

DEFORMING THE \mathbb{R} -FUCHSIAN (4,4,4)-TRIANGLE GROUP INTO A LATTICE

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ABSTRACT. We prove that the last discrete deformation of the \mathbb{R} -Fuchsian (4,4,4)-triangle group in $PU(2, 1)$ is a cocompact arithmetic lattice. We also describe an experimental method for finding the combinatorics of a Dirichlet fundamental domain, and apply it to the lattice in question.

1. INTRODUCTION

A lot of interest for complex hyperbolic geometry has been generated by Mostow's work in the late 1970's, exhibiting the first examples of nonarithmetic lattices in $PU(n, 1)$ (in fact the current list of examples is only slightly larger, and all known examples in dimension four or higher are arithmetic).

The major difficulty to construct such groups is to find efficient methods for proving directly that a given group, for instance given by a number of generators, acts discretely on complex hyperbolic space. The construction of fundamental domains is much more complicated than in spaces with constant sectional curvature, since there are no totally geodesic real hypersurfaces. In particular there is no canonical choice for faces of a polyhedron, and the bare hands proofs of discreteness appeared to this day rely on using various particular types of hypersurfaces, adapted to the situation at hand (bisectors in [M1], \mathbb{C} -spheres in [FK], hybrid cones in [S1], cones over totally geodesic subspaces in [DFP]).

In this paper, we shall focus on Dirichlet domains, where the faces of the fundamental polyhedra are on bisectors, i.e. hypersurfaces equidistant between two points in complex hyperbolic space.

Given a group $\Gamma \subset PU(2, 1)$, the Dirichlet domain centered at p_0 is the set

$$(1.1) \quad F_\Gamma = \{x \in H_{\mathbb{C}}^2 : d(x, p_0) \leq d(x, \gamma p_0), \forall \gamma \in \Gamma\}$$

As discussed in [D1], one needs to be extremely cautious when using experimental methods to determine the combinatorics of Dirichlet

domains, and in fact several errors can be found in the founding paper [M1] (see also [D2]).

We describe a method for doing this using a theorem of Giraud, but it depends on having a computer with infinite precision, so it is of course not efficient in practice, even though it can be used to guess whether a given group is discrete, as well as to guess what a fundamental polyhedron should look like. Giving an actual proof of these guesses would then involve a somewhat prohibitive amount of numerical analysis.

On the other hand, our method based on computer experimentation turns out to be efficient for proving that a given group is cocompact, knowing that it is discrete (the latter can sometimes be checked by using arithmetics). The point is that we only need to show that some compact domain contains a fundamental domain, which leaves room for approximate constructions.

We apply the method to a particular deformation of an \mathbb{R} -Fuchsian triangle group in $PU(2, 1)$ and show that, at some point in the deformation, the group becomes a cocompact lattice. Note that cohomological dimension arguments do not preclude this from happening, since the corresponding representation of the triangle group is not faithful.

To put this in perspective, we mention that R. Schwartz has studied many deformations of triangle groups, and found ingenious ways to prove discreteness, but since his analysis is done by finding fundamental domains on the boundary $\partial\mathbb{H}_{\mathbb{C}}^2$, these methods do not allow to handle lattices in any straightforward way. In fact the group studied here was already alluded to in [S2], and it was already known to be discrete. It is part of a larger family of groups conjectured to be discrete (see [S1]).

The group $G(4, 4, 4; 5)$ is generated by three complex reflections of order two, I_1 , I_2 and I_3 , whose mirrors make an angle $\pi/4$. There is one additional parameter that expresses how far we are from the \mathbb{R} -Fuchsian situation. More precisely, denoting by e_1 , e_2 and e_3 some unit vectors polar to the mirrors of these three reflections, the angle condition between the mirrors translates into $|\langle e_i, e_j \rangle| = 1/\sqrt{2}$. The additional parameter can be chosen to be the triple Hermitian inner product $\langle v_1, v_2, v_3 \rangle = \langle v_1, v_2 \rangle \langle v_2, v_3 \rangle \langle v_3, v_1 \rangle$.

Choosing the parameter so that $I_1 I_2 I_3 I_2$ is elliptic of order n gives a family of groups $G(4, 4, 4; n)$, of which the case $n = 7$ corresponds to the group studied in [S1], which is not a lattice (it has infinite covolume).

Our main theorem is the following:

Theorem 1.1. *The group $G(4, 4, 4; 5)$ is a cocompact lattice in $PU(2, 1)$.*

The main original aspect of our work is the method for proving that the group is cocompact, even though some parts of the general study of

Dirichlet domains in complex hyperbolic geometry seem not to be well known (even after W. Goldman's detailed study of bisectors, cf. [G]).

We give a conjectural picture of the Dirichlet domain for our group, but it is conceivable (though unlikely) that more refined computer analysis would reveal that we have missed some of its faces (in a situation parallel to the one described in [D1] for Mostow's groups).

Finally we mention that this phenomenon seems to be quite rare among deformations of \mathbb{R} -Fuchsian triangle groups (and presumably also of other \mathbb{R} -Fuchsian groups). The author was informed by J. Parker that the group $G(5, 5, 5; 5)$ has the same property - that group can be checked to be a subgroup of index 60 in Mostow's lattice $\Gamma(5, 7/10)$, by using an explicit presentation (see [Pa]). No such simple description is known for $G(4, 4, 4; 5)$. It is the author's impression that these two deformed triangle groups are the only lattices among all the $G(n, n, n; p)$.

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2. DESCRIBING THE GROUP

We consider the deformation space of the \mathbb{R} -Fuchsian (4, 4, 4)-triangle group, generated by three complex reflections I_1, I_2 and I_3 that preserve a totally real plane in $H_{\mathbb{C}}^2$. As an abstract group, it is given by

$$\langle \iota_1, \iota_2, \iota_3 \mid \iota_j^2 = (\iota_j \iota_k)^4 = 1 \rangle$$

The order four relation imposes that we maintain the angle between the mirrors of the reflections at $\pi/4$ throughout the deformation.

We exclude the \mathbb{C} -Fuchsian representation, since it cannot be deformed (see [To]). Except for the latter representation, the mirrors are not orthogonal to a common complex geodesic, hence their orthogonal complements are linearly independent, and we may choose unit vectors e_1, e_2 and e_3 as basis vectors for \mathbb{C}^3 , in such a way that the mirror of I_j is e_j^\perp . We must have

$$|\langle e_i, e_j \rangle| = r = \cos(\pi/4) = 1/\sqrt{2}$$

and we are free to choose the argument of this complex number. By rescaling the vectors e_i , we may assume, without loss of generality, that

$$\langle e_1, e_2 \rangle = \langle e_2, e_3 \rangle = \langle e_3, e_1 \rangle = -r\varphi$$

The minus sign is included to stick with the notations in [M1]. We summarize the above discussion in the following:

Lemma 2.1. *The deformation space of the \mathbb{R} -Fuchsian $(4, 4, 4)$ triangle group is one-dimensional, parameterized by the argument of the triple Hermitian product $\langle e_1, e_2, e_3 \rangle = \langle e_1, e_2 \rangle \langle e_2, e_3 \rangle \langle e_3, e_1 \rangle$*

In the basis adapted to the mirrors of the generators, the Hermitian form is given by

$$(2.1) \quad H = \begin{bmatrix} 1 & -r\varphi & -r\bar{\varphi} \\ -r\bar{\varphi} & 1 & -r\varphi \\ -r\varphi & -r\bar{\varphi} & 1 \end{bmatrix}$$

Note that this is a real matrix when $\varphi = 1$, which corresponds to the \mathbb{R} -Fuchsian case. The matrix has determinant

$$1 - 3r^2 - r^3(\varphi^3 + \bar{\varphi}^3) = -\frac{1}{2} - \frac{1}{\sqrt{2}} \cos t$$

where we denote by t the argument of φ^3 , i.e. $\varphi^3 = e^{it}$. The form has signature $(2, 1)$ if and only if its determinant is negative, which is equivalent to

$$(2.2) \quad |t| < \frac{3\pi}{4}$$

This gives an explicit interval parameterizing the deformation space of the \mathbb{R} -Fuchsian $(4, 4, 4)$ -triangle group. Similar descriptions are of course easily obtained for other triangle groups. For completeness we write down the matrix of one of the generating reflections, the other two being easily deduced by symmetry:

$$(2.3) \quad I_1 = \begin{bmatrix} 1 & -2r\bar{\varphi} & -2r\varphi \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

The other two generators are obtained by symmetry, $I_2 = JI_1J^{-1}$, $I_3 = J^2I_1J^{-2}$ where J is the isometry corresponding to the natural 3-cycle on the standard basis vectors, $Je_i = e_{i+1}$ (indices modulo 3).

For convenience, we shall use shortcut notations for words in the generators, since they are described by giving a sequence of integers. We shall write I_{1232} or sometimes even simply 1232 for $I_1I_2I_3I_2$, etc.

Conjecture 2.2 (Schwartz). *The deformed representation is discrete and faithful if and only if $\cos t \geq -\frac{1}{2\sqrt{2}}$. In the remainder of the parameter space, there is a countable collection of values where the representation is discrete but not faithful.*

The representation is hence conjecturally discrete and faithful for $|t| \leq t_0$, $t_0 = 1.93216345\dots$, this interval corresponding to parameters

where $I_1 I_2 I_3 I_2$ is not elliptic. Indeed, it is completely elementary to write down the trace of this element, either by computing the matrix for $I_1 I_2 I_3 I_2$ explicitly, or by using the formula from [Pr]. One checks that

$$(2.4) \quad \text{Tr}(I_1 I_2 I_3 I_2) = 5 + 4\sqrt{2} \cos t$$

If $|t| > t_0$, in order to get a discrete representation, one needs to impose that $I_1 I_2 I_3 I_2$ have finite order. This leads to define the following parameter values:

Definition 2.3. $G(4, 4, 4; n)$ is the image of the representation ρ_n of the (4, 4, 4)-triangle group such that

$$(2.5) \quad \text{Tr}(I_1 I_2 I_3 I_2) = 1 - 2 \cos \frac{2\pi}{n}$$

This is equivalent to

$$(2.6) \quad \cos t = -\frac{2 + \cos \frac{2\pi}{n}}{2\sqrt{2}}$$

and one checks that this makes sense only for $n \geq 5$.

The group $G(4, 4, 4; 7)$ is studied in great detail in [S1]. It has infinite covolume, and Schwartz shows that its manifold at infinity has a real hyperbolic structure. Note that the main technique there is to construct a fundamental domain on the boundary.

The groups corresponding to $n = 5$ and $n = 6$ are mentioned in [S2] as being difficult to analyze, even though they are known to be discrete. It is natural to suspect the difficulty to come from the fact that their limit set is the whole boundary $\partial H_{\mathbb{C}}^2$. In fact this is the case only for $n = 5$, which turns out to yield a cocompact lattice (this is the main result of the present paper, see Theorem 3.1).

From this point on, we focus on analyzing the group $G(4, 4, 4; 5)$. In that case, using the fact that $\cos(2\pi/5) = (\sqrt{5} - 1)/4$, we have

$$(2.7) \quad t = \arccos\left(-\frac{9 - \sqrt{5}}{8\sqrt{2}}\right) \approx 2.211616098$$

and

$$(2.8) \quad \varphi^3 = -\frac{9 - \sqrt{5}}{8\sqrt{2}} + i\sqrt{3}\frac{3 + \sqrt{5}}{8\sqrt{2}}$$

One can get an explicit expression for φ itself, by rewriting

$$(2.9) \quad \varphi^3 = -\bar{\omega}\frac{\sqrt{5} + 3i\sqrt{3}}{4\sqrt{2}}$$

and noting that

$$(2.10) \quad -\frac{\sqrt{5} + 3i\sqrt{3}}{4\sqrt{2}} = \left(\frac{\sqrt{5} - i\sqrt{3}}{2\sqrt{2}} \right)^3$$

Accordingly, we write

$$(2.11) \quad \varphi = -\bar{\omega} \frac{\sqrt{5} - i\sqrt{3}}{2\sqrt{2}} e^{\pi i/9} = \frac{\sqrt{5} - i\sqrt{3}}{2\sqrt{2}} e^{4\pi i/9}$$

One could of course multiply this value by ω without changing the group, which is really determined by φ^3 (the above value is chosen so that its argument is closest to 0).

Recall that the integers in $\mathbb{Q}(\sqrt{5})$ are given by $\mathbb{Z} + \mathbb{Z}\frac{1 + \sqrt{5}}{2}$, so that φ^3/r^3 is an algebraic integer. We shall see that more can be said along those lines (see Proposition 2.5).

Note that the phase shift is not rational (in the terminology of [M1]), i.e. φ is not a root of unity or, in other words, t is not a rational multiple of π . Perhaps surprisingly, other well chosen mirrors of complex reflections in the group do have rational phase shift, as we now justify (see Proposition 2.4).

We first compute

$$(2.12) \quad I_1 I_2 I_3 = \begin{bmatrix} -3 - 2\sqrt{2}\bar{\varphi}^3 & -2\varphi^2 - \sqrt{2}\bar{\varphi} & \sqrt{2}\varphi + 2\bar{\varphi}^2 \\ -\sqrt{2}\varphi - 2\bar{\varphi}^2 & -1 & \sqrt{2}\bar{\varphi} \\ -\sqrt{2}\bar{\varphi} & -\sqrt{2}\varphi & 1 \end{bmatrix}$$

Its characteristic polynomial is

$$(2.13) \quad \lambda^3 - \frac{3 + \sqrt{5}}{2}\omega\lambda^2 + \frac{3 + \sqrt{5}}{2}\bar{\omega}\lambda - 1$$

$$(2.14) \quad = \bar{\omega}(\lambda - \omega) \left[(\bar{\omega}\lambda)^2 - \frac{1 + \sqrt{5}}{2}(\bar{\omega}\lambda) + 1 \right]$$

Noting that $\frac{1 + \sqrt{5}}{2} = 2 \cos \frac{\pi}{5}$, we find that the eigenvalues of I_{123} are

$$(2.15) \quad \omega, \omega e^{\pi i/5}, \omega e^{-\pi i/5}$$

In particular, $(I_{123})^5$ is a complex reflection, whose mirror appears in the fundamental domain for our group. One checks that the mirror corresponds to v_{123}^\perp , where

$$(2.16) \quad v_{123} = [e^{-\pi i/9}, 1, e^{\pi i/9}]^T$$

The latter vector is an eigenvector for I_{123} with eigenvalue ω . Its other eigenvectors can be written as

$$(2.17) \quad u_{123} = [e^{-2\pi i/45}, 1, e^{2\pi i/45}]^T, \quad w_{123} = [e^{-38\pi i/45}, 1, e^{38\pi i/45}]^T$$

The negative vector among the three is u_{123} , giving the isolated fixed point of I_{123} in hyperbolic space (it has eigenvalue $e^{\pi i/5}\omega$ for I_{123}). This point will be one of the vertices of our fundamental domain.

Proposition 2.4.

$$(1) \quad \frac{|\langle v_{123}, v_{231} \rangle|}{\sqrt{\langle v_{123}, v_{123} \rangle \langle v_{231}, v_{231} \rangle}} = \frac{1}{2 \sin \frac{\pi}{10}}$$

$$(2) \quad \arg(\langle v_{123}, v_{231} \rangle \langle v_{231}, v_{312} \rangle \langle v_{312}, v_{123} \rangle) = -\frac{2\pi}{3}$$

Proof: The result follows from a tedious calculation.

$$\begin{aligned} \langle v_{123}, v_{123} \rangle &= 3 - 2\Re\{2r\varphi e^{-\pi i/9} + r\bar{\varphi} e^{-2\pi i/9}\} \\ &= 3 - 2\Re\{2r\bar{\varphi} e^{\pi i/9} (2 - \omega)\} \\ &= 3 + 2\Re\left\{\frac{\sqrt{5} + i\sqrt{3}}{2}(2\omega - \bar{\omega})\right\} \\ &= \frac{3 - \sqrt{5}}{4} = 2 \sin^2 \frac{\pi}{10} \end{aligned}$$

Now we compute

$$\begin{aligned} \langle v_{123}, v_{231} \rangle &= 2e^{\pi i/9} + e^{-2\pi i/9} - 3r\varphi - r\bar{\varphi}(2e^{-\pi i/9} + e^{2\pi i/9}) \\ &= e^{\pi i/9}(2 - \omega) + 3\bar{\omega} \frac{\sqrt{5} - i\sqrt{3}}{4} e^{\pi i/9} + \omega \frac{\sqrt{5} + i\sqrt{3}}{4} e^{-2\pi i/9} (2 - \bar{\omega}) \\ &= e^{\pi i/9} \left\{ 2 - \omega + 3\bar{\omega} \frac{\sqrt{5} - 1}{4} + 3\bar{\omega} \left(-\frac{\omega}{2}\right) + (2 + 3\omega) \frac{\sqrt{5} - 1}{4} - (2 + 3\omega) \frac{\bar{\omega}}{2} \right\} \\ &= e^{\pi i/9} \left\{ 2 - \omega - \frac{3}{2} - \bar{\omega} - \frac{3}{2} + \frac{\sqrt{5} - 1}{4} (2 + 3\omega + 3\bar{\omega}) \right\} \\ &= e^{\pi i/9} \left\{ -1 - \omega + 1 + \omega + \frac{\sqrt{5} - 1}{4} (-1) \right\} \\ &= \frac{1 - \sqrt{5}}{4} e^{\pi i/9} \end{aligned}$$

This means that $|\langle v_{123}, v_{231} \rangle| = \frac{\sqrt{5} - 1}{4} = \sin \frac{\pi}{10}$, from which the result of the proposition easily follows. Note that the triple Hermitian inner product is

$$(2.18) \quad \left(-\frac{\sqrt{5} - 1}{4} e^{\pi i/9}\right)^3 = \left(\frac{\sqrt{5} - 1}{4}\right)^3 e^{-2\pi i/3}$$

The angle between the two complex geodesics v_{123}^\perp and v_{231}^\perp implies that reflections of order 10 along those mirrors would braid. We do not have such reflections in our group, but I_{123} is regular elliptic and its fifth power is a reflection of order 2. In other words, our group shares a subgroup with Mostow's group $\Gamma(10, 2/3)$. The latter is commensurable to a hypergeometric monodromy group Γ_μ , where $\mu = (1, 6, 6, 6, 11)/15$ (see [DM] or [Th]). Note that this Γ_μ is not discrete, see [M2] (in fact, it is easy to find a non-discrete triangle subgroup). In particular the common subgroup has infinite index in one of the two groups. \square

In fact more can be said about coordinates given by the v_{ijk} :

Proposition 2.5. *The group $G(4, 4, 4; 5)$ can be conjugated to consist of matrices with entries in the ring of integers in $\mathbb{Q}(\sqrt{5}, \omega)$.*

Proof: This follows from a direct calculation, based on Proposition 2.4. We shall write $\tau = \frac{\sqrt{5}-1}{2}$. In the basis $\sqrt{\frac{2}{\tau}}\{v_{312}, e^{\pi i/9}v_{123}, e^{2\pi i/9}v_{231}\}$, one verifies that I_1 , I_2 and I_3 are given respectively by the matrices

$$(2.19) \quad \begin{bmatrix} -1 & 0 & 0 \\ \tau\bar{\omega} & 0 & -1 \\ -\tau\bar{\omega} & -1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & \tau\omega & \bar{\omega} \\ 0 & -1 & 0 \\ \omega & \tau\bar{\omega} & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 & -\tau\omega \\ -1 & 0 & \tau\omega \\ 0 & 0 & -1 \end{bmatrix}$$

and the Hermitian form is

$$(2.20) \quad \begin{bmatrix} \tau & -1 & \omega \\ -1 & \tau & -1 \\ \bar{\omega} & -1 & \tau \end{bmatrix}$$

Observe that the entries of the above two matrices are algebraic integers. Indeed the integers in $\mathbb{Q}(\sqrt{5})$ are given by $\mathbb{Z}[\tau]$, and the integers in $\mathbb{Q}(\omega)$ by $\mathbb{Z}[\omega]$. \square

Corollary 2.6. *The group $\Gamma = G(4, 4, 4; 5)$ is discrete.*

Proof: Observe that $\tau^2 = 1 - \tau$, and compute the determinant of the Hermitian matrix (2.20), which is

$$(2.21) \quad \tau^3 - 1 - 3\tau = -2 - \tau = \frac{-5 - \sqrt{5}}{2} < 0$$

Note in particular that the form H has signature $(2, 1)$, since $\det(H) < 0$ and there exist vectors that are positive for H (the diagonal entries are equal to $\tau > 0$).

The determinant remains negative when applying $\sqrt{5} \mapsto -\sqrt{5}$, so the conjugate form H^σ has signature either $(0, 3)$ or $(2, 1)$. To distinguish

between those two cases, we consider the span of the first two basis vectors (let us call these v_1 and v_2).

$$(2.22) \quad \left\| v_2 - \frac{\langle v_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 \right\|^2 = \frac{\langle v_1, v_1 \rangle \langle v_2, v_2 \rangle - |\langle v_1, v_2 \rangle|^2}{\langle e_1, e_1 \rangle}$$

For either Hermitian form H or H^σ , the latter expression is equal to $\frac{\tau^2-1}{\tau} = \frac{(\tau^\sigma)^2-1}{\tau^\sigma} = -1$. For H^σ , this exhibits two orthogonal negative vectors, hence H^σ is negative definite.

The integers of $\mathbb{Q}(\sqrt{5}, \omega)$, embedded in $\mathbb{C} \times \mathbb{C}$ by the product of two non complex embeddings $\mathbb{Q}(\sqrt{5}, \omega)$, is a discrete set, hence $\Gamma \times \Gamma^\sigma$ is discrete in $U(H) \times U(H^\sigma)$. The group $U(H^\sigma)$ is compact, hence the projection onto $U(H)$ maps discrete subsets to discrete subsets. \square

Remark 2.7. (1) *The above shows that Γ , if a lattice, is in fact an arithmetic subgroup of $U(H) \approx U(2, 1)$.*

(2) *The matrices (2.19) make it easy to test equality between words in the generators simply by performing algebra in $\mathbb{Z}[\tau, \omega]$, which reduces to repeated use of the relations $\tau^2 = 1 - \tau$, $\omega^2 = -1 - \omega$. We shall not make much use of this observation in the present paper.*

3. PROOF OF COCOMPACTNESS

Theorem 3.1. *The group $\Gamma = G(4, 4, 4; 5)$ is a cocompact arithmetic lattice.*

We construct a bounded polyhedron that contains a fundamental domain for the action of the group. Since we know that the group is discrete, the Dirichlet construction is guaranteed to provide a fundamental domain (possibly with infinitely many faces, see [GP]).

We shall use the coordinates given by Hermitian form (2.1), where generators have matrices of the form (2.3). We take for the center p_0 of the domain the natural point fixed by the symmetry of order three given by J , i.e. $p_0 = [1, 1, 1]^T$.

$$(3.1) \quad F = \{x \in H_{\mathbb{C}}^2 : d(x, p_0) \leq d(x, \gamma p_0) \forall \gamma \in \Gamma\}$$

It is quite difficult however to find the combinatorics of the polyhedron F , and we shall only state the results without proof in section 7. Observe however that F is of course contained in any ‘‘partial’’ Dirichlet domain

$$(3.2) \quad F_W = \{x \in H_{\mathbb{C}}^2 : d(x, p_0) \leq d(x, \gamma p_0) \forall \gamma \in W\}$$

where W is any finite set of group elements.

We shall choose a certain small set of words W , making the analysis of the combinatorics more tractable. The following result clearly implies 3.1, since $F \subset F_W$.

Theorem 3.2. *Let W consist of all group elements obtained from words of length three in the generators I_1, I_2 and I_3 . Then the polyhedron F_W is bounded.*

One advantage of using the set W is that it is much smaller than the actual set of elements needed for the Dirichlet fundamental domain (see section 7). More importantly, it turns out that all intersections of bisectors defining F_W are transverse, which simplifies matters greatly to determine the combinatorics.

Recall that the bisector $\hat{\gamma}$ corresponding to a group element γ is the hypersurface in $H_{\mathbb{C}}^2$ equidistant between p_0 and γp_0 :

$$(3.3) \quad \hat{\gamma} = \{x \in H_{\mathbb{C}}^2 : d(x, p_0) = d(x, \gamma p_0)\}$$

Their intersections can be quite complicated, see [G], and are in general not connected nor transverse.

Theorem 3.2 can be reformulated as follows.

Proposition 3.3. *All the 2-faces of the polyhedron F_W are bounded.*

We first prove that the theorem is indeed a consequence of the proposition. **Proof:** The polyhedron is defined by a number of closed inequalities, so we only need to show that it is bounded. It is enough to show that for each $\gamma \in W$ the 3-face $\tilde{\gamma} := \hat{\gamma} \cap F_W$ is bounded. Indeed, suppose that this is the case, and consider $Z = F_W \cap \partial H_{\mathbb{C}}^2$. It is closed by definition, and it is also open by the hypothesis that the 3-faces are bounded. Now Z is clearly not the whole of $\partial H_{\mathbb{C}}^2 \simeq S^3$, so it must be empty.

Note that the bisector $\hat{\gamma}$ is diffeomorphic to an open ball, and its trace on the boundary is a smooth 2-sphere. The corresponding 3-face is a polyhedron in that 3-ball, defined by a finite number of closed inequalities (corresponding to the half spaces bounded by the other $\hat{\alpha}$, $\alpha \in W$). The same topological argument as above shows that the 3-dimensional polyhedron is bounded if and only if all of its 2-faces are bounded. \square

Remark 3.4. *Note that the argument shows that it suffices to prove that all the 1-faces are bounded. This in turn can be done by analyzing all possible intersections $\hat{\alpha} \cap \hat{\beta} \cap \hat{\gamma}$, which then becomes a simple problem about whether finitely many intervals are compact (having the advantage that it amounts to finitely many verifications). We shall explain a similar approach in section 6.3.*

We defer the proof of 3.3 to section 5, where we shall use the computer to analyze the 2-faces of F_W individually, each of which is contained in the intersection of two bisectors $\widehat{\gamma}_1 \cap \widehat{\gamma}_2$. Since there are twelve group elements in W and two types of 3-faces, we need to check the claim only for twenty-two such intersections.

Moreover, there is a natural symmetry between the generators, given by antiholomorphic involutions σ_{ij} that conjugates I_i into I_j (and I_k into itself). For instance,

$$(3.4) \quad \sigma_{12} : [x_1, x_2, x_3]^T \mapsto [\bar{x}_2, \bar{x}_1, \bar{x}_3]^T$$

This is clearly an isometry in view of the Hermitian form (2.1). Note that the product of any two such involutions is given by the 3-cycle J or its inverse.

All objects we define will be indexed by integers so that the above symmetries simply corresponds to permutations of the indices $\{1, 2, 3\}$. The finite set W is preserved by the symmetry, hence the polyhedron F_W only has two isometry type of faces, contained on bisectors for I_{iji} and I_{ijk} respectively (here and in what follows we mean the indices i , j and k to be distinct integers between 1 and 3).

4. INTERSECTIONS OF BISECTORS

In order to analyze the 2-faces of Dirichlet polyhedron, we describe a convenient set of coordinates on them, deduced from the slice decomposition of bisectors.

Consider the bisector

$$(4.1) \quad B = \{x \in H_{\mathbb{C}}^2 : d(x, p_0) = d(x, p_1)\}$$

The equality of distances can be rewritten as

$$(4.2) \quad |\langle x, p_0 \rangle| = |\langle x, p_1 \rangle|$$

as long as we choose the vectors p_i to have the same square norm (we always do so in the remainder of this section).

In particular, the (non necessarily negative) vectors in \mathbb{C}^3 that satisfy (4.2) are in the orthogonal complement of one and only one vector of the form $p_0 - \alpha p_1$ where $|\alpha| = 1$. This gives a foliation of B by complex geodesics, given by the linear hyperplanes $(p_0 - \alpha p_1)^\perp$, where α ranges through the arc of the unit circle where the square norm of $p_1 - \alpha p_2$ is positive. This is the so-called **slice decomposition** of bisectors, described in [M1] or [G]. The circle of vectors $p_0 - \alpha p_1$ intersects the ball in a geodesic that is contained in B , called the **real spine** of B . The complex geodesic that contains it is the **complex spine** of B .

Under some genericity assumption to be discussed shortly, we can parameterize the intersection of two bisectors by their respective spinal coordinates. More precisely, consider the equidistant loci between p_0 , p_1 and p_2 . Let T denote its extension to projective space, which is the projectivization of the set of nonzero vectors that satisfy

$$(4.3) \quad |\langle x, p_0 \rangle| = |\langle x, p_1 \rangle| = |\langle x, p_2 \rangle|$$

There is a map $T \rightarrow S^1 \times S^1$ that sends $[x]$ to the pair (u_1, u_2) , where $x \in (p_0 - u_1 p_1)^\perp \cap (p_0 - u_2 p_2)^\perp$. This is well defined, and in fact

$$(4.4) \quad u_j = \frac{\langle x, p_j \rangle}{\langle x, p_0 \rangle}$$

The denominator cannot be zero if the three points are not in a common complex geodesic, i.e. if the three vectors p_0 , p_1 and p_2 are linearly independent in \mathbb{C}^3 . Indeed, if any of the inner products in equation (4.3) is zero, then all of them are, hence x is zero. Note that (u_1, u_2) give inhomogeneous coordinates on the ball.

One can recover the homogeneous coordinates, since the u_j determine the three inner products $\langle x, p_j \rangle$ up to a multiple, which in turn determines the complex line $[x]$. Concretely, writing P for the matrix that passes from the standard basis to $\{p_0, p_1, p_2\}$, we can write

$$(4.5) \quad x = (\overline{P}^T \overline{H})^{-1} U$$

where $U^T = [1, u_1, u_2]$. This corresponds to normalizing the homogeneous coordinates so that $\langle x, p_0 \rangle = 1$. We shall sometimes write the above change of coordinates as

$$(4.6) \quad x = \mathbf{a}_0 + \mathbf{a}_1 u_1 + \mathbf{a}_2 u_2$$

so that the vectors \mathbf{a}_j are the columns of $A = (\overline{P}^T \overline{H})^{-1}$.

4.1. Cospinal bisectors. If the three points are in a common geodesic, then the complex spines of the two bisectors coincide, and they are called **cospinal** bisectors. In that case it is easy to see that the bisectors intersect if and only if their real spines intersect, in which case the intersection consists of one complex geodesic, orthogonal to their common complex spine, through the intersection of the real spines.

Since this situation occurs in the context of Dirichlet domains, it is important to be able to describe the intersection of a complex geodesic C with another bisector (or with the corresponding half spaces). This is done in detail in [G], and we only review some of the results that we need.

We take an orthonormal basis $\{v_1, v_2\}$ for the complex 2-plane that projects to the geodesic, so that $\langle v_1, v_2 \rangle = 0$, $\langle v_1, v_1 \rangle = -1$ and

$\langle v_2, v_2 \rangle = 1$. This identifies the complex geodesic with the unit disk $|z| < 1$ by taking vectors to be of the form $v_1 + zv_2$.

It is then easy to describe the intersection $B(p_0, p_3) \cap C$ of any bisector with C . Indeed in projective space one simply has the following equation

$$(4.7) \quad |\langle v_1, p_0 \rangle + z\langle v_2, p_0 \rangle| = |\langle v_1, p_3 \rangle + z\langle v_2, p_3 \rangle|$$

which yields a circle (in degenerate cases, it could be empty, or a line, or the whole complex plane). In general this circle is not a geodesic, but it is a hypercycle, i.e. it is at a constant distance from a certain geodesic (see [G]). One checks that it is a geodesic if and only if C intersects the real spine of the bisector, possibly in projective space (see [DFP]).

The 2-faces of a Dirichlet domain F_W that lie on complex geodesics are obtained by intersecting a number of regions delimited by arcs of circles in the unit disk (note that in general such a region is not convex in the hyperbolic metric, contrary to the claim in Lemma 3.3.1 of [M1]). We shall see an example when describing a fundamental domain for our lattice.

4.2. Generic intersections. If the three points are not in a common complex geodesic, we shall say that the two bisectors have **generic intersection**. The above map is then a diffeomorphism from T onto the torus, and the intersection of T with complex hyperbolic space is a smooth, non totally geodesic disk (the fact that the intersection is connected is not obvious, see Proposition 4.3).

Definition 4.1. *We call T a **Giraud torus** and its intersection with complex hyperbolic space $B_1 \cap B_2 = T \cap H_{\mathbb{C}}^2$ a **Giraud disk**.*

The terminology comes from the fact that these disks appear in the theorem of Giraud, see Theorem 4.4, which turns out to be crucial in analyzing Dirichlet domains.

Concretely, this disk can be found explicitly by solving $\langle x, x \rangle \leq 0$ with $x = \mathbf{a}_0 + \mathbf{a}_1 u_1 + \mathbf{a}_2 u_2$ and $|u_1| = |u_2| = 1$ (see (4.6)). We shall write

$$(4.8) \quad f(u_1, u_2) = \langle \mathbf{a}_0 + \mathbf{a}_1 u_1 + \mathbf{a}_2 u_2, \mathbf{a}_0 + \mathbf{a}_1 u_1 + \mathbf{a}_2 u_2 \rangle$$

It is easy to see that, for each u_1 , there is a (possibly empty) interval of values of u_2 that satisfy $f(u_1, u_2) \leq 0$.

One way to check this is to use the following elementary fact, that will be used several times throughout the paper.

Lemma 4.2. *Let $\beta \in \mathbb{R}$. The equation*

$$(4.9) \quad 2\Re(\alpha z) = \beta$$

has a solution on the unit circle $|z| = 1$ if and only if

$$(4.10) \quad |\beta| \leq 2|\alpha|$$

There is exactly one solution when equality holds, and the solution is arbitrary if and only if $\alpha = \beta = 0$. Otherwise there are two values of z , corresponding to intersecting two circles in the plane.

The solutions can be obtained for instance by rewriting equation (4.9) as the following quadratic equation

$$(4.11) \quad \alpha z^2 - \beta z + \bar{\alpha} = 0$$

whose solutions are on the unit circle if and only if condition (4.10) holds. One gets

$$(4.12) \quad z = \frac{\beta \pm i\sqrt{4|\alpha|^2 - \beta^2}}{2\alpha}$$

For a given value of u_1 , the values of u_2 for which $f(u_1, u_2) = \|\mathbf{a}_0 + \mathbf{a}_1 u_1 + \mathbf{a}_2 u_2\|^2 = 0$ are obtained by solving an equation of the form (4.9), where

$$(4.13) \quad \alpha = \alpha(u_1) = -\langle \mathbf{a}_2, \mathbf{a}_0 + \mathbf{a}_1 u_1 \rangle$$

$$(4.14) \quad \beta = \beta(u_1) = -\|\mathbf{a}_0 + \mathbf{a}_1 u_1\|^2 - \|\mathbf{a}_2\|^2$$

Hence, for each u_1 , there is an interval of values of u_2 for which $f \leq 0$. It can be checked that, when nonempty, this interval yields a hypercycle in the corresponding complex slice of B_1 .

Determining the Giraud disk amounts to finding the values of u_1 such that the interval is non empty. Using (4.10), this translates into an inequality of the form

$$(4.15) \quad |\mu_0 + \mu_1 u_1 + \bar{\mu}_1 \bar{u}_1| \leq 2|\nu_0 + \nu_1 u_1|$$

where

$$(4.16) \quad \mu_0 = \langle \mathbf{a}_0, \mathbf{a}_0 \rangle + \langle \mathbf{a}_1, \mathbf{a}_1 \rangle + \langle \mathbf{a}_2, \mathbf{a}_2 \rangle$$

$$(4.17) \quad \mu_1 = \langle \mathbf{a}_1, \mathbf{a}_0 \rangle$$

$$(4.18) \quad \nu_0 = \langle \mathbf{a}_0, \mathbf{a}_2 \rangle$$

$$(4.19) \quad \nu_1 = \langle \mathbf{a}_1, \mathbf{a}_2 \rangle$$

After squaring both sides, this can be rewritten as

$$(4.20) \quad \tau_0 + \tau_1 u_1 + \bar{\tau}_1 \bar{u}_1 + \tau_2 u_1^2 + \bar{\tau}_2 \bar{u}_1^2 \leq 0$$

For completeness, we give a formula for the coefficients that appear:

$$(4.21) \quad \tau_0 = \mu_0^2 + 2|\mu_1|^2 - 4|\nu_0|^2 - 4|\nu_1|^2$$

$$(4.22) \quad \tau_1 = 2\mu_0\mu_1 - 4\bar{\nu}_0\nu_1$$

$$(4.23) \quad \tau_2 = \mu_1^2$$

The left hand side of (4.20) is a degree 2 expression in the real and imaginary parts of u_1 , so equality holds along a conic. It is quite clear from the geometry of our situation that the conic will not be the unit circle, so that it intersects $|u_1| = 1$ in at most four points.

In particular, there are at most two intervals for u_1 where $f \leq 0$ has solutions, and each interval corresponds to a connected component of the intersection of the bisectors (indeed, the intersection fibers over each interval, with intervals as fibers).

Proposition 4.3. *The intersection of two coequidistant bisectors is a smooth disk (in particular it is connected).*

This result is proved by Goldman in [G], by analyzing in detail the possible tangencies between bisectors. Note that the above analysis makes it clear that, for general bisector intersections, there are at most two connected components, and that the number of components can be found by plotting the graph of $\Delta(u_1) = \beta(u_1)^2 - 4|\alpha(u_1)|^2$ (once again, for a pair of coequidistant bisectors, we know this has at most two zeros on the unit circle).

For concreteness, we now describe one specific example, for $p_1 = I_{121}p_0$ and $p_2 = I_{131}p_0$. We write $u_1 = e^{2\pi it_1}$, $0 \leq t_1 \leq 1$ and plot the graph of the discriminant as a function of t_1 in Figure 1. The minimum

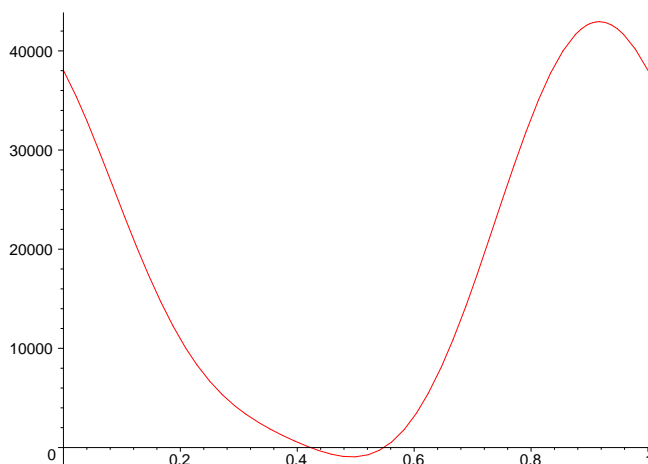


FIGURE 1. The graph of $\beta^2 - 4|\alpha|^2$, as a function of $\frac{1}{2\pi}\arg(u_1)$.

and maximum values of t_1 on the boundary of the ball, up to five decimal places, are given by $t_1^{min} = 0.42219\dots$ and $t_1^{max} = 0.54845\dots$

The trace of the boundary of the ball on the torus T can be plotted by using formula (4.12), and plotting the two branches $u_2 = \phi_{\delta}^{\pm}(u_1)$. For convenience replace u_j by $t_j = \arg(u_j)/(2\pi)$, with a little care needed in order to choose the argument in a continuous fashion (in the case at hand it is easier to choose the argument between 0 and 2π).

One obtains the graph of $\widehat{121} \cap \widehat{131} \cap \partial H_{\mathbb{C}}^2$, given in Figure 2. Note

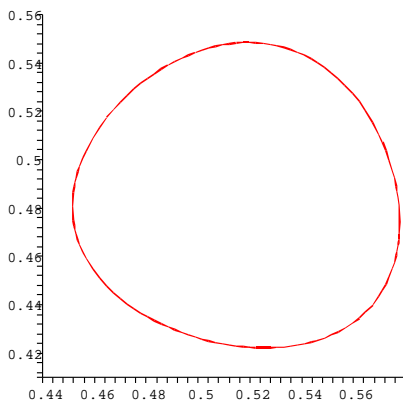


FIGURE 2. The trace on the torus $u_1 = e^{2\pi it_1}$, $u_2 = e^{2\pi it_2}$ of the boundary of the ball.

that the maximum value of ϕ^+ and the minimum value of ϕ^- can be obtained by repeating the above process, now writing u_1 as a function of u_2 . One finds that $t_2^{min} = 0.45155\dots$ and $t_2^{max} = 0.57781\dots$

4.3. Intersection of three bisectors. Given a third bisector $B_3 = B(p_0, p_3)$, it is clear that the intersection $T \cap B_3$ has an equation of the form

$$(4.24) \quad |\delta_0 + \delta_1 u_1 + \delta_2 u_2| = 1$$

where $\delta_j \in \mathbb{C}$ are simply obtained by expressing $p_3 = \sum_{j=0}^2 \bar{\delta}_j p_j$. The corresponding region closer to p_0 than p_3 is given by the inequality $|\delta_0 + \delta_1 u_1 + \delta_2 u_2| \geq 1$.

The solution set of equation (4.24) in the torus $|u_1| = |u_2| = 1$ is a curve, unless p_3 is equal to one of the three points p_0 , p_1 and p_2 (i.e. only one of the δ_j is nonzero). This is one way to formulate Giraud's theorem (see [G]):

Theorem 4.4. *Suppose p_0, p_1 and p_2 are not contained in a common geodesic. Then their disk of intersection is not totally geodesic, and it is contained in precisely three bisectors, namely $\mathcal{B}(p_0, p_1)$, $\mathcal{B}(p_1, p_2)$, $\mathcal{B}(p_2, p_0)$.*

The curves (4.24) can have somewhat interesting topology, and in particular they are often reducible. We shall describe in detail how to plot them in section 4.4.

The curves $u_1 = c$ are by definition the intersection of the torus T with the complex slices of B_1 . Of course the roles of u_1 and u_2 are symmetric, so $u_2 = c$ gives the slices of B_2 . The geometric interpretation also suggests that u_1, u_2 behave similarly as $\bar{u}_1 u_2 = u_2/u_1$. Indeed, the curves $u_2 = cu_1$, $|c| = 1$, are in the complex slices of the third bisector $B(p_1, p_2)$.

Definition 4.5. *We shall refer to intersections of T with the complex slices of any of the three bisectors intersecting in T (see Proposition 4.3) as the complex slices of T . More specifically, we shall refer to the curves $u_1 = c$, $u_2 = c$ and $u_2 = cu_1$ with $|c| = 1$ as vertical, horizontal and diagonal complex slices of T .*

In order to determine the 2-faces of the Dirichlet domain F_W , we look at the pairs w_1, w_2 of group elements in W such that the corresponding bisectors are not cospinal (the cospinal case is much easier, and was discussed in the previous section). The 2-face is contained in the disk $\hat{w}_1 \cap \hat{w}_2$, and it is delimited by the various \hat{w} , $w \in W, w \neq w_1, w_2$. Note that the 2-dimensional region delimited on our disk is piecewise smooth, but not necessarily connected.

The corresponding 2-face of F_W are obtained by solving a system of inequalities, written as

$$(4.25) \quad |\delta_0^{(w)} + \delta_1^{(w)}u_1 + \delta_2^{(w)}u_2| \geq 1$$

for each $w \in W$, $w \neq w_1, w_2$.

This can be done by plotting the various curves $|\delta_0^{(w)} + \delta_1^{(w)}u_1 + \delta_2^{(w)}u_2| = 1$ on the torus, and analyzing the various connected components cut out on the corresponding Giraud disk. We shall describe a number of examples in section 5.

4.4. Parameterizing the 1-faces. As discussed in the previous section, the intersection of three coequidistant bisectors two of which are not cospinal can be written in the natural torus coordinates $|u_1| = |u_2| = 1$ in the form

$$(4.26) \quad |\delta_0 + \delta_1 u_1 + \delta_2 u_2| = 1$$

We now explain in detail how plotting 4.26 on the torus amounts to solving a family of quadratic equations.

4.4.1. *Degenerate case.* We first consider the special case where one of the δ_j is equal to zero, which occurs precisely when one pair of bisectors among the three is cospinal (see equation 4.24).

Say for instance that $\delta_2 = 0$ (we assume also that none of the other two δ_j is zero, otherwise two of the bisectors are equal). Geometrically this means that first and third bisectors are cospinal, hence their intersection is either a complex geodesic or empty, depending on whether their real spines intersect or not.

On the level of projective space, it is clear that $|\delta_0 + \delta_1 u_1| = 1$ yields 0, 1 or 2 values $u_1 = u_1^{(i)}$ such that u_2 is arbitrary, obtained by intersecting the circle $\delta_0 + \delta_1 u_1$ with the unit circle. At most one of the circles $u_1 = u_1^{(i)}$ can intersect the ball, because of the above geometric interpretation.

4.4.2. *General case.* Suppose now that $\delta_j \neq 0$ for all j . We concentrate on plotting u_2 as a function of u_1 . For each u_1 , we need to find the u_2 's that satisfy

$$(4.27) \quad 2\Re((\bar{\delta}_0 + \bar{\delta}_1 \bar{u}_1)\delta_2 u_2) = 1 - |\delta_0 + \delta_1 u_1|^2 - |\delta_2|^2$$

Solving the corresponding quadratic equation as in 4.11 yields

$$(4.28) \quad u_2 = \frac{\beta \pm i\sqrt{4|\alpha|^2 - \beta^2}}{2\alpha}$$

where

$$(4.29) \quad \alpha = (\bar{\delta}_0 + \bar{\delta}_1 \bar{u}_1)\delta_2$$

$$(4.30) \quad \beta = 1 - |\delta_0 + \delta_1 u_1|^2 - |\delta_2|^2$$

The solution is on the unit circle if and only if the expression under the squareroot is positive.

Formula (4.28) makes sense only as long as $\delta_0 + \delta_1 u_1 \neq 0$, which can occur for at most one value of u_1 , namely $u_1 = -\delta_0/\delta_1$. This is on the torus if and only if $|\delta_0| = |\delta_1|$, and one then gets a “vertical line” on the graph (i.e. u_2 is arbitrary for that value of u_1) if and only if $|\delta_2| = 1$. The latter case can be given a geometric interpretation, considering the definition of the coefficients δ_j . Indeed, one has

$$\delta_0 p_0 + \delta_1 p_1 = p_3 - \delta_2 p_2$$

which means that the two real spines of $B(p_0, p_1)$ and $B(p_2, p_3)$ intersect. Since it comes up somewhat frequently in Dirichlet fundamental domains, we push the analysis a little further. Note however that for

the partial Dirichlet domain used defined in the previous section, using only words of length three, all intersections are completely generic so we do not need to handle this difficulty in order to get Theorem 3.1.

Lemma 4.6. *Suppose $|\delta_0| = |\delta_1|$ and $|\delta_2|=1$. Then the circle $u_1 = -\delta_0/\delta_1$ is contained in the curve (4.26). Moreover,*

$$\lim_{u_1 \rightarrow (-\delta_0/\delta_1)^+} u_2^+ = \lim_{u_1 \rightarrow (-\delta_0/\delta_1)^-} u_2^- = -\frac{\delta_0}{|\delta_0|} \frac{1}{\delta_2}$$

$$\lim_{u_1 \rightarrow (-\delta_0/\delta_1)^-} u_2^+ = \lim_{u_1 \rightarrow (-\delta_0/\delta_1)^+} u_2^- = +\frac{\delta_0}{|\delta_0|} \frac{1}{\delta_2}$$

Note that it is clear from equation (4.28) that there are indeed two determinations of u_2 near the value $u_1 = -\delta_0/\delta_1$, which we denote by u_2^\pm . The meaning of the symbol $\lim_{u_1 \rightarrow u^\pm}$, where $|u_1| = 1$, stands for a one-sided limit with decreasing or increasing argument, respectively.

Proof: With the hypotheses of the lemma, equation (4.28) becomes

$$(4.31) \quad u_2^\pm = \frac{-|\delta_0 + \delta_1 u_1|^2 \pm i\sqrt{4|\delta_0 + \delta_1 u_1|^2 - |\delta_0 + \delta_1 u_1|^4}}{2(\bar{\delta}_0 + \bar{\delta}_1 u_1)\delta_2}$$

$$(4.32) \quad = \frac{\delta_0 + \delta_1 u_1}{|\delta_0 + \delta_1 u_1|} \cdot \frac{-|\delta_0 + \delta_1 u_1| \pm i\sqrt{4 - |\delta_0 + \delta_1 u_1|^2}}{2\delta_2}$$

The second fraction clearly tends to $\pm i/\delta_2$, and the second one is analyzed as follows.

Let us write $\delta_1 = \delta_0 e^{i\mu}$ and $u_1 = e^{i\theta}$, and compute

$$(4.33) \quad \frac{\delta_0 + \delta_1 u_1}{|\delta_0 + \delta_1 u_1|} = \frac{\delta_0}{|\delta_0|} \cdot \frac{1 + e^{i(\mu+\theta)}}{|1 + e^{i(\mu+\theta)}|}$$

$$(4.34) \quad = \frac{\delta_0}{|\delta_0|} \cdot \frac{1 + \cos(\mu + \theta) + i \sin(\mu + \theta)}{\sqrt{2}\sqrt{1 + \cos(\mu + \theta)}}$$

The result follows at once from the fact that

$$(4.35) \quad \lim_{x \rightarrow \pi^\pm} \frac{1 + \cos x}{\sqrt{1 + \cos x}} = 0$$

$$(4.36) \quad \lim_{x \rightarrow \pi^\pm} \frac{\sin x}{\sqrt{1 + \cos x}} = \mp\sqrt{2}$$

□

We end this section with a useful observation, that implies that none of the graphs we plot can have any complicated oscillations. Denote by Λ the set $|\delta_0 + \delta_1 u_1 + \delta_2 u_2| = 1$ on the torus $|u_1| = |u_2| = 1$.

Lemma 4.7. *Suppose that Λ does not contain any complex slice of the torus T . Then on any complex slice of T , there are at most two points of Λ .*

Proof: The fact that vertical complex slices contain at most two points follows at once from the fact that, for a fixed u_1 , one can solve for u_2 by solving an equation of degree at most two.

For horizontal slices one solves for u_1 in terms of u_2 , and for diagonal ones, one solves for either u_1 or u_2 in terms of $\bar{u}_1 u_2$. \square

5. 2-FACES OF F_W

We now describe the combinatorics of the various 2-faces of F_W , where W consists of all groups elements written as length three words in the generators. The verification for each 2-face relies on computer use quite heavily, and it is highly recommended for the reader to test our claims on their own machine. Computer code that does this in efficiently is available on the author's webpage, see [D2].

It is enough to describe the faces corresponding to $I_1 I_2 I_1$ and $I_1 I_2 I_3$, since the other ones are isometric to one of them by applying the natural symmetry between the generators.

It turns out the face $\widehat{121} \cap F_W$ has eight 2-faces, contained in the intersection with the bisector corresponding to all words in W except 232, 313 and 323 (see Figure 3). The bisectors $\widehat{323}$ does not intersect $\widehat{121}$ (although their analytic continuation to projective space do intersect). The other two intersections $\widehat{121} \cap \widehat{232}$ and $\widehat{121} \cap \widehat{313}$ are both disks in complex hyperbolic space, but their intersection with F_W is empty.

The face $\widehat{123} \cap F_W$ has ten 2-faces, corresponding to all words except 213. All the 2-faces given by the intersection with a bisector corresponding to a word of the form iji is isometric to some 2-face of the face $\widehat{121} \cap F_W$. For instance the isometry σ_{12} , cf. equation (3.4), clearly maps $\widehat{123} \cap \widehat{212}$ to $\widehat{213} \cap \widehat{121}$. We list all the isometries in the table below.

$123 \cap 121$	\xleftrightarrow{Id}	$121 \cap 123$
$123 \cap 212$	$\xleftrightarrow{\sigma_{12}^2}$	$121 \cap 213$
$123 \cap 232$	$\xleftrightarrow{J^{-1}}$	$121 \cap 312$
$123 \cap 323$	$\xleftrightarrow{\sigma_{13}^2}$	$121 \cap 321$
$123 \cap 131$	$\xleftrightarrow{\sigma_{23}^2}$	$121 \cap 132$
$123 \cap 313$	\xleftrightarrow{J}	$121 \cap 231$

Note also that two 2-faces of $\widehat{123} \cap F_W$ are isometric, since J sends $\widehat{123} \cap \widehat{312} \cap F_W$ to $\widehat{231} \cap \widehat{123} \cap F_W$.

Figure 4 gives a picture of the remaining three 2-faces of $\widehat{123} \cap F_W$.

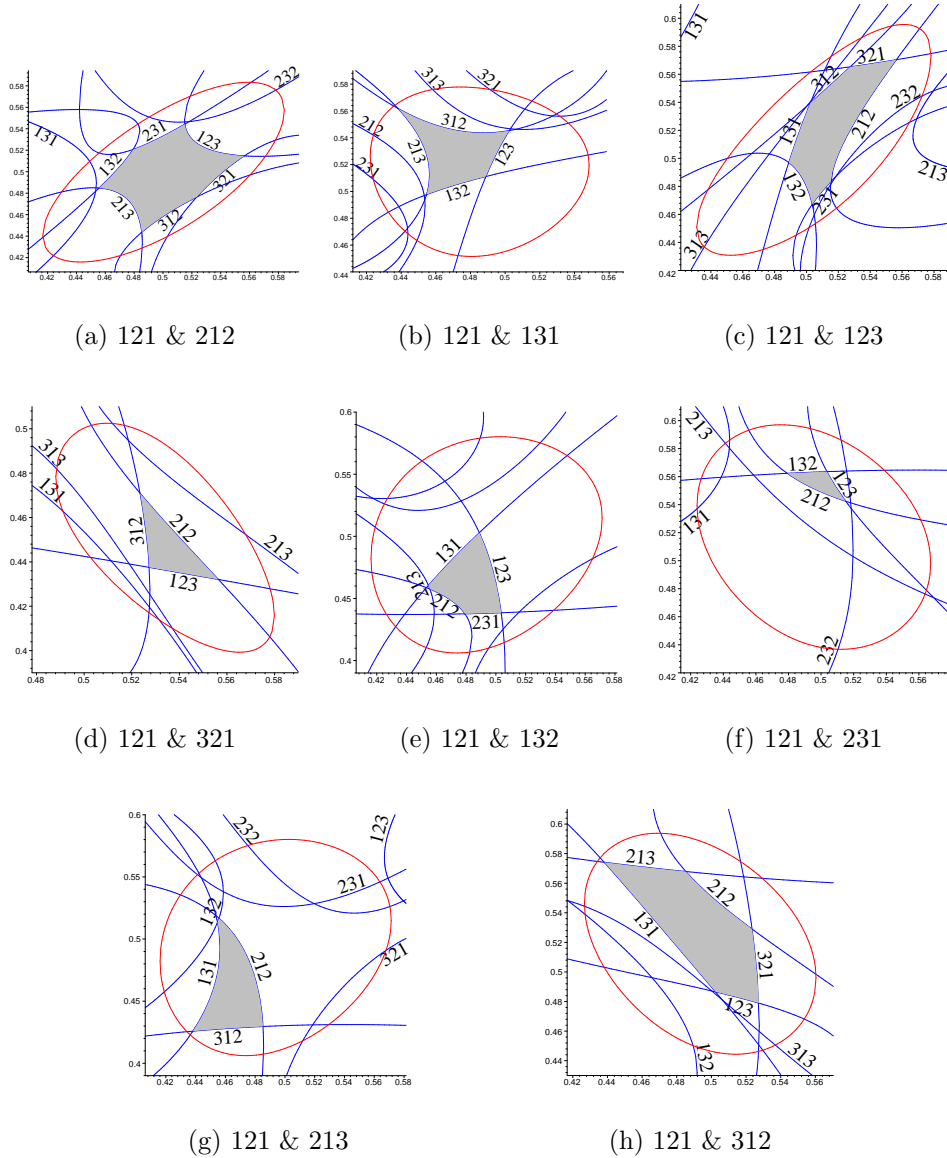


FIGURE 3. The 2-faces of $\widehat{121} \cap F_W$, drawn in spinal coordinates.

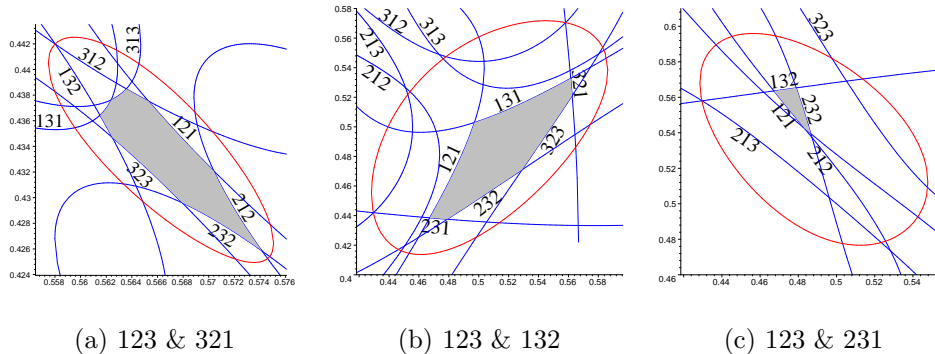


FIGURE 4. The 2-faces of $\widehat{123} \cap F_W$, drawn in spinal coordinates.

6. ISSUES OF PRECISION

The results from the preceding section are satisfactory only as long as we can verify that our pictures of the 2-faces of F_W , produced using computer calculations, are precise enough not to be misleading. We list a couple of natural approaches, the most efficient one being presented in section 6.3.

6.1. Interval calculus. One possible way is to use interval arithmetic throughout the calculations, and replace each plotted point by an interval, large enough that we are sure that the actual graph is contained in it.

There are mainly two difficulties with this approach. Note that we are after a precise enough description of the connected components of the complement of a number of curves in the plane. The interval arithmetic would have the effect of “fattening” each curve (in the vertical direction), and it is not completely clear how this affects the connected components of the complement. In particular, it is not at all clear even that the number of complementary regions would be preserved.

The other difficulty is that we need to obtain some control over possible jumps or oscillations in the graphs, in order for the finite number of values sample points on the horizontal axis to be representative of the behavior of the entire graph (note however that complicated oscillation behaviors are prevented by Lemma 4.7).

We do not discuss the details, but simply remark that using so called “double precision” would suffice by far to plot the above graphs, and in fact it is the level of precision used in [D2].

6.2. Large precision software. Another option is to use mathematical software like Maple, for instance, that allows to perform the calculations with arbitrary precision. This is how the graphs in this paper were produced. The price to pay consists of longer running times for the computer programs (note that the graphs in the paper were all produced in a matter of minutes, on a conventional workstation).

There are still a couple of issues with this approach. The first one is that Maple gives a plot based on a finite number of sample points, and technically one would need to justify that these are representative of the general behavior of the graph.

The other issue is that one needs to be careful when dividing by numbers very close to zero (here the relevant result is Lemma 4.6, see section 4.4). We do not comment on this issue here because it does not happen for any of the graphs relevant to F_W , hence has no bearing on our main theorem. One would have to analyze this carefully in order to justify our conjectural picture of the complete fundamental domain for the group.

6.3. Using topology and basic calculus. A better approach is the following. We have an explicit parameterization of the intersection with the boundary of the ball of a pair of coequidistant bisectors (or rather of the corresponding extors). Recall from section 4 that this is a smooth circle, which we want to prove is completely disjoint from F_W .

It is enough to cover this circle by a finite number of intervals, each of which is entirely outside of some bisector \widehat{w} , $w \in W$, where “outside” means closer to $w p_0$ than to p_0 . Such intervals were of course implicitly defined by Figures 3 and 4, but the advantage of the present formulation is that it now relies on finitely many numerical verifications.

We shall verify all the details for the 2-face corresponding to 121 and 131 , drawn in Figure 3(b). The other 2-faces are of course entirely similar. We saw in section 4 how to parameterize the intersection of the two extors extending $\widehat{121}$ and $\widehat{131}$ as $|u_1| = |u_2| = 1$, as well as the curves on the torus corresponding to the boundary of the ball and the intersection with other bisectors.

Recall that for each $w \in W$, the intersection of \widehat{w} with the torus can be written in the form (4.12), hence it has a parameterization $u_2 = \phi_w^\pm(u_1)$, valid on at most two intervals of values of u_1 . In principle there could be values of u_1 where the denominator vanishes, but we disregard this issue here as it does not come up in any of the 2-faces of F_W .

Recall that the above expression parameterizes not only the curves $\widehat{121} \cap \widehat{131} \cap \widehat{w}$, but also their extension to projective space. In particular,

one can find with arbitrarily large precision all the intersections of these curves with the boundary of the ball. Concretely, one solves the equations

$$(6.1) \quad \|\mathbf{a}_0 + \mathbf{a}_1 u_1 + \mathbf{a}_2 \phi_w^\pm(u_1)\|^2 = 0$$

for u_1 , with arbitrarily large precision. In turn, this breaks the topological circle $\widehat{121} \cap \widehat{131} \cap \partial H_{\mathbb{C}}^2$ into a number of intervals that are either closer to or further from p_0 than to $w p_0$.

A schematic picture of those intervals is drawn in Figure 5. Table 1 gives all the intersections with the boundary for $w = 123, 132, 213$ and 312 . Observe that finding the solutions amounts to finding the

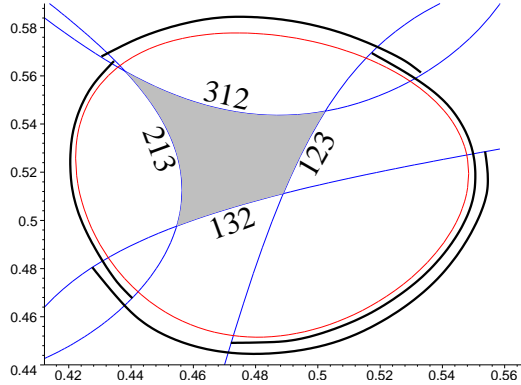


FIGURE 5. Each thick curve corresponds to an interval in the topological circle $\widehat{121} \cap \widehat{131} \cap \partial H_{\mathbb{C}}^2$ that is entirely outside of some bisector \widehat{w} for some $w \in W$.

w	(t_1, t_2)	ϕ_{∂}^+ or ϕ_{∂}^-
123	(0.47286, 0.45218)	–
	(0.51546, 0.56734)	+
132	(0.43267, 0.48454)	–
	(0.54783, 0.52714)	+
213	(0.44224, 0.47035)	–
	(0.43641, 0.56427)	+
312	(0.43578, 0.56359)	–
	(0.52965, 0.55776)	+

TABLE 1. The coordinates of the endpoints on the boundary of some curves $\widehat{121} \cap \widehat{131} \cap \widehat{w}$, with 5 decimal places. The functions ϕ_{∂}^\pm correspond to the top and bottom halves of the boundary circle, respectively.

roots of a polynomial (even though this is not completely apparent from equation (6.1)), which reduces the problem to a basic calculus problem.

We close this section with a remark about empty 2-faces. As mentioned in section 5, the intersection $\widehat{121} \cap \widehat{323}$ (or more generally $\widehat{iji} \cap \widehat{kjk}$) is empty. Recall from section 4 that the connected components of their intersections correspond to intervals on the unit circle $|u_1| = 1$ where an inequality of the form

$$\tau_0 + \tau_1 u_1 + \bar{\tau}_1 \bar{u}_1 + \tau_2 u_1^2 + \bar{\tau}_2 \bar{u}_1^2 \leq 0$$

holds. The fact that the two bisectors are in fact disjoint can be checked by verifying that the expression (4.20) is strictly positive for any u_1 . A Maple plot of its graph is given in Figure 6.

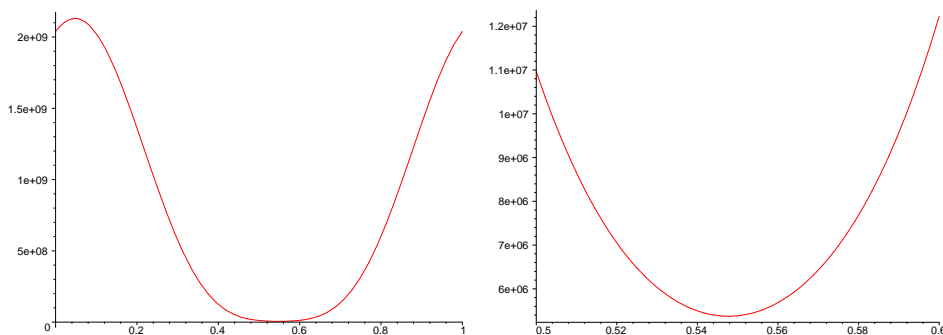


FIGURE 6. The graph of $\tau_0 + 2\Re\{\tau_1 e^{2\pi it} + \tau_2 e^{4\pi it}\}$ for $\widehat{121} \cap \widehat{323}$. The right part of the figure zooms around the minimum value (and makes it apparent that the function stays above 5,000,000).

The other empty 2-faces (like $\widehat{121} \cap \widehat{232}$ for instance, see section 5) are of a different nature, since the corresponding bisectors do intersect (in a Giraud disk). Notice that for our main theorem, what matters is that the 2-face does not approach the boundary, and that can be checked just like in the beginning of this section. Figure 7 illustrates why the 2-face is actually empty (the arrow on each curve points to the side where one gets closer to p_0).

7. DESCRIPTION OF A FUNDAMENTAL DOMAIN

It is quite clear that the polyhedron F_W described in the previous section, using only words of length 3, cannot be a fundamental domain

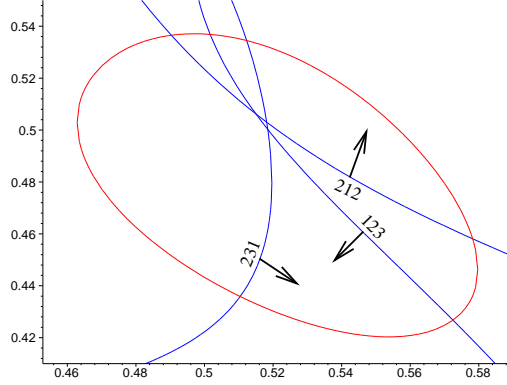


FIGURE 7. $\widehat{121} \cap \widehat{232}$ is a disk, but its intersection with F_W is empty.

for our group. We verify this precisely since it suggests an algorithm for extending the set W to obtain a fundamental domain.

Proposition 7.1. *Suppose that no nontrivial group element in W fixes p_0 . If the Dirichlet polyhedron F_W has side pairings (i.e. each w maps w^{-1} isometrically onto \tilde{w}), then*

- (1) W is symmetric, $w^{-1} \in W$ whenever $w \in W$.
- (2) W is Giraud-closed, i.e. for any non totally geodesic 2-face of F_W corresponding to a pair of elements $w_1, w_2 \in W$, the word $w_1^{-1}w_2$ is in W .

Proof: It follows from the definition of bisectors that

$$(7.1) \quad w : \widehat{w^{-1}} \rightarrow \widehat{w}$$

and also that

$$(7.2) \quad w_1^{-1} : \widehat{w_1} \cap \widehat{w_2} \rightarrow \widehat{w_1^{-1}} \cap \widehat{w_1^{-1}w_2}$$

If F_W has side pairings, then the latter intersection must be contained in some intersection $\widehat{w_1^{-1}} \cap \widehat{w}$ for some $w \in W$. By Giraud's theorem (see 4.4), for any such w , $w p_0$ must be one of p_0 , $w_1^{-1}p_0$ or $w_1^{-1}w_2 p_0$. Since we assumed no nontrivial element of W fixes p_0 , this implies $w = w_1^{-1}w_2 \in W$. \square

The set W of words of length 3 is symmetric but not Giraud-closed, hence the polyhedron F_W does not have side pairings. On the other hand, Proposition 7.1 gives a natural way to algorithmically extend a given set of words to obtain a Dirichlet fundamental domain.

Algorithm:

- (1) Start with a symmetric set of generators W_0 and determine the combinatorics of F_{W_0} . If $\widehat{w} \cap F_{W_0}$ has dimension strictly smaller than three for some $w \in W_0$, remove w from the list.
- (2) Take W_1 to be the union of W_0 and the group elements of the form $w_1^{-1}w_2$ and their inverses, where $\tilde{w}_1 \cap \tilde{w}_2$ is a non totally geodesic 2-face of F_{W_0} . Find the combinatorics of F_{W_1} , and throw away any w such that $\widehat{w} \cap F_{W_1}$ has dimension smaller than three.
- (3) If W_i is not Giraud-closed, repeat the above process to obtain W_{i+1} from W_i .
- (4) If W_i is Giraud closed, check whether the side pairings send the complex 2-faces to the appropriate complex 2-faces.

Remark 7.2. (1) *Along this process, we could find some relations in the group. Indeed, at each stage one needs to verify whether $w_1^{-1}w_2$ is already in the previous set of words. For our particular group of interest $G(4, 4, 4; 5)$, this can be done algorithmically, using (2.19) and algebra in $\mathbb{Z}[\tau, \omega]$.*

- (2) *It is not hard to take care of cases where some elements of W fix p_0 . As long as the group is discrete, the isotropy group of a point is a finite group. One needs to stay aware of this at each stage of the above algorithm. For the particular group we analyze in this paper, no nontrivial group element fixes p_0 .*
- (3) *Note that a priori this process could never stabilize, and it is possible that none of the W_i ever becomes Giraud-closed. Presumably, such pathological situations can happen for certain infinite covolume discrete groups.*
- (4) *The reader will notice that we do not set any condition about the geodesic faces in the fundamental domain. The heuristic reason not to worry about them is that, given any complex geodesic 2-face, its neighbors cannot be totally geodesic (two complex geodesics that share a curve are equal). It is of course possible to write a refinement of Proposition 7.1 to yield a necessary and sufficient condition for F_W to have side-pairings. We avoid doing so for the sake of saving space.*

A major difficulty with this algorithm is that it is very difficult to determine the combinatorics of F_W even for a finite set W . Since our methods are numerical, it would be quite difficult to verify our detailed claims about the combinatorics, even though they can be checked with as much precision as we want. Note also that this lack of precision is not a major issue when one only wants to prove that a group is cocompact (see the discussion in section 6).

Applying the above algorithm to our deformed triangle group, starting with $W_0 = \{I_1, I_2, I_3\}$, and assuming that our computer pictures are not misleading, and obtained that the W_k stabilizes to the finite set of words of the form:

Length	Word
1	i
2	ij
3	iji, ijk
4	$ijik, ijkj$
5	$ijkij$
6	$ijikji, ijkiji, ijikij, ijkjij$
7	$ijkjij, ijikiji, ijkjijk$
9	$k(ijik)^2, j(ijkj)^2$
10	$ik(ijik)^2$

Along this process, we gather the following observations:

- (1) Since the generators have order two, there are two faces contained in the same bisector \widehat{I}_i , intersecting along the mirror of I_i . Note that these two faces meet at an angle π , and the cycle transformation for their intersection is trivial. A similar phenomenon occurs for all involutive side pairings, which are precisely the conjugates of I_i , namely I_{iji} , $I_{ijikiji}$ and $I_{k(ijik)^2}$.
- (2) There are other geodesic 2-faces, contained in the intersection of the bisectors corresponding to the pairs of words of the form $ijkij$ and $kjikj$. These bisectors intersect in the mirror of the complex reflection I_{ijk}^5 , which can also be described as v_{ijk}^\perp (see 2.16). Note that the fact that I_{ijk}^5 is indeed a complex reflection was already discussed in section 2, see (2.15).
There are three such 2-faces, and they get identified by the side pairings. Note also that J clearly permutes these three 2-faces, but it is not equal to the side pairing - they differ by a complex reflection of order 6.
- (3) As stated in Remark 7.2, one finds certain relations in the group when enlarging the set of words. One of them is $ijiji = jij$, which is of course equivalent to saying that $I_i I_j$ has order four. Another non obvious relation is $ik(ijik)^2 = ji(kiji)^2$, which is equivalent to saying that $I_i I_j I_i I_k$ has order 5.
- (4) Several vertices of the fundamental domain (but not all) are given by the isolated fixed points of certain elliptic elements in the group.

The combinatorics of the faces of the conjectural Dirichlet domain are given in Figures 8-11. Note that we only list one face for each

isometry type of faces. The ones that are omitted can all be obtained by applying the natural symmetries σ_{ij} .

We also obtain a presentation for the group (which is also conjectural since it depends on the accuracy of our Dirichlet domain):

$$(7.3) \quad G(4, 4, 4; 5) \simeq \langle \iota_1, \iota_2, \iota_3 | \iota_i^2, (\iota_i \iota_j)^4, (\iota_i \iota_j \iota_k)^{10}, (\iota_i \iota_j \iota_k \iota_j)^5 \rangle$$

Out of the geometry of the Dirichlet domain, one easily notices that there are commuting complex reflections in the group, corresponding to $(ijk)^5$ and $jkjijkj$. We leave it as an exercise to the reader to verify that this commuting relation is indeed a consequence of the presentation (7.3).

8. TOTALLY GEODESIC FACES

Recall that 3-faces of our polyhedra cannot be totally geodesic, since there are no totally geodesic real hypersurfaces in complex hyperbolic space.

It is an interesting feature of Dirichlet domains that totally geodesic 2-faces can only be in complex geodesics, i.e. they cannot be totally real. Indeed, if $B(p_0, p_1) \cap B(p_0, p_2)$ were to contain a Lagrangian plane L , the involution fixing L would exchange p_0 and p_1 , as well as p_0 and p_2 , which can only happen if $p_1 = p_2$.

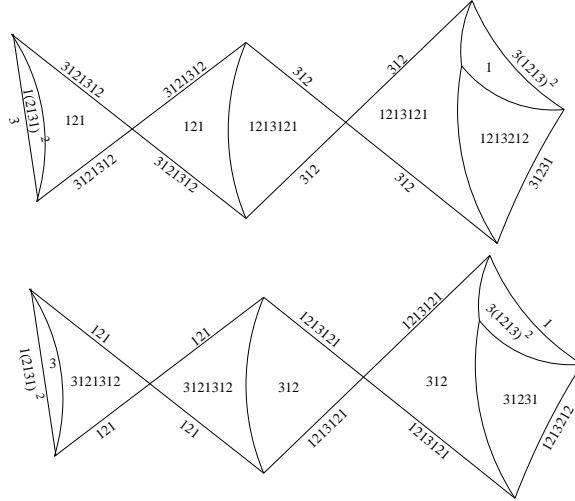
In the fundamental domain for our group $G(4, 4, 4; 5)$, there are four isometry classes of complex 2-faces, on the mirrors of I_i , I_{ij} , I_{ijkiji} , $I_{k(ijik)^2}$ and finally on the mirror of I_{ijk}^5 . The first four types are what Giraud called “hidden faces”, in the sense that the two 3-faces that contain it meet at an angle π , i.e. they are on the same bisector.

Note that the hidden 2-faces are bounded by non-geodesic hypercycles, and the angles between their 1-faces are not all rational multiples of π .

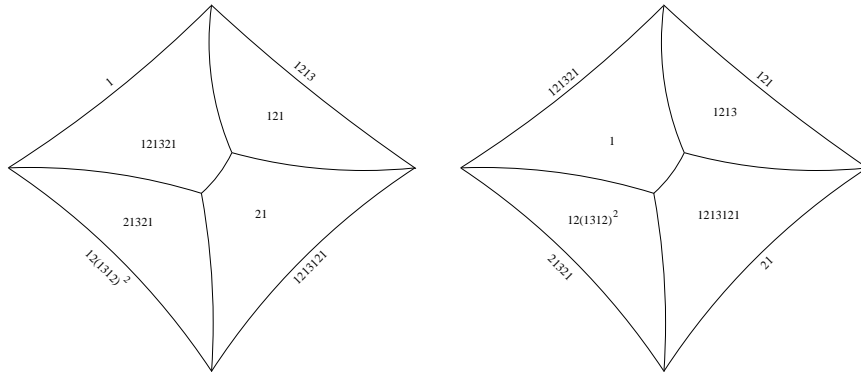
It can be checked that the other three 2-faces, which are on the v_{ijk}^\perp , are bounded by geodesic quadrilaterals with angles $2\pi/5$, $\pi/2$, $\pi/3$, $\pi/2$.

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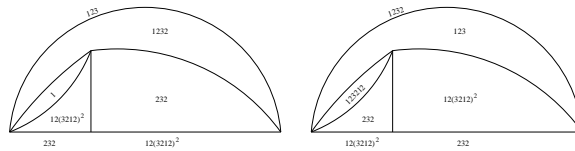
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(a) $12(1312)^2$

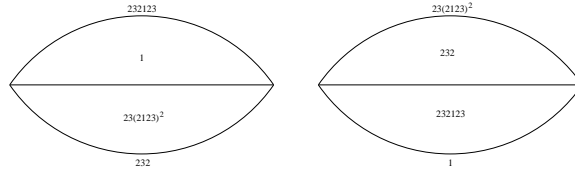


(b) 1213212

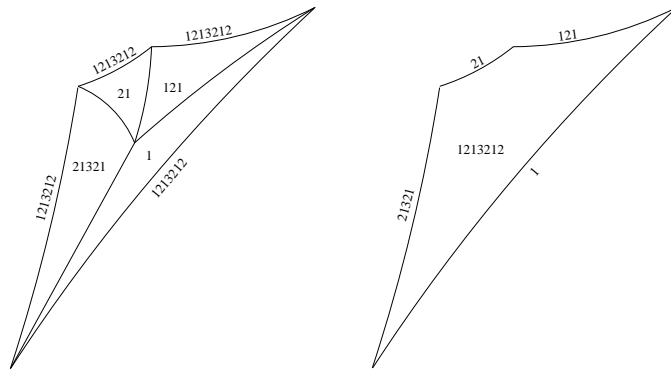


(c) 1232123

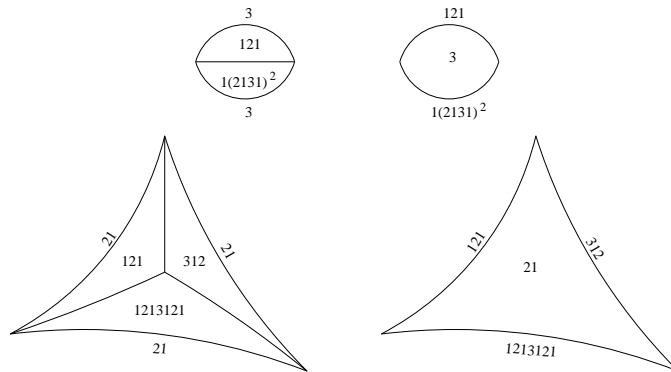
FIGURE 10. The combinatorics of faces of the Dirichlet domain. Note that $F \cap \overline{12(1312)^2}$ is not a topological ball, one needs to break it up into three separate faces.



(a) $2(3212)^2$



(b) 121321



(c) 121312

FIGURE 11. The combinatorics of faces of the Dirichlet domain. Note that $\widehat{121312} \cap F_W$ has two connected components, so there are two different faces on that bisector.

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