1. Let $\text{hom} : \mathbb{C}[x, y] \to \mathbb{C}[x, y, z]$ be the homogeneization function (mapping a polynomial $f(x, y)$ to its homogeneization $F(x, y, z)$), and let $\text{dehom} : \mathbb{C}[x, y, z] \to \mathbb{C}[x, y]$ be the de-homogeneization function. Is $\text{dehom}$ a left inverse of $\text{hom}$? Is it a right inverse?

2. Apply the Fundamental Theorem of Algebra to prove that every homogeneous polynomial $F(x, y)$ in two variables, with complex coefficients, factors as a product of linear polynomials $F(x, y) = \prod (\alpha_i x + \beta_i y)$.

3. Let $L$ be a line in $\mathbb{P}^2_\mathbb{R}$, defined by $ax + by + cz = 0$. Find a projectivity $\phi : \mathbb{P}^2_\mathbb{R} \to \mathbb{P}^2_\mathbb{R}$ that maps $L$ into the line of equation $z = 0$. (That is, find a $3 \times 3$ matrix $M$ representing the projectivity $\phi$.) Conclude that if $L$ is any line in $\mathbb{P}^2_\mathbb{R}$, then the complement $U = \mathbb{P}^2_\mathbb{R} \setminus L$ is equal to $\mathbb{R}^2$.

4. Let $C$ be the conic in $\mathbb{P}^2_\mathbb{R}$ defined by $y^2 = xz$. Sketch $C$ with respect to the “triangle” in $\mathbb{P}^2_\mathbb{R}$ given by the three coordinate lines $x = 0$, $y = 0$, $z = 0$. Find the equations of three lines $L_1$, $L_2$, and $L_3$ in $\mathbb{P}^2_\mathbb{R}$ such that, denoting $U_i = \mathbb{P}^2_\mathbb{R} \setminus L_i$ (for $i = 1, 2, 3$), $C$ restricts to:
   - a parabola in $U_1$,
   - an ellipse in $U_2$,
   - a hyperbola in $U_3$.

5. Prove that any non-degenerate conic $C$ in $\mathbb{P}^2_\mathbb{C}$ is defined by $y^2 = xz$ for a suitable choice of homogeneous coordinates $(x : y : z)$. Is the same true in $\mathbb{P}^2_\mathbb{R}$? Explain.

6. Let $P_1, P_2, P_3$ be three distinct points on a line $L$ in $\mathbb{P}^2_\mathbb{C}$. We say that the points are collinear. Use Bertini Theorem (in the form already proved in class) to show that if a conic $C$ passes through these three points, then it contains the whole line. Deduce that $C$ is degenerate.

7. We saw in class that the space of conics in $\mathbb{P}^2_\mathbb{C}$ passing through four general points $P_1, P_2, P_3, P_4$ is 1-dimensional. Show that if the four points are collinear (that is, on a line), then the space of conics through them is 2-dimensional.

8. In the book of M. Reid, subsection (1.12) (pages 20–21) there is a list of all possible ways in which two conics may intersect. In each case, the two conics generate a 1-dimensional family of conics. Write down equations to show that each possibility really occurs. Find all the singular conics in each family. [Hint: In each case, make a smart choice of homogeneous coordinates.]

9. Prove that a curve of degree 4 with four or more singular points is reducible.

10. Consider the curve $C$ of equation $x^6 + y^6 - x^2 y^2 = 0$.
   - Establish whether $C$ is irreducible or not.
   - Determine all singular points of $C$ and their multiplicity.
   - Sketch $C$ in $\mathbb{R}^2$.
   [Hint: Use the symmetries of $C$ in your arguments.]