Problem # 1
Let $Z = \mathbb{Z}^+$ and $\mathbb{R} = \mathbb{R}^+$ be the additive groups, and let $S^1$ denote the circle group. Let $\phi: \mathbb{R} \to \mathbb{C}^\times$ be the homomorphism defined by $\phi(t) = e^{2\pi it}$.

(a) Apply the First Isomorphism Theorem to prove that the quotient group $\mathbb{R}/\mathbb{Z}$ is isomorphic to $S^1$.

(b) Give a reason why $\mathbb{R}$ is not isomorphic to the product $\mathbb{Z} \times S^1$.

Problem # 2
Prove Fermat’s Little Theorem: For any prime $p$ and any integer $a$,

$$a^p \equiv a \mod p.$$ 

Problem # 3
Let $V = P_{\leq 3}(F)$ be the vector space of polynomials $p(x)$ of degree $\leq 3$ with coefficients in a field $F$.

(a) What is the dimension of $V$? Explain.

(b) Let $D: V \to V$ denote derivation. That is, $D$ is the linear operator defined by $D(p(x)) = p'(x)$ for all $p(x) \in V$. Prove that $D$ is not diagonalizable. [Hint: What are the eigenvectors of $D$?]

Problem # 4
(a) Mark the Jordan blocks in the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

(b) Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by the above matrix $A$. Describe all proper $T$-invariant subspaces of $\mathbb{R}^3$. 