Nonlinear Noise Excitation

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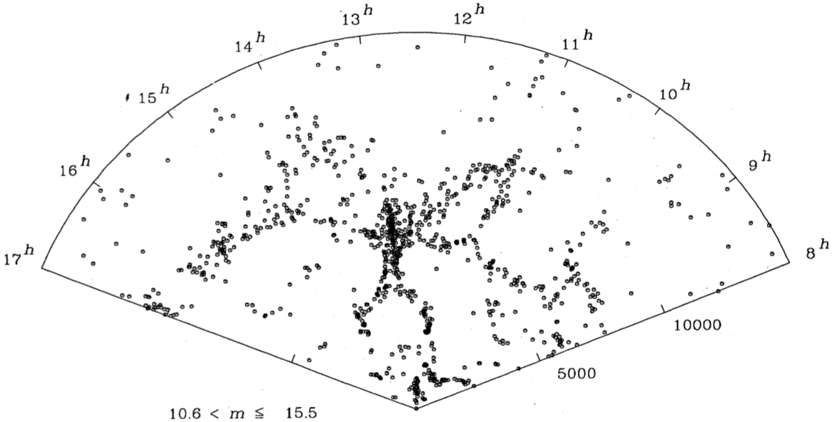
Large-scale structure of galaxies

S. F. Shandarin and Ya B. Zeldovich, *Rev. Modern Phys. (1989)*



A simple model for intermittency $[\dot{u}_t(x) = (1/2)u_t''(x) + \lambda u_t(x)\eta_t, u_0(x) = 1]$

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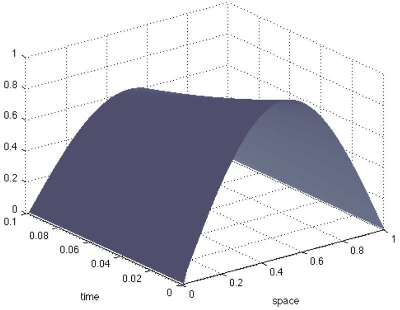
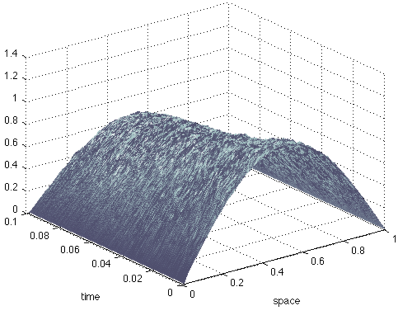
- $\mathrm{E}(u_t^2) = \exp\left\{\lambda^2 t\right\} \to \infty$ (fast!) as $\lambda \to \infty$

A SHE simulation $[\dot{u}_t(x) = (1/2)u_t''(x) + \lambda u_t(x)\eta_t(x),$ $u_0(x) = \sin(\pi x),\ 0 \leqslant x \leqslant 1;\ u_t(0) = u_t(1) = 0.]$

$\lambda = 0$ ***(left;*** $u_t(x) = \sin(\pi x)\exp(-\pi^2 t/2)$***)*** ***and*** $\lambda = 0.1$ ***(right)***





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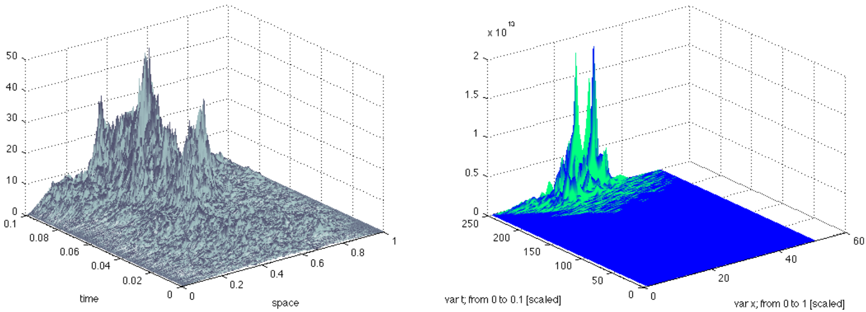
$\lambda = 2$ *(left) and* $\lambda = 6$ *(right)*



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- Example of what is to come. “The noise excitation index q, when it $\exists$, is a topological invariant.”

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- Can add drift to the SPDE in order to get all Itô processes, but we will not

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- Can be easily extended to $G = \mathbf{Z}_n$

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- $G = \mathbf{Z}^d$ and $\mathscr{L} = \kappa\Delta_{\mathbf{Z}^d}$—the semi-dicrete stochastic heat equation

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  - X is "Gaussian"

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Theorem (essentially due to Dalang, 1999)

Consider the linear SPDE $\sigma \equiv 1$. Then our SPDE has a function solution if and only if

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- (D) iff $X_t Y_t^{-1}$ has local times, where Y is an indept copy of X [essentially due to Hawkes 1986]; see also Foondun–K–Nualart (2011) and Eisenbaum–Foondun–K (2011)

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Condition (D): $\int_{G^*}(1 + \text{Re}\Psi(\chi))^{-1}\, m_{G^*}(d\chi) < \infty$

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- We will need the linear solution to have a function solution in order to be able to apply variation of parameters to the non-linear equation;
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  - This is a first example of how the structure of G alone can matter: When G is discrete the linear SPDE always has a function solution

Existence and Uniqueness

$\partial_t u_t(x) = \mathscr{L} u_t(x) + \lambda\sigma(u_t(x))\xi_t(x)$

Theorem (K–Kim)

Suppose that σ *is Lipschitz continuous, and either* $\sigma(0) = 0$ *or* G *is compact. If, in addition,* $u_0 \in L^2(G)$ *is non random, then our SPDE has a solution that satisfies the following energy inequality for some* $c \in (0\,, \infty)$*:*

$$\mathscr{E}_t(\lambda)^2 := \mathrm{E}\left(\|u_t\|^2_{L^2(G)}\right) \leqslant c\exp(ct) \qquad \textit{for all } t \geqslant 0.$$

$\exists$ *uniqueness among solutions that have an energy inequality.*

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Linear noise excitation

$\partial_t u_t(x) = \mathscr{L} u_t(x) + \lambda\sigma(u_t(x))\xi_t(x), \quad \mathscr{E}_t(\lambda) := \sqrt{\mathrm{E}(\|u_t\|^2_{L^2(G)})}$

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- Wish to understand the noise excitation of such SPDEs

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- $\Rightarrow$ consequences for numerical analysis of shocks

Outline of proof

$\partial_t u_t(x) = \mathscr{L} u_t(x) + \lambda\sigma(u_t(x))\xi_t(x)$, $\mathscr{E}_t(\lambda) := \sqrt{\mathrm{E}(\|u_t\|^2_{L^2(G)})}$

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$$\Upsilon(\beta) := \int_{G^*} \left(\frac{1}{\beta + \mathrm{Re}\Psi(\chi)} \right) m_{G^*}(d\chi) \quad \text{for } \beta \gg 1$$

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Outline of proof: The discrete case

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- Use this formula in the abstract bounds
- The connected case is more interesting because we do not have formulas for the behavior of Υ

Reduction principle 1: Group invariance

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- For α-stable processes on $\mathbf{R}$, $\log \mathscr{E}_t(\lambda) \asymp \lambda^{4/(\alpha-1)}$, for all $\alpha \in (1\,,2]$

Back to $[0\,,1]$ *with Dirichlet 0-boundary conditions*

Two asides:

Theorem (Foondun–Joseph, 2014)

If u solves $\partial_t u = u'' + \sigma(u)\xi$ on $[0\,,1]$ with $u_t(0) = u_t(1) = 0$ and nice I.C., then $\log \mathcal{E}_t(\lambda) \asymp \lambda^4$ for all $\lambda \geqslant 1$.

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