

## RESEARCH STATEMENT

Daniel J. Hernández

**Overview.** My work is in the area of commutative algebra, and is motivated by the connections between algebra and geometry. Commutative algebra is the study of commutative rings (e.g. polynomial rings over fields and their quotients) and modules over these rings, while algebraic geometry is the study of finitely many polynomial equations in finitely many unknowns. The set of all solutions to such a system of equations gives rise to a geometric object, often called a variety. Basic examples of a variety include lines (e.g., solutions to the equation  $y = x$ ) and parabolas (e.g., solutions to the equation  $y = x^2$ ), though a variety described by many equations in many unknowns can be quite complicated. Commutative algebra and algebraic geometry are intimately related, as many geometric properties of an algebraic variety  $X$  can be measured by the ring theoretic properties of the ring of functions on  $X$ . For example, the geometric notion of smoothness is equivalent to the algebraic notion of regularity. In a very broad sense, both commutative algebra and algebraic geometry deal with the theory of equations; as such, both are closely related to many areas of math, including differential geometry, representation theory, and number theory. Furthermore, the fields of computational commutative algebra and algebraic geometry form the basis for many “real world” applications. These applications are quite diverse, and include contributions to epidemiology, biology, computer science, and coding theory.

My research is in the subfield of positive characteristic commutative algebra. For nearly forty years, mathematicians have used the Frobenius (or  $p^{\text{th}}$ -power) map to investigate phenomena in commutative algebra, algebraic geometry, representation theory, and number theory. Notable applications include the key role of “Frobenius purity” (or  $F$ -purity, for short) in the proof of the well-known Hochster-Roberts theorem (which states that rings of invariants are Cohen-Macaulay) [HR74]. Another important application appears in the work of the Indian school of researchers studying algebraic groups, who introduced the concept of “Frobenius splittings,” and used it to deduce important vanishing theorems for Schubert varieties [MR85]. Today, the notions of  $F$ -purity and Frobenius splittings are understood to be very closely related [Smi00a]. Furthermore, emerging connections with the singularities of hypersurfaces over  $\mathbb{C}$  (typically defined via  $L^2$ -methods, or via resolution of singularities) appearing in the so-called minimal model program have led to a renewed interest in the theory of  $F$ -purity (and some of its natural generalizations). My research aims to shed light on the connection; in particular, I seek to understand the connection between certain invariants of singularities defined using Frobenius in characteristic  $p$ , and invariants defined using  $L^2$ -methods (and more generally, resolution of singularities) over  $\mathbb{C}$ .

**Singularities via Frobenius.** We briefly recall the role of Frobenius in the study of singularities: Let  $R$  be a domain of characteristic  $p > 0$  (e.g., a quotient of a polynomial ring over  $\mathbb{F}_p$ , the finite field with  $p$  elements, by an irreducible polynomial). Recall that the image of the Frobenius (or  $p^{\text{th}}$  power) map  $R \xrightarrow{F} R$  is the subring  $R^p \subseteq R$  consisting of all  $p^{\text{th}}$  powers. Consequently,  $R$  is naturally an  $R^p$ -module, and it follows from a theorem of Kunz that this module structure can be used to detect singularities [Kun69]. For example, if  $g$  is an irreducible polynomial in  $\mathbb{F}_p[x_1, \dots, x_n]$ , then the variety in  $(\mathbb{F}_p)^n$  defined by the zeroes of  $g$  is smooth (i.e., non-singular) if and only if  $R = \mathbb{F}_p[x_1, \dots, x_n]/(g)$  is a flat (i.e., locally free)  $R^p$ -module. Thus, singular rings exhibit pathological behavior with respect to Frobenius, and motivated by this, one may define new classes of singularities by imposing conditions on the structure of  $R$  as an  $R^p$ -module. For example, we say that  $R$  is  $F$ -pure (or  $F$ -split) if the inclusion  $R^p \subseteq R$  splits as a map of  $R^p$ -modules [HR76]. Recently, using more subtle applications of Frobenius to define singularities (e.g., through tight closure theory) has led to new classes of so-called  $F$ -singularities (e.g.,  $F$ -regularity and  $F$ -rationality). Via the process of reduction to positive characteristic, these  $F$ -singularities are

closely related to the singularity types of varieties defined over  $\mathbb{C}$  (e.g., log canonical and rational singularities) that appear in the so-called minimal model program, a major area of research in birational geometry [Fed83, Smi97a, Har98, Smi00b, KM98]. My research is especially motivated by the connection between  $F$ -purity and log canonical singularities.

**Outline.** The research discussed here is organized into the following sections:

**Section 1:  $F$ -pure thresholds versus log canonical thresholds.** The research discussed in Section 1 revolves around an important conjecture relating  $F$ -purity and log canonicity: We seek to investigate when (and how often)  $F$ -pure and log canonical thresholds agree, as well as to investigate arithmetic properties of the  $F$ -pure threshold whenever it differs from the log canonical threshold. We conclude this section by discussing a project motivated by Elkies' theorem on the existence of infinitely many supersingular primes for an elliptic curve over  $\mathbb{Q}$ .

**Section 2: Explicit Formulas via congruence classes.** Here, we seek to understand when explicit formulas for  $F$ -pure thresholds (and related characteristic  $p > 0$  invariants) exist; in the case that any formulas may be calculated, we propose to study in what ways these formulas depend on the class of  $p$  modulo some integer. As an application, we discuss projects related to a recently-observed connection with Bernstein-Sato polynomials in characteristic zero.

**Section 3: Test ideals and  $F$ -jumping exponents.** In this section, we propose to study *test ideals*, invariants in characteristic  $p$  that may be thought of as analogs of the *multiplier ideals*, typically defined over  $\mathbb{C}$ . One group of projects involves an associated sequence of numerical invariants called  *$F$ -jumping exponents*, while others focus on producing algorithms for computing test ideals and  $F$ -jumping exponents.

## 1. $F$ -PURE THRESHOLDS VERSUS LOG CANONICAL THRESHOLDS

Given a polynomial  $f_{\mathbb{Q}}$  over  $\mathbb{Q}$ , one may use resolution of singularities to construct the birational invariant  $\mathbf{lct}(f_{\mathbb{Q}})$ , called the *log canonical threshold* of  $f_{\mathbb{Q}}$  [Laz04]. Note that  $\mathbf{lct}(f_{\mathbb{Q}}) \in \mathbb{Q} \cap (0, 1]$ , and as the name suggests, log canonical thresholds may also be defined using the notion of log canonical singularities, a prominent singularity type in the minimal model program.

For  $p \gg 0$ , one may reduce the coefficients of  $f_{\mathbb{Q}} \bmod p$  to obtain polynomials  $f_p \in \mathbb{F}_p[x_1, \dots, x_n]$ ; one upshot of reducing to characteristic  $p$  is that one may now use Frobenius to study  $f_p$ . If we set  $\nu_{f_p}(p^e) := \max \left\{ N : f_p^N \notin (x_1^{p^e}, \dots, x_n^{p^e}) \right\}$ , then the limit  $\mathbf{fpt}(f_p) := \lim_{e \rightarrow \infty} p^{-e} \cdot \nu_{f_p}(p^e)$  exists, and is called the  $F$ -pure threshold of  $f_p$  [TW04, MTW05]. Like  $\mathbf{lct}(f_{\mathbb{Q}})$ , we have that  $\mathbf{fpt}(f_p) \in \mathbb{Q} \cap (0, 1]$  [BMS08]. In fact, the connection between these two invariants runs far deeper [MTW05]:

$$(1) \quad \mathbf{fpt}(f_p) \leq \mathbf{lct}(f_{\mathbb{Q}}) \text{ for } p \gg 0, \text{ and } \lim_{p \rightarrow \infty} \mathbf{fpt}(f_p) = \mathbf{lct}(f_{\mathbb{Q}}).$$

**Example 1.** If  $f_{\mathbb{Q}} = x^3 - y^2 \in \mathbb{Q}[x, y]$ , then  $f_p = x^3 - y^2 \in \mathbb{F}_p[x, y]$ . Furthermore, for  $p \geq 5$ ,

$$\mathbf{lct}(f_{\mathbb{Q}}) = \frac{5}{6}, \text{ and } \mathbf{fpt}(f_p) = \begin{cases} \mathbf{lct}(f_{\mathbb{Q}}) & p \equiv 1 \pmod{6} \\ \mathbf{lct}(f_{\mathbb{Q}}) - \frac{1}{6p} & p \equiv 5 \pmod{6} \end{cases}.$$

**Conjecture 2.** [MTW05, Conjecture 3.6]  $\mathbf{fpt}(f_p) = \mathbf{lct}(f_{\mathbb{Q}})$  for infinitely many  $p$ .

Conjecture 2 represents one of the most important open problems in the field, and is closely related to other open arithmetic problems [MS11, Mus12]. Note that Conjecture 2 holds in Example 1 by Dirichlet's Theorem on primes in arithmetic progressions. However, Conjecture 2 can be quite subtle in general. In the case that  $f_{\mathbb{Q}} \in \mathbb{Q}[x, y, z]$  defines an elliptic curve  $E \subseteq \mathbb{P}_{\mathbb{Q}}^2$ , Conjecture 2 is closely related to the result which states that the reduction  $E_p$  is supersingular for infinitely many  $p$ ; see [MTW05, Example 4.6] for more details.

**Project 3.** Address Conjecture 2 (and its natural generalizations).

The author has recently identified some special cases in which Conjecture 2 is “attackable.” Indeed, given a polynomial  $f_{\mathbb{C}}$  over  $\mathbb{C}$ , one may apply the process of “reduction to characteristic  $p$ ” to obtain models  $f_p$  over finite fields of characteristic  $p \gg 0$  [Smi97b]; in the case that  $f_{\mathbb{C}}$  has coefficients in  $\mathbb{Q}$ , this process recovers the one discussed earlier. In this setting, one may ask whether  $\mathbf{fpt}(f_p) = \mathbf{lct}(f_{\mathbb{C}})$  for infinitely many  $p$ , and I have answered this question in the affirmative whenever the coefficients of the polynomial  $f_{\mathbb{C}}$  are algebraically independent over  $\mathbb{Q}$  [Herb]. Moreover, I suspect that one may be able to use these methods to address a version of Conjecture 2 for more general varieties. Note that one first step in this direction was recently taken by Shunsuke Takagi, who generalized the methods I developed in [Herb] to obtain a positive answer to a variation of Conjecture 2 in the setting of an ambient space with mild singularities [Tak11].

**Project 4.** Recall that  $\mathbf{fpt}(f_p) \in \mathbb{Q} \cap (0, 1]$ , and suppose that  $\mathbf{fpt}(f_p) \neq \mathbf{lct}(f_{\mathbb{Q}})$ .

- (1) (Karl Schwede) Must  $p$  divide the denominator of  $\mathbf{fpt}(f_p)$ ?
- (2) When is the denominator of  $\mathbf{fpt}(f_p)$  a power of  $p$ ?

In ongoing joint work with Luis Nuñez Betancourt, Emily Witt, and Wenliang Zhang, we have generalized Bhatt and Singh’s argument in [BS] to obtain the following.

**Theorem 5.** [HNBWZ] Let  $f_{\mathbb{Q}} \in \mathbb{Q}[x, y]$  be quasi-homogeneous with isolated singularity, and suppose  $\mathbf{fpt}(f_p) \neq \mathbf{lct}(f_{\mathbb{Q}})$ . If  $\mathbf{lct}(f_{\mathbb{Q}}) = \sum_{s=1}^{\infty} \beta_s \cdot p^{-s}$  is the base  $p$  expansion of  $\mathbf{lct}(f_{\mathbb{Q}})$ , then there exists an integer  $L_p \geq 1$  such that  $\mathbf{fpt}(f_p) = \sum_{s=1}^{L_p} \beta_s \cdot p^{-s}$  (i.e., a truncation of  $\mathbf{lct}(f_{\mathbb{Q}})$ , base  $p$ ).

In the context of Theorem 5, we conclude that  $\mathbf{fpt}(f_p)$  is a truncation of the base  $p$  expansion of  $\mathbf{lct}(f_{\mathbb{Q}})$ , and this addresses the second point of Project 4. In fact, the denominator of  $\mathbf{fpt}(f_p)$  is always a power of  $p$  whenever  $f_{\mathbb{Q}}$  is quasi-homogeneous with isolated singularity and  $\mathbf{fpt}(f_p) \neq \mathbf{lct}(f_{\mathbb{Q}})$ , even though  $\mathbf{fpt}(f_p)$  need not be a truncation of  $\mathbf{lct}(f_{\mathbb{Q}})$  [HNBWZ]. I am very interested in investigating under what conditions  $\mathbf{fpt}(f_p)$  is simply a truncation, base  $p$ , of  $\mathbf{lct}(f_{\mathbb{Q}})$ , and also seek to establish Theorem 5 in the absence of the quasi-homogeneous assumption.

In the case that  $f_{\mathbb{Q}}$  defines an elliptic curve  $E \subseteq \mathbb{P}_{\mathbb{Q}}^2$ , then Elkies’ famous result that the reduction  $E_p$  is supersingular for infinitely many  $p$  can be translated as follows:  $\mathbf{fpt}(f_p) \neq \mathbf{lct}(f_{\mathbb{Q}}) = 1$  for infinitely many  $p$ . Motivated by this, we propose the following project.

**Project 6.** For which  $f_{\mathbb{Q}}$  must  $\mathbf{fpt}(f_p) \neq \mathbf{lct}(f_{\mathbb{Q}})$  for infinitely many  $p$ ?

Very recently, I have identified the following method for attacking a special case of Project 6: In [Her12], I demonstrated that the base  $p$  expansions of numbers of the form  $\mathbf{fpt}(f_p)$  must satisfy very strong conditions. Using these conditions, I expect to be able to demonstrate that, for almost all possible values of  $\mathbf{lct}(f_{\mathbb{Q}})$ , one *cannot* have that  $\mathbf{fpt}(f_p) = \mathbf{lct}(f_{\mathbb{Q}})$  for all  $p \gg 0$  (and thus address Project 6 for these general values of  $\mathbf{lct}(f_{\mathbb{Q}})$ ). However, we do not expect our method to apply when  $\mathbf{lct}(f_{\mathbb{Q}}) = 1$  (and more generally, when  $\mathbf{lct}(f_{\mathbb{Q}}) = \frac{1}{n}$  for some  $n \geq 1$ ). As Elkies’ methods suggests, more subtle techniques would be needed to address Project 6 for these special values of  $\mathbf{lct}(f_{\mathbb{Q}})$ .

## 2. EXPLICIT FORMULAS VIA CONGRUENCE CLASSES

As before, let  $f_{\mathbb{Q}}$  be a polynomial over  $\mathbb{Q}$  vanishing at the origin, and  $f_p$  denote its reduction to characteristic  $p > 0$ . In Example 1, the value of  $\mathbf{fpt}(f_p)$  is expressed as a function of the class of  $p$  modulo some integer  $N$  (in this case,  $N = 6$ ). However, the example of the elliptic curve shows that the variance of  $\mathbf{fpt}(f_p)$  as a function of  $p$  can be extremely subtle, and need not depend on the class of  $p$  modulo some integer  $N$  in any meaningful way.

**Problem 7.** [MTW05, Problem 3.10] Let  $f_{\mathbb{Q}}$  be a polynomial over  $\mathbb{Q}$ .

- (1) When can  $\mathbf{fpt}(f_p)$  may be expressed as a function of the class of  $p$  modulo some  $N$ ?

- (2) Recall that  $\mathbf{fpt}(f_p)$  is defined as the limit of the normalized sequence  $\{p^{-e} \cdot \nu_{f_p}(p^e)\}_{e \geq 1}$ . When does there exist an integer  $N$  and a family of polynomials  $H_{m,e}(t) \in \mathbb{Q}[t]$  such that  $\nu_{f_p}(p^e) = H_{m,e}(p)$  for all  $p \equiv m \pmod{N}$ ?

Computing  $F$ -pure thresholds and the terms  $\nu_{f_p}(p^e)$  is quite difficult in general. In fact, the only explicit computations in the literature are for the polynomials  $x^3 + y^2$ ,  $x^2 + y^7$  and  $x^5 + y^4 + x^3y^2$  [MTW05, Examples 4.3, 4.4, 4.5]. In particular, nothing is known regarding families of polynomials satisfying the conditions in Problem 7, motivating the following project.

**Project 8.** Identify families of polynomials satisfying the conditions in Problem 7.

We now outline a possible approach to Project 8: In [Herb], I associate to  $f_{\mathbb{Q}}$  a rational polytope  $P$  (not the Newton polyhedron) contained in  $[0, 1]^m$  for some  $m \geq 1$ . Using geometric properties of the polytope  $P$ , I was able to provide an affirmative answer to Conjecture 2 in a special case [Herb]. Moreover, I was able to use the polytope  $P$  to construct algorithms for computing  $F$ -pure thresholds for binomial and diagonal hypersurfaces [Hera, Her11].

**Project 9.** Work out Project 8 in the case that the associated polytope  $P$  has nice combinatorial properties. In particular, work out Problem 7 for binomial and diagonal hypersurfaces.

We now recall a connection between  $F$ -pure thresholds and Bernstein-Sato polynomials. Given a polynomial  $f_{\mathbb{Q}}$  over  $\mathbb{Q}$ , one may use the theory of holonomic  $D$ -modules (a special class of modules over rings of differential operators) to construct a polynomial  $b_{f_{\mathbb{Q}}}(s) \in \mathbb{Q}[s]$ , typically called the Bernstein-Sato polynomial of  $f_{\mathbb{Q}}$  [Cou95]. The roots of the Bernstein-Sato polynomial  $b_{f_{\mathbb{Q}}}(s)$  have long been studied, and it was shown by Kashiwara in [Kas77] that the roots of  $b_{f_{\mathbb{Q}}}(s)$  are all negative rational numbers. Furthermore, the roots of  $b_{f_{\mathbb{Q}}}(s)$  are closely related to many invariants of singularities. For example, conjectures due to Denef, Loeser, and of Igusa relate the roots of Bernstein-Sato polynomials to the poles of (topological, motivic, and  $p$ -adic) zeta functions. In a result that is closely related to this proposal and which builds on earlier work, Ein, Lazarsfeld, Smith, and Varolin show that  $-\mathbf{lct}(f_{\mathbb{Q}})$ , and more generally, the negative of every jumping exponent of  $f_{\mathbb{Q}}$ , is a root of  $b_{f_{\mathbb{Q}}}(s)$  [ELSV04]; see Section 3 for a brief description of the jumping exponents of  $f_{\mathbb{Q}}$ . The following describes a key connection between  $F$ -pure thresholds and roots of Bernstein-Sato polynomials.

**Theorem 10.** [MTW05] If  $f_{\mathbb{Q}}$  is a polynomial over  $\mathbb{Q}$  and  $f_p$  denotes its reduction to characteristic  $p \gg 0$ , then  $b_{f_{\mathbb{Q}}}(\nu_{f_p}(p^e)) \equiv 0 \pmod{p}$  for every  $e \geq 1$ . Furthermore, if  $f_{\mathbb{Q}}$  satisfies the second condition of Problem 7, then  $H_{m,e}(0) \in \mathbb{Q}$  is a root of  $b_{f_{\mathbb{Q}}}(s)$  (over  $\mathbb{Q}$ ) for every  $m$  and  $e \geq 1$ .

**Project 11.** Suppose that  $f_{\mathbb{Q}}$  satisfies the conditions in Problem 7. Determine which roots of  $b_{f_{\mathbb{Q}}}(s)$  that can be obtained using the method described above. In the specific case that  $f_{\mathbb{Q}}$  is either a diagonal or binomial hypersurface, we propose the following.

- (1) Completely characterize which roots of  $b_{f_{\mathbb{Q}}}(s)$  are determined as above by giving explicit formulas for said roots as a function of the class of  $p$  modulo some integer  $N$ .
- (2) Use these explicit formulas to give lower bounds for the number of roots of  $b_{f_{\mathbb{Q}}}(s)$ . As far as I am aware, the degree of  $b_{f_{\mathbb{Q}}}(s)$  is quite mysterious, and so any explicit examples would be insightful. (especially in the case of binomial hypersurfaces)
- (3) As described above, the roots of  $b_{f_{\mathbb{Q}}}(s)$  are closely related to many other invariants of singularities. Relate any explicit formulas obtained via completing Project 11 to these other invariants of singularities.

### 3. TEST IDEALS AND $F$ -JUMPING EXPONENTS

Once again, let  $f_{\mathbb{Q}}$  denote a polynomial over  $\mathbb{Q}$ . Via (log) resolution of singularities, one may construct a family of ideals  $\{\mathcal{I}(\lambda \bullet f_{\mathbb{Q}})\}_{\lambda \in \mathbb{R}_{\geq 0}}$  called the *multiplier ideals* of  $f_{\mathbb{Q}}$ ; see [BL04, Laz09] for

an introduction to the theory of multiplier ideals, and [Laz04] for a more in-depth description. Using these ideals, one may extract from  $f_{\mathbb{Q}}$  a sequence of numerical invariants  $\{\xi_1(f_{\mathbb{Q}}), \dots, \xi_k(f_{\mathbb{Q}})\}$  contained in  $[0, 1]$  called the *jumping exponents* of  $f_{\mathbb{Q}}$ ; see [ELSV04] for a detailed description of these important invariants of singularities of  $f_{\mathbb{Q}}$ .

Now, let  $f_p$  denote the reduction of  $f_{\mathbb{Q}}$  to characteristic  $p > 0$ . Using Frobenius, one may construct a family of ideals  $\{\tau(\lambda \bullet f_p) : \lambda \in \mathbb{R}_{\geq 0}\}$  called the test ideals of  $f_p$  [HH90, HY03]. We refer the reader to [BMS08] for a down-to-earth description of test ideals, and to the survey [ST12] for an overview of the state of the art. As with the multiplier ideals, the test ideals of  $f_p$  may be used to define a sequence of invariants  $\{\xi_1(f_p), \dots, \xi_{N_p}(f_p)\} \subseteq [0, 1]$  of  $f_p$ , called the *F-jumping exponents* of  $f_p$ . Amazingly, test ideals and multiplier ideals are shown to agree after reduction to characteristic  $p > 0$ , and the relationship between  $\mathbf{lct}(f_{\mathbb{Q}})$  and  $\mathbf{fpt}(f_p)$  described above may be viewed as a consequence of this fact [Smi00b, HY03]. The projects proposed here focus on better understanding the relationship between *F-jumping exponents* of  $f_p$  and the jumping exponents of  $f_{\mathbb{Q}}$ . In particular, we propose the following.

**Project 12.** Given  $f_{\mathbb{Q}}$ , bound the number of *F-jumping exponents* of  $f_p$ . In particular:

- (1) Given consecutive jumping exponents  $\xi_k(f_{\mathbb{Q}}) < \xi_{k+1}(f_{\mathbb{Q}})$  of  $f_{\mathbb{Q}}$ , bound the number of *F-jumping exponents* of  $f_p$  contained in  $(\xi_k(f_{\mathbb{Q}}), \xi_{k+1}(f_{\mathbb{Q}})]$  (which is non-empty by [MZ11]).
- (2) Note that  $\mathbf{lct}(f_{\mathbb{Q}})$  is the smallest positive jumping exponent of  $f_{\mathbb{Q}}$ . As a special case, bound the number of *F-jumping exponents* in  $(0, \mathbf{lct}(f_{\mathbb{Q}})]$ .

At present, the only explicit computation of higher jumping numbers for a family of polynomials were the ones I presented in [Her11]. As a consequence of these computations, we have the following:

**Theorem 13.** [Her11] Let  $f_{\mathbb{Q}}$  be a Fermat hypersurface over  $\mathbb{Q}$  of degree  $d$ .

- (1) For every choice of  $d$ , the only jumping exponent of  $f_{\mathbb{Q}}$  in  $[0, 1]$  is 1.
- (2) For every  $k \in \mathbb{N}$ , there exists  $p$  and  $d$  with the following property: Every Fermat hypersurface  $f_p$  of degree  $d$  over  $\mathbb{F}_p$  has at least  $k$  distinct *F-jumping exponents* in  $(0, 1)$ .

Thus, as this theorem illustrates, one cannot expect to naively bound the number of *F-jumping exponents* of  $f_p$  by the number of jumping exponents of  $f_{\mathbb{Q}}$ . However, as the following theorem of mine suggests, it may be possible to produce a bound for the number of jumping exponents over  $\mathbb{F}_p$  sandwiched between consecutive jumping coefficients over  $\mathbb{Q}$  as a function of the dimension of the ambient polynomial ring.

**Theorem 14.** [Her11] Suppose  $f_{\mathbb{Q}}$  is a Fermat hypersurface of degree  $d$  over  $\mathbb{Q}$ , and let  $f_p$  denote its reduction to characteristic  $p$ . If  $p > d$ , then every *F-jumping exponent* of  $f_p$  is in  $\frac{1}{p} \cdot \mathbb{N}$ . Moreover, for every  $p$ , the number of *F-jumping exponents* in  $(0, 1) = (0, \mathbf{lct}(f_{\mathbb{Q}}))$  is bounded above by  $d$ .

At the moment, the only general progress on Project 12 appearing in the literature a result, due to Katzman, Lyubeznik, and Zhang, which bounds the number of *F-jumping exponents* in  $[0, 1]$  in the isolated singularity case [KLZ11]. However, the computations in [Her11] suggest that these bounds are much larger than necessary. In addition, the methods I developed suggest a strategy at obtaining sharper bounds, which will be highly relevant in the following project.

**Project 15.** Given a polynomial  $f_p$  over  $\mathbb{F}_p$ , produce algorithms for computing the test ideals  $\tau(\lambda \bullet f_p)$  for all  $\lambda \in (0, 1)$ . In other words, produce an algorithm for computing each *F-jumping exponent*  $\xi(\cdot)$  of  $f_p$ , and each associated test ideal  $\tau(\xi \bullet f_p)$ .

I plan to work on Project 15 jointly with Luis Nùñez Betancourt and Emily E. Witt (at present, this collaboration is in its early stages). Our main approach to Project 15 is to deduce explicit stabilization results for intersections of test ideals of the form  $\bigcap_{0 < \epsilon < \lambda} \tau(\epsilon \bullet f_p)$ . In doing so, obtaining sharper bounds for the number of *F-jumping exponents* of  $f_p$  might allow one to obtain

effective algorithms (i.e., ones that might possibly be implemented in standard computer algebra programs) in special cases. As  $F$ -invariants are difficult to compute, any such algorithms would be of great value to the community.

## REFERENCES

- [BL04] Manuel Blickle and Robert Lazarsfeld. An informal introduction to multiplier ideals. In *Trends in commutative algebra*, volume 51 of *Math. Sci. Res. Inst. Publ.*, pages 87–114. Cambridge Univ. Press, Cambridge, 2004. [4](#)
- [BMS08] Manuel Blickle, Mircea Mustață, and Karen E. Smith. Discreteness and rationality of  $F$ -thresholds. *Michigan Math. J.*, 57:43–61, 2008. Special volume in honor of Melvin Hochster. [2](#), [5](#)
- [BS] Bhargav Bhatt and Anurag Singh.  $F$ -pure thresholds of calabi-yau hypersurfaces. *preprint*. [3](#)
- [Cou95] S. C. Coutinho. *A primer of algebraic D-modules*, volume 33 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1995. [4](#)
- [ELSV04] Lawrence Ein, Robert Lazarsfeld, Karen E. Smith, and Dror Varolin. Jumping coefficients of multiplier ideals. *Duke Math. J.*, 123(3):469–506, 2004. [4](#), [5](#)
- [Fed83] Richard Fedder.  $F$ -purity and rational singularity. *Trans. Amer. Math. Soc.*, 278(2):461–480, 1983. [2](#)
- [Har98] Nobuo Hara. A characterization of rational singularities in terms of injectivity of Frobenius maps. *Amer. J. Math.*, 120(5):981–996, 1998. [2](#)
- [Hera] Daniel J. Hernández.  $F$ -pure thresholds of binomial hypersurfaces. *to appear in Proc. Amer. Math. Soc.* [4](#)
- [Herb] Daniel J. Hernández.  $F$ -purity versus log canonicity for polynomials. *preprint*. [3](#), [4](#)
- [Her11] Daniel J. Hernández.  $F$ -invariants of diagonal hypersurfaces. *to appear in Proc. Amer. Math. Soc.*, 2011. [4](#), [5](#)
- [Her12] Daniel J. Hernández.  $F$ -purity of hypersurfaces. *Math. Res. Lett.*, 19(02):1–13, 2012. [3](#)
- [HH90] Melvin Hochster and Craig Huneke. Tight closure, invariant theory, and the Briançon-Skoda theorem. *J. Amer. Math. Soc.*, 3(1):31–116, 1990. [5](#)
- [HNBWZ] Daniel J. Hernández, Luis Núñez-Betancourt, Emily E. Witt, and Wenliang Zhang.  $F$ -pure thresholds of quasi-homogeneous polyomials. *in final preparations*. [3](#)
- [HR74] Melvin Hochster and Joel L. Roberts. Rings of invariants of reductive groups acting on regular rings are Cohen-Macaulay. *Advances in Math.*, 13:115–175, 1974. [1](#)
- [HR76] Melvin Hochster and Joel L. Roberts. The purity of the Frobenius and local cohomology. *Advances in Math.*, 21(2):117–172, 1976. [1](#)
- [HY03] Nobuo Hara and Ken-Ichi Yoshida. A generalization of tight closure and multiplier ideals. *Trans. Amer. Math. Soc.*, 355(8):3143–3174 (electronic), 2003. [5](#)
- [Kas77] Masaki Kashiwara.  $B$ -functions and holonomic systems. Rationality of roots of  $B$ -functions. *Invent. Math.*, 38(1):33–53, 1976/77. [4](#)
- [KLZ11] Mordechai Katzman, Gennady Lyubeznik, and Wenliang Zhang. An upper bound on the number of  $F$ -jumping coefficients of a principal ideal. *Proc. Amer. Math. Soc.*, 139(12):4193–4197, 2011. [5](#)
- [KM98] János Kollár and Shigefumi Mori. *Birational geometry of algebraic varieties*, volume 134 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original. [2](#)
- [Kun69] Ernst Kunz. Characterizations of regular local rings for characteristic  $p$ . *Amer. J. Math.*, 91:772–784, 1969. [1](#)
- [Laz04] Robert Lazarsfeld. *Positivity in algebraic geometry. II*, volume 49 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2004. Positivity for vector bundles, and multiplier ideals. [2](#), [5](#)
- [Laz09] Robert Lazarsfeld. A short course on multiplier ideals. 01 2009. [4](#)
- [MR85] V. B. Mehta and A. Ramanathan. Frobenius splitting and cohomology vanishing for Schubert varieties. *Ann. of Math. (2)*, 122(1):27–40, 1985. [1](#)
- [MS11] Mircea Mustață and Vasudevan Srinivas. Ordinary varieties and the comparison between multiplier ideals and test ideals. *Nagoya Math. J.*, 204:125–157, 2011. [2](#)
- [MTW05] Mircea Mustață, Shunsuke Takagi, and Kei-ichi Watanabe.  $F$ -thresholds and Bernstein-Sato polynomials. In *European Congress of Mathematics*, pages 341–364. Eur. Math. Soc., Zürich, 2005. [2](#), [3](#), [4](#)
- [Mus12] Mircea Mustață. Ordinary varieties and the comparison between multiplier ideals and test ideals II. *Proc. Amer. Math. Soc.*, 140(3):805–810, 2012. [2](#)
- [MZ11] Mircea Mustata and Wenliang Zhang. Estimates for  $f$ -jumping numbers and bounds for hartshorne-speiser-lyubeznik numbers. 10 2011. [5](#)

- [Smi97a] Karen E. Smith.  $F$ -rational rings have rational singularities. *Amer. J. Math.*, 119(1):159–180, 1997. [2](#)
- [Smi97b] Karen E. Smith. Vanishing, singularities and effective bounds via prime characteristic local algebra. In *Algebraic geometry—Santa Cruz 1995*, volume 62 of *Proc. Sympos. Pure Math.*, pages 289–325. Amer. Math. Soc., Providence, RI, 1997. [3](#)
- [Smi00a] Karen E. Smith. Globally  $F$ -regular varieties: applications to vanishing theorems for quotients of Fano varieties. *Michigan Math. J.*, 48:553–572, 2000. Dedicated to William Fulton on the occasion of his 60th birthday. [1](#)
- [Smi00b] Karen E. Smith. The multiplier ideal is a universal test ideal. *Comm. Algebra*, 28(12):5915–5929, 2000. Special issue in honor of Robin Hartshorne. [2](#), [5](#)
- [ST12] Karl Schwede and Kevin Tucker. A survey of test ideals. In *Progress in commutative algebra 2*, pages 39–99. Walter de Gruyter, Berlin, 2012. [5](#)
- [Tak11] Shunsuke Takagi. Adjoint ideals and a correspondence between log canonicity and  $f$ -purity. 05 2011. [3](#)
- [TW04] Shunsuke Takagi and Kei-ichi Watanabe. On  $F$ -pure thresholds. *J. Algebra*, 282(1):278–297, 2004. [2](#)