr{L}\{g(t)\}$. Combining this with Euler's identity

$$e^{iat} = \cos(at) + i \cdot \sin(at),$$

we were able to use our knowledge that $\mathscr{L}\left\{e^{at}\right\} = \frac{1}{s-a}$ for $s > a$ to compute that $\mathscr{L}\{\cos(at)\} = \frac{s}{s^2+a^2}$. The trick behind this was to write $\cos(at)$ using only e^{iat} and e^{-iat}. For fun, try a similar trick to show that $\mathscr{L}\{\sin(at)\} = \frac{a}{s^2+a^2}$.

Lecture 18: Tuesday, 11/5

Today, we considered n^{th}-order inhomogeneous linear ODEs

$$a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_2 y^{(2)} + a_1 y^{(1)} + a_0 y = f(x),$$

where $a_1, \cdots, a_n$ are constants, and $f(x)$ is allowed to be any function of x. We recalled that once one has a *particular solution* y_P of this equation, then *any* solution is of the form $y_P + y_H$, where the *complimentary solution* y_H is the general solution to the homogeneous equation

$$a_n y^{(n)} + \cdots + a_1 y^{(1)} + a_0 y = 0,$$

which we learned how to solve in the last lecture. We then commented that it is difficult to find a particular solution y_P, though there is a method that will allow one to made an educated guess whenever the function $f(x)$ is a linear combination of products of polynomials, exponential functions e^{ax} and the trig functions $\sin(bx)$ and $\cos(bx)$. We outline this method, called the *method of undetermined coefficients*, below.

Step 1: Verify that $f(x)$ is a linear combination of products of $e^{ax}, \sin bx, \cos bx$ and $p(x)$, where $p(x)$ is a polynomial.

Step 2: Write $f(x)$ as $f_1(x) + \cdots + f_d(x)$, where each $f_i(x)$ is of the form $e^{ax}\cos(bx)p(x)$ or $e^{ax}\sin(bx)p(x)$. For each $f_i(x)$, make a list of all of the functions needed to write $f_i(x)$ and all of its derivatives. This list should be finite.

Step 3: Compute y_H, a solution to the associated homogeneous equation.

Step 4: Check to see if any of the functions appearing in y_H appear in any of the lists you found in in Step 2. If there is overlap, you must multiply the **entire** offending list by the *smallest* power of x so that there is no overlap. **Note:** You only need to modifying the offending (i.e., overlapping) list, and can leave the other lists alone.

Step 5: Once you have modified your lists in this way (so there is no overlap), then guess that y_P (a particular solution) is a linear combination of the functions appearing in the modified lists.

Step 6: Substitute back into the original differential equation to solve for the coefficients in y_P. **Note:** This last step can often be computationally demanding.

We spent the rest of the class going through examples of how to run this guessing process.

Lecture 17: Thursday, 10/31

The first half of the class was spent reviewing for and taking the super quiz. We then spent some time considering complex valued functions. In particular, we though a lot about e^{ix}, where i is the imaginary number (i.e., $i^2 = -1$). Using power series expansions and the fact that all of the powers of i can be expressed using only $i, -1, -i, 1$, we deduced Euler's beautiful formula

$$e^{ix} = \cos(x) + i \cdot \sin(x).$$

As an aside, we noted that setting $x = \pi$ in this equation produces the identity $e^{i\pi} = -1$, which when restated shows that

$$e^{i\pi} + 1 = 0.$$

Note that every important constant (i.e., $e, i, \pi, 1, 0$) that you know appears in this subtle and beautiful identity! Using these identies, along with the fact that 1) the sum of two solutions to an n^{th} order linear ODE is a solutions and 2) any multiple of a solution is a solution, we were able to fill in the last step in the following result: **Algorithm** Consider an n^{th} order homogeneous ODE

$$a_n y^{(n)} + \cdots + a_1 y^{(1)} + a_0 y = 0$$

with associated characteristic polynomial

$$a_n r^n + \cdots + a_1 y + a_0$$

(1) If r is a real root (of multiplicity one), the e^{rx} is a solution.
(2) More generally, if r is a real root of multiplicity k, then $e^{rx}, xe^{rx}, \cdots, x^{k-1}e^{rx}$ are k solutions.
(3) If $r = a \pm ib$ is a complex solution (of multiplicity 1), then $e^{ax} \cdot \cos(bx)$ and $e^{ax}\sin(bx)$ are solutions.

We didn't discuss the case of multiple complex roots, since it won't come up in applications. Using these results, we are now able to solve homogeneous linear equations with constant coefficients. Next time, we'll see what to do if when faced with *non-homogeneous* linear equations.

Lecture 16: Tuesday, 10/29

Today, we continued our discussion of n^{th}-order linear ODEs; in particular, we focused largely on homogeneous n^{th}-order linear ODEs, i.e., equations of the form

$$(\dagger) \qquad a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y^{(1)} + a_0(x)y = 0$$

where $y^{(k)}$ denotes the k^{th} derivative of y with respect to x. We were amazed by the following important fact. **Theorem:** The set of all solutions to ($\dagger$) is a vector space of dimension n. We spent a lot of time parsing this statement, and pointed out the following consequence: **IF** we can find n linearly independent solutions $y_1, \cdots, y_n$ to ($\dagger$), then any other solution to ($\dagger$) must be of the form

$$c_1 \cdot y_1 + \cdots + c_n \cdot y_n$$

for some constants $c_1, \cdots c_n$. This raised two questions: **Q1:** How do we find n solutions? **Q2:** How can we check whether a given collection of functions is linearly independent? We ended up addressing **Q1** after the break, and stated the following way to address **Q2**.

Given any functions $y_1, \cdots, y_n$ defined on an interval I, let

$$W = W(y_1, \cdots, y_n) = \det \begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1^{(1)} & y_2^{(1)} & \cdots & y_n^{(1)} \\ & \cdots & & \\ y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)} \end{bmatrix}$$

denote the **Wronskian** of the functions $y_1, \cdots, y_n$. We then saw the following possibilities.

(1) If $y_1, \cdots, y_n$ are *not* linearly independent, then the Wronskian is zero for all values of x.
(2) If the Wronskian is not zero for some x value in I, then $y_1, \cdots, y_n$ are linearly independent functions.

With this tool, we were able to check whether or not a certain collection of (given) solutions to a homogeneous linear ODE were linearly independent. We then went over a couple of examples, and used this principle to solve a couple of IVPs. Before the break, we stressed the following **Theorem**: Given any *particular* solution y_p to the n^{th}-order linear ODE

$$(*) \qquad a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y^{(1)} + a_0(x)y = b(x)$$

we can find *all* solutions to $(*)$. In other words, any solutions to $(*)$ is of the form

$$y = y_p + y_h,$$

where y_h is a solution to the associated homogeneous system

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y^{(1)} + a_0(x)y = 0.$$

Note that we spent the whole class studying what the general form of such a y_h is.

After the break, we then started learning how to solve n^{th}-order linear ODEs with **constant coefficients**. We once again focused on the case where 0 appears on the right hand side. Seeking solutions of the form $y = e^{rx}$ for some r led us to consider the so-called **characteristic equation** of our ODE, and we saw how roots of this equation gave us solutions to our original ODE. We discussed the case of repeated real roots (and non-repeated real roots), but did not give a full explanation for what happens when one gets *complex* roots. We'll pick up here next time.

Lecture 15: Thursday, 10/24

We continue our discussion of vector spaces and related topics. We recalled the definition of **linear combinations** and **linear independence** of vectors in a vector space. We also recalled what it means for a collection of vectors to be a **basis** for a vector space, and also recalled the notion of **dimension** of a vector space. We gave many examples of different bases for the same vector space, and really emphasized the following point: Roughly stated, a collection of vectors forms a basis if and only if using them, one can express every element of your vector space in a unique way! In other words, a basis can be thought of as giving you a coordinate system. We stressed that there is no natural choice of basis (or coordinate system) for a given vector space (even $\mathbb{R}^n$); indeed, it is possible that if there are aliens somewhere, they would probably choose a different basis than what you and I might choose (e.g., the standard basis for $\mathbb{R}^n$ given by the vectors $\mathbf{e}_1, \cdots, \mathbf{e}_n$).

When giving examples, we computed the basis for the kernel of a matrix A, and showed how this technique will work in general. **NOTE**: The book prefers to use non-standard terminology to describe the kernel of a matrix A. In their terminology, the kernel of a matrix A, which we denote by $\ker A$, is called the **solution space** to the homogeneous system of equations $A\mathbf{x} = \mathbf{0}$. As pointed out by Jason, the following holds: **Theorem:** The dimension of $\ker A$ is always equal to the number of free variables given by the reduced echelon form of A. That being said, though this tells you the dimension, you still need to work through the whole process to find an actual basis!

We concluded our discussion of bases/vector spaces by touching on this week's lab. In particular, if V denotes the set of all polynomials of degree less than equal to 3, we brielfy showed (you'll need to add more detail in your lab) that V is a vector space. We also gave an argument showing that $\{1, x, x^2, x^3\}$ is a basis for V by explicitly checking that every polynomial in V is a linear combination of these four guys, and by also checking (*explicitly*, using the definition) that these guys were also linearly indepdent. The real content of the lab is to show that *another* set of 4 polynomials (namely, the set $\{1, x, 3x^2 - 1, 5x^3 - 3x\}$) is also a basis.

Students who remained confused about these topics (i.e., bases, linear combinations, linear independent, etc) should stop by and see me, or talk to me after class!

After the quiz, we discussed higher order linear ODEs. Though it may not seem like it, this topic is closely related to the last. We defined n^{th}-order linear ODEs and their associated initial value problems (IVPs), and talked about when these types of IVPs have a (unique) solution.

Lecture 14: Tuesday, 10/22

We began by recalling the definition of a **vector space**. Roughly speaking, a vector space V is a collection of objects than can be added together (this sum must stay in V) and multiplied by a real number (this product must also stay in V); additionally, we require that there be an element $\mathbb{0}$ in V with the property that $\mathbf{u} + \mathbb{0} = \mathbb{0} + \mathbf{u} = \mathbf{u}$ for every $\mathbf{u}$ in V, and we ask that these operations are nice (we omit the precise lists here, and instead refer students to their notes, or to the definition on page 240). We went over *many* examples of vector spaces, the most important being the vector space $\mathbb{R}^n$. We also gave some non-examples of a vector space. The second most important example of a vector space is the kernel of a matrix. We recall the definition here: **Definition:** If A is an $m \times n$ matrix, then the kernel of A, denoted $\ker A$, is the subset of $\mathbb{R}^n$ consisting of all vectors sent to zero (i.e., killed by) the matrix A. In math notation, we have that

$$\ker A = \{\mathbf{x} \text{ in } \mathbb{R}^n \text{ such that } A\mathbf{x} = \mathbb{0}\ .\}$$

We showed in great detail that $\ker A$ is always a subspace of $\mathbb{R}^n$ (i.e., is a vector space with the same operations of $\mathbb{R}^n$ sitting inside of $\mathbb{R}^n$.) We then started a running example in which we explicitly described the kernel of a given matrix A as the set of all linear combinations of two vectors.

After the break, we defined what it means for vectors to be **linearly independent**, and we recall the definition here. **Definition**: Vectors $\mathbf{v}_1, \cdots, \mathbf{v}_k$ are called linearly independent if the only way to write $\mathbb{0}$ as a linear combination of these vectors is to have the coefficients in this combo be zero themselves (i.e., the so-called *trivial* way). More precisely, $\mathbf{v}_1, \cdots, \mathbf{v}_k$ are linearly independent if

$$c_1 \cdot \mathbf{v}_1 + \cdots + c_k \cdot \mathbf{v}_k = \mathbb{0}$$

has only the trivial solution $c_1 = \cdots = c_k = 0$. We then pointed out the following equivalent characterization: If $\mathbf{v}_1, \cdots, \mathbf{v}_k$ are linearly independent, and

$$a_1 \cdot \mathbf{v}_1 + \cdots + a_k \cdot \mathbf{v}_k = \mathbf{w} = b_1 \cdot \mathbf{v}_1 + \cdots + b_k \cdot \mathbf{v}_k$$

are two ways of writing a vector $\mathbf{w}$ as a linear combination of $\mathbf{v}_1, \cdots, \mathbf{v}_k$, then these must be the same combination, i.e. $a_1 = b_1, \cdots a_k = b_k$. We pointed out by showing (by hand) that the standard basis vectors $\mathbf{e}_1, \cdots, \mathbf{e}_n$ in $\mathbb{R}^n$ are linearly independent, and gave other examples as well (recall that $\mathbf{e}_i$ is the vector with 1 in the i^{th} spot and zeroes elsewhere). We then stated the following **important criteria**

How to check when vectors in $\mathbb{R}^n$ are linearly independent: Let $\mathbf{v}_1, \cdots, \mathbf{v}_k$ be k vectors in $\mathbb{R}^n$, and let $A = \begin{bmatrix}\mathbf{v}_1 & \cdots & \mathbf{v}_k\end{bmatrix}$, i.e., A the matrix whose i^{th} column is $\mathbf{v}_i$. Note that A will be an $k \times n$ matrix.

- If $k > n$, then $\mathbf{v}_1, \cdots, \mathbf{v}_k$ are *never* linearly independent.
- If $k = n$, then $\mathbf{v}_1, \cdots, \mathbf{v}_k$ are linearly independent if and only if the determinant of A is non-zero (note, since $k = n$, A will actually be a square matrix in this case)
- If $k < n$, then $\mathbf{v}_1, \cdots, \mathbf{v}_k$ are linearly indepedent if and only if **any** matrix obtained by taking k rows of A has non-zero determinant.

We stress the term *any* in the last point, because what the theorem is saying is that as long as one of these $k \times k$ matrices has non-zero determinant, we are OK (i.e., the vectors are linearly independent). In particular, if one such determinant is zero, you have to keep checking to see if you can find a single non-zero one. If you can, the vectors are linearly independent; if you can't, they are *not* linearly independent. To emphasize this point, we gave an example where some of these determinants was zero, but others were not.

Finally, we defined the terms basis and **dimension** of a vector space V, we recall the definition here: **Definition:** A collection of vectors $\mathbf{v}_1, \cdots, \mathbf{v}_n$ is called a basis for the vector space V whenever the following two conditions are satisfied.

- Every element of V can be written as a linear combination of the vectors $\mathbf{v}_1, \cdots, \mathbf{v}_n$.
- The vectors $\mathbf{v}_1, \cdots, \mathbf{v}_n$ are linearly independent.

Another alternative definition (based on earlier observations) is the following: A collection of vectors $\mathbf{v}_1, \cdots, \mathbf{v}_n$ is called a basis for V if given any element of w of V, there are unique real numbers $a_1, \cdots, a_n$ such that

$$w = a_1 \cdot \mathbf{v}_1 + \cdots + a_n \cdot \mathbf{v}_n.$$

We made a big deal of pointing out that $\mathbf{e}_1, \cdots, \mathbf{e}_n$ form a basis for $\mathbb{R}^n$ (called the standard basis for $\mathbb{R}^n$), and also found a basis our running example, a vector space given by the kernel of a matrix A. We discussed at length how a given vector space can have more than one basis, and how there is no natural choice for a basis. On the other hand, each basis must have the same number of elements in it, and this number of elements is called the **dimension** of a vector space.

More precisely, we have the following definition: **Definition:** If $\mathbf{v}_1, \cdots, \mathbf{v}_n$ is a basis for a vector space V, then we say that V is n-dimensional (or has dimension n), and we write $\dim V = n$. We talked more about dimension and bases for vector spaces, and will talk next time about how to find a basis for more exotic vector spaces (i.e., vector spaces of solutions to linear ODEs).

Lecture 13: Thursday, 10/10

Lecture today was covered by Priscilla; HW has been added to the course website. Lecture was started by defining $\mathbb{R}^n$ as the collection of all column vectors (of size n); we also recalled how to add vectors, an dhow to multiply a vector by a real number. We then defined the concept of a linear combination of a collection of vectors, and explicitly saw how to write a given vector as a linear combination of other vectors (and we also saw how to show establish when doing so is impossible). Many of our examples focused on vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$. We then took the quiz, and then ended the course by stating the defining properties of a **vector space.** We'll pick up here next time. **Note:** In order not to fall behind during the break, students are urged to do the assigned reading before our next meeting.

Lecture 12: Tuesday, 10/8

We began by recalling the basic facts of determinants. Note that many of the properties of determinants that we've been discussing are far from obvious. Among the properties of matrices that we discussed are the following:

(1) When computing determinants, you may expand down any row and column that you like (keeping track of the signs of the terms appearing).
(2) When computing, it is better to simplify your matrices. In this case, it is crucial to remember the way in each of the elementary row ops affects the value of the determinant you are trying to compute. In what follows, suppose that B is obtained from A by a single elementary row operation, so that $A \sim B$, where "$\sim$" represents a single row op.
    - If "$\sim$" is swapping rows, then $\det B = -\det A$.
    - If "$\sim$" is multiplying a row by a non-zero constant α, then $\det B = \alpha \cdot \det A$.
    - If "$\sim$" is adding a non-zero multiple of one row to another row, then $\det B = \det A$.

In addition to these, there are other importannt properties of matrices(e.g., the fact that the determinant of a matrix is zero whenever the matrix contains two identical rows, etc.). We then discussed the **transpose** of a matrix, and saw that the determinant of a matrix A equals the determinant of its transpose A^T. Since row ops on A^T correspond to column ops on A, we see that the above list holds whenever "row" is replaced by "column." After discussing further properties of matrices (e.g., the **amazing** fact that $|AB| = |A| \cdot |B|$) and some consequences (e.g., that $|A^{-1}| = |A|^{-1}$), we concluded this discussion by giving a general formula for the inverse of a matrix in terms of its minors that generalizes the well-known formula for 2×2 matrices:

$$A^{-1} = \frac{1}{\det A} \cdot \left[(-1)^{i+j} M_{i,j}\right],$$

where $M_{i,j}$ denotes the i, j^{th} minor of the matrix A.However, this formula is computationally demanding, and so students may prefer to compute inverses of matrices using the algorith covered in Lecture 10. Of course, you'll only notice the difference when working with large matrices, and if you find you like this formula, feel free to use it.

NOTE: Priscilla Elizondo (your TA) will be covering Lecture 13 on Thursday. We will be covering more abstract material, and so it is very important that students read §4.1 and 4.2 *before* coming to lecture.

Lecture 11: Thursday, 10/3

We listed 6 different characterizations of what it meant for a square matrix to be invertible, which motivated our discussion of **determinants**. We recalled the definition of a determinant of a 2×2 matrix, and then gave an *inductive* definition for the determinant of a general $n \times n$ matrix. Note, by *inductive*, we mean the following: In order to compute the determinant of an $n \times n$ matrix, you need to know how to compute the determinant of all $2 \times 2, 3 \times 3, \cdots, (n-1) \times (n-1)$ matrices. As an example, we used the definition to show how to compute the determinant of a 3×3 matrix in terms of determinants of 2×2 matrices. As noted by some students, this process can get quite tedious and involve many computations, and we ended by mentioning that there are ways to "prepare" a matrix so that the process of computing its determinant is easier. We didn't get very far, and the rest of the day was spent taking **Midterm 1**. Note: solutions have been posted, and students are urged to look the solutions over **before** class on Tuesday.

Lecture 10: Tuesday, 10/1

We began the lecture by recalling a lot of the odd behavior that one sees when working with matrices that one does not see when working with the more familiar number systems (i.e., the $\mathbb{R}$-world). Among the examples we saw was the failure of the cancellation property, as well as an example where $AB = 0$ with neither A nor B being equal to zero. We also saw a couple of other strange examples (some of these will come up again in the HW). In the midst of pointing out this strange behavior, we made the following analogy:

- $\mathbb{0}$, the zero matrix, is to matrix addition as 0 is to addition of real numbers.
- $\mathbb{1}$, the identity matrix, is to matrix multiplication as 1 is to multiplication of real numbers.

Motivated by the situation over the real numbers, we made the following **Definition**: A square matrix A is called **invertible** if there exists a matrix B such that $AB = BA = \mathbb{1}$. In this case, we call B the **inverse** of A, and we often write $B = A^{-1}$. After going through some examples, we saw that certain matrices are not invertible, and so we were led to consider when a matrix is invertible, and in the case that an inverse exists, we asked whether the inverse was unique. To this end, we proved the following **Theorem:** If A is invertible, then its inverse is unique. Moreover, if A and B are invertible, then AB is also invertible, and $(AB)^{-1} = B^{-1}A^{-1}$. The order here really matters!

Note that all of the "proofs" we gave were straightforward, and not very complicated (fortunately). We then started to search for a criteria for checking whether a given matrix to be invertible. In order to justify our criteria, we defined an **elementary** matrix to be a matrix obtained from $\mathbb{1}$ via one elementary row operation, and we noted the following crucial point: If B is obtained from A via a single elementary row op, and E is the elementary matrix corresponding to this op, then $B = EA$; in other words, elementary row operations on A can be achieved through matrix multiplication by elementary matrices! Having observed this useful fact, we concluded with the following **Theorem:** A is invertible if and only if the reduced form of A is equal to $\mathbb{1}$. Moreover, to actually compute A^{-1} in this case, all we have to do is track the elementary row operations needed to turn A into $\mathbb{1}$. In particular, we deduced the following **Algorithm**: If A is invertible, then to compute A^{-1}, form the matrix $[A \mid \mathbb{1}]$ and perform elementary row ops until it is in the form $[\mathbb{1} \mid B]$. In this case, B will be the inverse of A. Finally, we concluded with pointing out the usefulness of inverses; e.g., if $n \times n$ coefficient matrix A associated to a linear system is invertible, then system $A\mathbf{x} = \mathbf{b}$, where $\mathbf{x} = (x_1, \cdots, x_n)$ is a column vector of variables and $\mathbf{b}$ is a column vector of real numbers, has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$. We referred to this as "solving for $\mathbf{x}$" in class.

Lecture 9: Thursday, 9/26

We started by finishing up with §3.3. We begin by recalling the definitions of echelon form and reduced echelon form. We then used reduced echelon form to study **homogeneous** systems of linear equations (i.e., systems of linear equations with zeroes on the right hand side). In particular, we noted the following: **Theorem:** A homogeneous system with more variables than equations always has infinitely many solutions. Furthermore, a homogeneous system with as many variables as equations has a unique solution (i.e., the **trivial solution**, consisting of setting all the variables equal to zero) if and only if the reduced echelon form of the associated coefficient matrix is the **identity matrix**. We then took the quiz.

After taking the quiz, we started discussing **matrix algebra**. In particular, we discussed how to add matrices (of the same side), and how to multiply matrices by real numbers (basically, everything is done component-wise). We also defined what it means to multiply a row times a column, and used this to show how to multiply an $m \times p$ matrix by an $p \times n$ matrix to get an $m \times n$ matrix. Knowing how to multiply matrices, we then started listing some of the properties of these operations. Very quickly, we started to notice that a lot of "weird" things happen with matrices that don't happen with real numbers.

Lecture 8: Tuesday, 9/24

We started off by recalling how we used so called "permissable actions" to manipulate a system of equations. After going over some examples, we got lazy and started omitting the variables and equality sign from these systems, thus ending up with the so called **coefficient matrix** and **augmented matrix** associated to a system of linear equations. We then noted that the permissable actions on equations correspond to the following actions on matrices (often called **elementary row operations**):

- Swap any two rows.
- Multiply a row by any non-zero constant.
- Add a multiple of a row to another row.

After this, we went through a bunch of examples; in each example, we illustrated how to run **Gaussian elimination** to put a given matrix in so-called **row echelon form**. This method is extremely important in this course, and students are advised to do the assigned HW (and extra problems, if they want to sharpen their skills). As we noted, a matrix may have *many* different row echelon forms (in fact, any matrix has infinitely many such forms). To remedy this, we defined *reduced echelon form* (or *reduced form*, for short). The process of taking a matrix and using elementary row and column operations to get it into reduced form is known as **Guass-Jordan elimination**. We formally described this process, and ended up going over many examples. One important benefit of considering *reduced* echelon form (as opposed to just echelon form) is the following uniqueness result: **Theorem**: Given any matrix A, A has only one (i.e., a unique!)

reduced echelon form. This is in analogy with fractions: there is many different ways to write a fraction, but only one way in lowest terms (i.e., in reduced form).

We saw that by considering the reduced echelon form of a matrix, we can often give a pretty good description of its solutions. For example, we saw the following: When an system of equations has more variables than equations, the reduced echelon form will consist of a matrix whose first rows have ones as their leading terms, and then whose remaining rows don't. The variables corresponding to rows with a leading term 1 will correspond to a so-called **leading variable**, while the remaining variables are **free**. We gave a very concrete example of this at the end of class, and will pick up next time considering this situation. The class is *urged* to do the assigned reading before the next class, and are encouraged to visit their TAs or the math lab if they have any questions between now and Thursday.

Lecture 7: Thursday, 9/19

Today, we begun the class by covering an old HW problem (§2.1, #9), which the grader mentioned that many students had trouble with or skipped. We then proctored the first **Super Quiz.** After this, we started discussing systems of linear equations. We did a bunch of examples, and noticed that there is a bad method to solve linear equations (i.e., immediately solving for a variable and then back substituting), and a better method involving manipulating systems of equations via the following *permissable actions*:

- Multiply an equation by a (non-zero !) constant.
- Switch the order of the equations.
- Add a multiple of one equation to another equation.

We emphasized that, though these processes change the system of equations, they don't actually change the solutions; the key point here is that each of these operations can be undone, and therefore solutions in one system and solutions in a system modified in these ways must remain the same. We saw graphically what is happening as well. In each case, we were looking at the intersection point of two lines, and we saw that, even though these permissable actions changed the lines we were looking at, the intersection point remained the same. After discussing this at length, we then saw a couple of examples of systems of equations (2 equations, 2 variables) that have no solution. We'll pick up here next time. **Note:** Students are urged to do the assigned reading before class, so that they are familiar and not surprised by the material and pace of the next few sections.

Lecture 6: Tuesday, 9/17

Today, we covered sections $2.4, 2.5$, and 2.6. Each section is dedicated to a different numerical method for solving ODEs (which, in practice, are quite important, as we are very limited in the types of differential equations we can solve explicitly). The main idea behind all three are the same. If you have an IVP

$$\frac{dy}{dx} = F(x,y), y(x_0) = y_0$$

then in order to estimate the value of y when x is increased from x_0 to $x_0 + h$ (here, h is the step size, and is thought of as "small"), you apply the FTIC (Fundamental Theorem of Integral Calculus) to see that

$$y(x_0 + h) = y_0 + \int_{x_0}^{x_0+h} F dx.$$

The tricky thing here is that the F in the integral sign involves both x and y, and so estimating this integral will be slightly more involved than one might expect. The methods we used to estimate this integral (left-hand rectangle rule, trapezoid rule, Simpson's rule) lead to different numerical approximations (e.g., Euler's method, hEuler's method, and the Runge-Kutta method). For the convenience of the student, we recall these below:

(1) Euler's method
  - $k = F(x_0, y_0)$
  - Conclude: $y(x_0 + h) \approx y_0 + hk$.

(2) hEuler's method (aka Improved Euler's Method)
  - $k_1 = F(x_0, y_0)$
  - $k_2 = F(x_0 + h, y + hk_1)$ (the Euler estimate)
  - $k = \frac{1}{2} \cdot (k_1 + k_2)$ (averaged slope)
  - Conclude: $y(x_0 + h) \approx y_0 + hk$

(3) RK method
  - $k_1 = F(x_0, y_0)$ (slope at initial condition)
  - $k_2 = F\left(x_0 + \frac{1}{2} \cdot h, y_0 + \frac{1}{2} \cdot h \cdot k_1\right)$ (the Euler estimate for slope at midpoint)
  - $k_3 = F\left(x_0 + \frac{1}{2} \cdot h, y_0 + \frac{1}{2} \cdot h \cdot k_2\right)$ (second guess for slope at midpoint)
  - $k_4 = F\left(x_0 + h, y_0 + h \cdot k_3\right)$ (estimate for slope at endpoint using k_3)
  - $k := \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$.
  - Conclude: $y \approx y_0 + hk$.

We then computed a bunch of examples, and saw how repeated applications of these steps allows us to divide up an interval to get better estimates. We also pointed out, via example, that though the RK method seems to take more steps, it is actually much less work to achieve high accuracy with RK than it is with Euler.

We concluded the class by starting a discussion of linear equations. Please make sure to read about this before next time.

Lecture 5: Thursday, 9/12

The quiz today had a major typo (due to my incorrect transcribing!). While this didn't cause issues math-wise, it made the problem much simpler than I intended! The second part should have read: Find a solution to the IVP

$$\frac{dx}{dt} = x(x-1) \text{ such that } x(0) = 2.$$

Moreover, determine where the blowup happens (if it happens at all).

Assignment: Add this problem as Problem 0 on the HW assigned for Lecture 5.

After taking the quiz, we started by reviewing our earlier velocity-position models for horizontal motion. First, we studied the (naive) case that F_T, the total force is constant, and hence equal to $-mg$. In this case, we re-derived the equations for velocity $v = \frac{dy}{dt}$ and position y. Later, we observed the following: In this model, the amount of time a projective spends ascending is the same amount of time it spends descending. Furthermore, the initial velocity and the velocity at impact are the same. Since these are not very realistic, we tried to do better.

We improved our model by considering the force of air resistance. In the first case, we considered the situation when F_R (the force of air resistance) was equal to $-k \cdot v$, for some positive constant k. Using this, Newton's law states that

$$mv' = F_T = F_G + F_R = -mg - kv.$$

We first noted that this was an autonomous equation, and using phase diagrams, were able to conclude that $v_\tau = -\frac{\rho}{g}$, where v_τ denotes the terminal velocity, and $\rho = k/m$. Noticing that this ODE could be turned into a first-order linear ODE, we were able to apply the techniques of Section 1.5 to solve explicitly for v and y. Having done so, we considered an example, and concluded that this model appears to be more "realistic" when compared to our naive one.

Our next model called for changing our formula for air resistance so that it was proportional to v^2. Since F_R should always have the opposite sign of v, this caused us to consider two cases:

- (Upward Motion) $F_R = -kv^2$, for some constant $k > 0$.

- (Downward Motion) $F_R = kv^2$, for some constant $k > 0$.

In the second case, we used phase diagrams to compute the terminal velocity $v_\tau = -\sqrt{\frac{g}{\rho}}$. Next, we examined the first case, and set up how to explicitly solve for $v(t)$ and $y(t)$. Using this outline and the Ingredients discussed in class, you shouldn't have too much trouble doing this (note: This is Section 2.3, Problems $13, 14$).

We then examined the force of gravity at large distances: If two bodies, of mass m and M, respectively, are a distance r apart, then the force of gravity is of the form $\pm\frac{GmM}{r^2}$, where G is a gravitational constant. Using this, we calculated that the escape velocity of the Earth. We concluded by pointing out the interesting fact that this velocity is a constant (that is, doesn't really depend on the object being launched into space).

Lecture 4: Tuesday, 9/10

We began by reviewing the deficiences of the natural growth model, and recalled the definition of the logistic equation for a population P, given by

$$\frac{dP}{dt} = kP(M - P) \text{ for some positive constants } M, k.$$

Separating variables, we were able to explicitly solve this equation, and we noticed that, regardless of the value of P_0, all solutions eventually tended towards the equilibrium solution $P = M$. We used this model to answer a question about the spread of rumors, but saw that it has many other applications. We continued our discussion by considering the slightly modified model

$$\frac{dP}{dt} = kP(P - M), \text{ where } M, k \text{ are positive constants,}$$

which we called the *explosion/extinction model*. We also solved this (and were once again careful with the signs), and used our solution to determine the following: If P_0 starts below $P = M$, then the population dies out (i.e., approaches zero). If P_0 starts above the line $P = M$, then the denominator in our expression for P has a t value where it becomes zero, and hence the function P blows up as t approaches this value. We saw that the point where the function blew up depended on P_0.

We next started §2.2, and began by defining an *autonomous* first order ODE (the two models considered above where in this class of ODEs). We discussed how, via drawing *phase diagrams* (I mistakenly called these phase portraits in class), we were able to understand a lot of the quantitative behavior of an autonomous ODE without actually solving it! We concluded by considering a logistic-harvesting example, i.e., an equation of the form $\frac{dx}{dt} = kx(M - x) - h$, where here k, M, h are all positive constants. The new addition is the harvesting constant h, which is typically small relative to k and M. We then outlined the following steps to solve a harvesting problem (you'll need this on your quiz).

(1) Factor the right hand side of your autonomous ODE (using the quadratic formula), so that it is of the form $\frac{dx}{dt} = -(x - A)(x - B)$ for some A, B. Here, you'll need that h is small so that these A and B actually exist (i.e., you get no imaginary roots from the quadratic equation).
(2) From here, you may draw a phase diagram to understand the qualititative behavior.
(3) If you need to explicitly solve this, you can separate variables!

Lecture 3: Thursday, 9/5

We started off by reviewing our solution to the general first-order linear ODE, and then stated a theorem regarding the uniqueness of these solutions. We then did some examples to warm up, and then the class took **Quiz 1**. After this, we discused input/output problems in some detail. Recall, of all of the relevant data (i.e., r_i, c_i, r_0, c_0, V) typically we have that r_i and c_i are constant, and whether or not V is constant or not is dictated by the context: assuming r_i and r_0 are constant, then V is constant if $r_i = r_0$, and otherwise $V = (r_i - r_0) \cdot t$. HOWEVER, c_0 is never constant, and is equal to $\frac{x(t)}{V}$ ($x(t)$ being the quantity of the material of interest at time t). In order to better digest this material, we went over a bunch of examples.

After the break, we begun discussing §2.1. The point was that the "natural growth" model that we used earlier to describe population changes isn't the most realistic (e.g., doesn't contain any non-zero horizontal asymptotes, assumes that birth/death rates are constant, etc). In seeking more realistic models (e.g., where the death rate was constant but the birth rate was a linear function of population), we arrived at the following equation

$$\frac{dP}{dt} = KP(M - P),$$

where K and M are constants. This equation is called the *logistic equation*. We saw some examples, and gave a hint at how to find the general solution (hint: partial fractions). We'll pick up here next time, and the class is *urged* to do their assigned reading before Tuesday.

Lecture 2: Tuesday, 9/3

We begun the class by recalling the definition of a separable ODE. After this, we continued our applications of separable ODEs; in particular, we worked through problems illustrating the use of the the natural growth / decay equation in the context of population estimate problems, half-life type problems, and problems relating to Newton's Law of Cooling/Heating. After going through these examples, we concluded our treatment of §1.4.

We began §1.5 by defining a *first order, linear ODE*, and presented the example $y' - y = \frac{11}{8} \cdot e^{-\frac{1}{3} \cdot x}$. After struggling to find/guess a solution for a while, I decided to multiply both sides of the equation by e^{-x}, and then we witnessed magic in the form of rewriting the LHS of the ODE using the product rule! During break, we pondered why I had chosen to multiply both sides by e^{-x}.

After this, we outlined a general method to solve first order linear ODEs

$$y' + P(x) \cdot y = Q(x),$$

which we recall here:

(1) First, compute $w(x) = \int P(x)dx$. Note that it is not necessary to include the constant (i.e., you may omit "$+C$" in this computation).
(2) Next, multiply both sides of your ODE by $e^{w(x)}$.
(3) Look hard, and rewrite the LHS of your ODE using the product rule.
(4) All you have to do is integrate, and then solve for y.

In fact, before describing this process, we actually proved it worked! Though the process itself may appear complicated, the only "ingredients" needed to derive it were: the product rule, a special instance of the chain rule, and the fundamental theorem of integral calculus (i.e., the fact that derivatives and integrals cancel each other out, so to speak). Though our presentation was a straightforward application of these fundamental facts, students should be aware that this derivation (or "proof") will **not** appear on any exams or quizzes. We concluded the class by working through a number of examples, and will pick up here next time. Students are advised to read ahead in Section 1.4 and start Section 2.1 reading.

Lecture 1: Thursday, 8/29

After some review, we started discussing the fact that we can't really solve many ODEs (even simple looking ones), and we then pointed out the usefulness of **slope fields**. Using slope fields, we were able to determine the qualitive behavior of some solutions to ODEs without knowing the formula for a solution. In case you are interested, the website I used to compute slope fields is

http://math.rice.edu/~dfield/dfpp.html;

note that you will need to have Java enabled on your browser to use this. Recall that slope fields may only be computed for ODEs of the form $y' = F(x, y)$, and so are limited in their application.

Continuing with §1.3, we recalled the examples we have seen, where ODEs have none/infinitely many/ a unique solution, and asked whether we can tell ahead of time when these possibilities might occur. In particular, we focused on the following **Question:** Given an IVP of the form $y' = F(x, y)$ with $(x, y) = (a, b)$, when does there exist a unique solution? We stated a theorem (see your text) which answered this question. We also spent a good amount of time pointing out the subtle fact that, even when we know that a solution exists, we don't know ahead of time what the domain of the solution will be. More precisely, when we know that a unique solution y to the ODE exists, all that we can say is that y is a function of x, and is defined in a neighboor of $x = a$ (our initial condition), but not much else. We then gave examples to illustrate this subtle point. We concluded §1.3 by giving an example of an IVP that had infinitely many solutions (note: this is the first example of this kind that we've seen so far in lecture).

After the break, we started §1.4, which examines separable ODEs, i.e., equations of the form

$$y' = g(x) \cdot h(y),$$

where here $g(x)$ is a function involving only x's, and $h(y)$ is a function involving only $y's$. Recall that, writing $y' = \frac{dy}{dx}$, and by multiplying/dividing as necessary, we are always able to isolate all expressions involving y on the LHS (left hand side) of the equation, and isolate all expressions involving x on the right hand side. At this point, all one needs to do is integrate. We saw the following types of behavior:

- Following this method, we are sometimes able to find a **explicit general solution**.
- Other times, we may arrive at an equality involving y and x (and not y'), but are still unable to solve for y, due to the complexity of the equation. In this situation, we have arrived at an **implicit solution**.
- Lastly, we observed how some initial solutions (which we call **singular solutions**) may disappear when we proceed with the method of solving by separating of variables. What is happening here is that information is lost when dividing (so be careful).

We concluded with some real-world applications. In particular, we derived the formula for exponential growth from a separable ODE, and talked about some consequences. **NOTE: Students are expected to continue reading the rest of** §1.4 **on their own (we will continue with this a bit next week).**

Lecture 0: Tuesday, 8/27

After some technical difficulties (!), we went over the syllabus and course overview for the course. After this, we got down to business and started with \$ 1.1. We began by defining the term "ODE" (ordinary differential equation), and gave a lot of examples and non-examples, to clarify things. Afterwards, we looked at some simple ODEs, and were able to guess solutions to some of them. However, we also saw that, even if the ODE looks simple, one may not always be able to guess a solution. In our exploration, we encountered ODEs that have infinitely many solutions, as well as ODEs that have no solution. We also noted that even if we can't (yet!) solve anything but the simplest ODEs in our head, we *can* always verify whether or not a given function is a solution. We then did more examples.

For the second half of lecture, we covered the material in §1.2. Our main observation here was that, via integration, we can (in theory) solve differential equations of the form

$$y' = f(x),$$

where here y is an unknown function of x. Repeated applications of this allowed us to deduce standard formulas (from physics) that describe the motion and velocity of a projectile with constant acceleration. Note: The case of non-constant acceleration is dealt with similarly, and will appear on one of your HW problems. In doing this, we noted that through repeated integration, we could solve differential equations of the form

$$y^{(n)} = f(x)$$

for any value of n, generalizing our intial approach.

Announcement regarding the text: I was able to verify with the Associate Chair of the math department that the 3rd edition of Edwards and Penney should suffice for this course (many of you asked about this in class). It still might be a good idea to get the text in the bookstore, as it will be used in future math classes (see the course syllabus for details).