Random Bits and Pieces

An Introduction to Symbolic Dynamics

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We work by examples, and in random order.

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  where $x_1, x_2, \ldots$ are either zero or one.
- If there are two ways of doing this [dyadic rationals] then opt for the non-terminating expansion.

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  - Infinite-option convention yields:

$$0.5 = [0, 1, 1, \ldots].$$

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- Why does this work? Hint:

$$y_1 = \sum_{j=1}^{\infty} \frac{x_{j+1}}{2^j}$$

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- Try it for $x = 0.5$ $\qquad 0.5 = [0\,,1\,,1\,,\ldots]$
- What if you split into $[0\,,0.5)$ and $[0.5\,,1]$ etc.?

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- $\#\mathscr{D}_n = 2^n$ (check!)

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- By (1), $\Pr\{X \in I\} = 2^{-n}$ for all $I \in \mathscr{D}_n$.

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- X is "distributed uniformly on $[0\,,1]$"

Borel's Strong Law of Large Numbers

- Recall $X_1, X_2, \ldots$ are independent, and

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- (Borel's Theorem, 1909) With probability one:

$$\lim_{n\to\infty} \frac{X_1 + \cdots + X_n}{n} = \lim_{n\to\infty} \frac{EX_1 + \cdots + EX_n}{n} = \frac{1}{2}.$$

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- Borel's theorem: Nonnormal numbers are of length zero.

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Normal numbers make sense also in base-ten arith. (or any other base ≥ 2 for that matter):

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- Borel's theorem: Almost every number is normal in base ten. In fact, almost every number is normal in all bases!

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- Is $\pi/10$ normal? How about $\sqrt{2}/10$?

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- Is it the case that X has the correct distribution?
- The binary digits $X_1, X_2, \ldots$ have lots of structure; so they need to pass various statistical tests (lots known)
- All RNG's will fail the true test of randomness: X_j's have to be normal in all bases.

Ternary Expansions

- Let $x = [0\,,1]$, and write uniquely,

$$x = \sum_{j=1}^{\infty} \frac{x_j}{3^j}, \qquad \text{where } x_j \in \{0\,,1\,,2\}.$$

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- If $\frac{1}{3} < x < \frac{2}{3}$ then $x \notin \mathscr{C}$ etc.

Ternary Expansions

- Let $x = [0\,,1]$, and write uniquely,

$$x = \sum_{j=1}^{\infty} \frac{x_j}{3^j}, \qquad \text{where } x_j \in \{0\,,1\,,2\}.$$

- The ternary Cantor set $\mathscr{C}$:

$$\mathscr{C} = \text{closure of } \{x \in [0\,,1] : \ x_j \in \{0\,,2\}\}$$

- $x = 1/3$ is in the Cantor set; in fact, $x = [0\,,2\,,2\,,\ldots]$
- If $\frac{1}{3} < x < \frac{2}{3}$ then $x \notin \mathscr{C}$ etc.
- $\mathscr{C}$ = The middle-thirds Cantor set

Devil's Staircase

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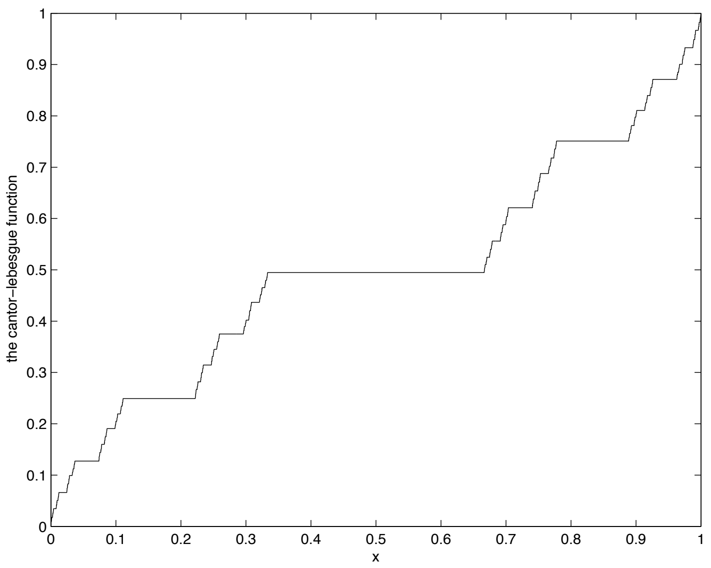
- Aka Cantor–Lebesgue function

The Cantor–Lebesgue Function



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Mind those technical conditions of theorems!

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Let S be a set in $\mathbf{R}^n$. Roughly speaking, its *Hausdorff dimension*

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(Frostman, 1935)

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- Let us prove that if $s < \log_3(2)$ then $E(|X - Y|^{-s}) < \infty$. This proves that $\dim_H \mathscr{C} \geq \log_3(2)$, and is in fact the harder bound.

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- If $s < \log_3(2)$ then

$$E\left(\frac{1}{|X - Y|^s}\right) \leq E\left(\frac{1}{3^{Ns}}\right) = \sum_{k=1}^{\infty} \frac{1}{3^{ks}} \times 2^{-k} < \infty.$$