Macroscopic Dimension

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Macroscopic Minkowski Dimension

- If $A \subset [0\,, \infty)$ is a set, then define

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- **Example.** Let $f(k) := k^p$ for $k \in \mathbb{N}$, where $p \ge 1$. Then,

$$\mathrm{Dim}_{\mathrm{M}}\left(f(\mathbb{N})\right) = \mathrm{Dim}_{\mathrm{M}}\left(\cup_{k=0}^{\infty}\{k^p\}\right) = p^{-1}.$$

Reason. $N_n(\{k^p\}_{k=0}^{\infty}) \asymp 2^{(n+1)/p} - 2^{n/p} \asymp 2^{n/p}$.

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  - The same for $Z := B^{-1}(\{0\})$ for $B :=$ a $\mathrm{BM}(\mathbb{R}^d)$.

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- There are natural ways to extend $\mathsf{Dim}_{\mathrm{M}}(A)$ for cases where $A \subseteq \mathbb{R}^d$, where $d \geq 1$. Here is one:

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- There is also a [more complicated] notion of macroscopic Hausdorff dimension (Barlow–Taylor, 1989; 1992. Naudts, 1988), denoted by $\mathsf{Dim}_{\mathrm{H}}$, which I will not define, in order to keep the exposition relatively simple. **Fact.** $0 \leq \mathsf{Dim}_{\mathrm{H}}(A) \leq \mathsf{Dim}_{\mathrm{M}}(A) \leq d$.

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- **Example.** $\mathrm{Dim}_{\mathrm{M}}(\mathbb{Z}^d) = \mathrm{Dim}_{\mathrm{M}}(\mathbb{N}^d) = \mathrm{Dim}_{\mathrm{M}}(\mathbb{R}^d) = d$.
- The main result of [Barlow–Taylor, 1992] is the fact that if $d \geq 2$ and X denotes a non-degenerate transient random walk on $\mathbb{Z}^d$ that is "stable-like" with index $0 < \alpha \leq 2$, then $\mathrm{Dim}_{\mathrm{M}}(\text{range of } X) = \mathrm{Dim}_{\mathrm{H}}(\text{range of } X) = \alpha$ a.s. The precise statement follows.

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Theorem (Barlow–Taylor, 1992)

Let $X :=$ *a transient walk on* $\mathbb{Z}^d$ *s.t.* $\exists\alpha \in (0\,,2]$ *with*

$$g(x) := \sum_{n=0}^{\infty} \mathrm{P}\{X(n) = x\} \asymp \|x\|^{-d-\alpha} \qquad \textit{for } \|x\| \gg 1.$$

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- The formula for $\mathrm{Dim}_{\mathrm{H}}(X(\mathbb{N}))$ is very complicated [Georgiou–K–Kim–Ramos, 2015]. I will point out only the formula for $\mathrm{Dim}_{\mathrm{M}}(X(\mathbb{N}))$ for politeness' sake [ibid.].

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Theorem (Georgiou–K–Kim–Ramos, 2015)

Let $X :=$ transient walk on $\mathbb{Z}^d$ with Green function $g(x) := \sum_{n=0}^{\infty} \mathrm{P}\{X(n) = x\}$. Then,

$$\mathrm{Dim}_{\mathrm{M}}(X(\mathbb{N})) = \inf\left\{\gamma \in (0\,,d) : \sum_{x \in \mathbb{Z}^d \setminus \{0\}} \frac{g(x)}{\|x\|^\gamma} < \infty\right\} \quad a.s.$$

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- There is a formula also for $\mathrm{Dim}_{\mathrm{H}}(X(\mathbb{N}))$ [Barlow–Taylor problem] but it is very complicated, and so I omit it.

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- Let $X :=$ a transient Lévy process on $\mathbb{R}^d$, char. exponent Ψ. Is there an explicit formula for $\text{Dim}_{\text{M}}(X(\mathbb{R}_+))$ in terms of Ψ?

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- A positive resolution has many consequences.

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- Consider the random set $\mathcal{L}_c^B := \left\{ t \geq 8 : \; B(t) > c\sqrt{2t \log\log t} \right\}$.

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- $\Rightarrow \text{Dim}_{\text{H/M}}\, \mathcal{L}_c^B = \begin{cases} 0 & \text{if } c > 1, \\ 1 & \text{if } c < 1. \end{cases}$ What about $\mathcal{L}_1^B$?

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  - The lower bound is slightly different from "standard" lower-bound methods … .

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- The proof of $\text{Dim}_{\text{H}}\, \mathcal{L}_c^X \geq 1 - \rho$ is only slightly more delicate. □

The Parabolic Anderson Model on $\mathbb{R}$

- Consider PAM on $\mathbb{R}$: $\xi :=$ space-time white noise;

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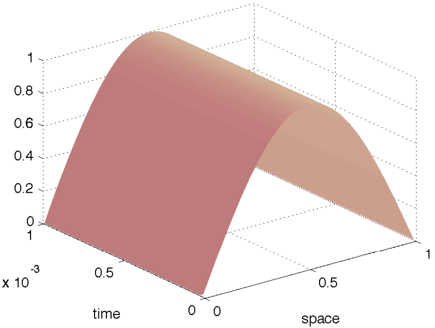

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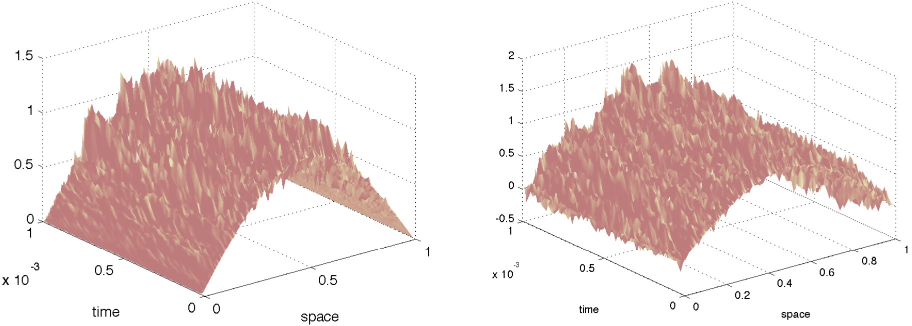

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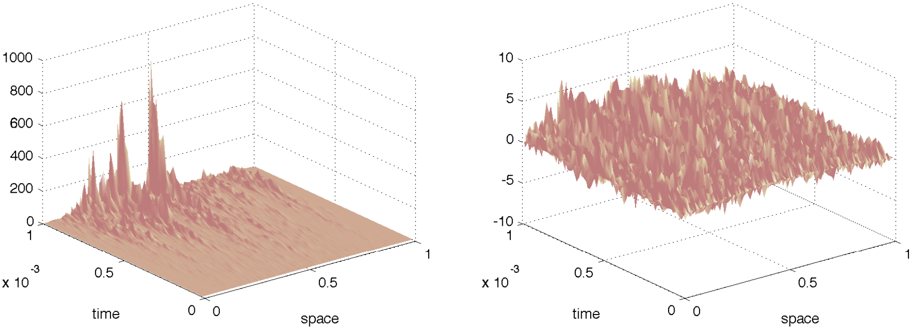

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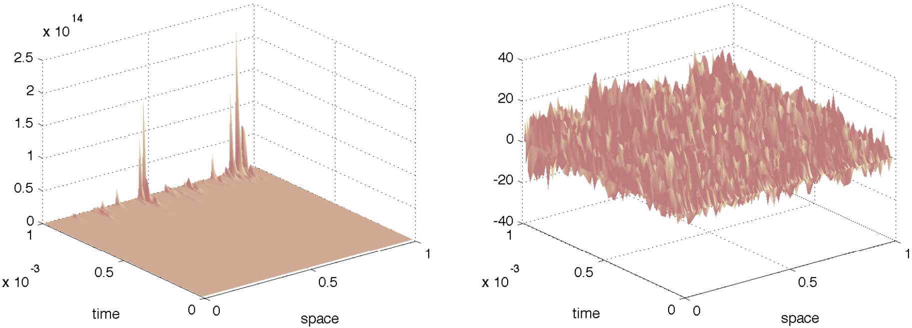

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