On the chaotic character of some parabolic SPDEs

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(joint with Daniel Conus, Mathew Joseph, and Shang-Yuan Shiu)

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Intermittency occurs when we multiply many roughly-independent r.v.'s; e.g., $\xi_1, \xi_2, \ldots$ i.i.d. with $P\{\xi_1 = 2\} = P\{\xi_1 = 0\} = 1/2$. Then $u_n := n \prod_{j=1}^{\infty} \xi_j = \{2^n \text{ with probab. } 2^{-n}, 0 \text{ with probab. } 1 - 2^{-n}\}$.

Conclusions:

$u_n = 0$ for all $n$ large a.s.; in particular, $u_n \rightarrow 0$ a.s.

$n - 1 \log E(u_k^n) \rightarrow \gamma_k := (k - 1) \log 2$ for all $k > 1$.

Now replicate this experiment.
A simple model for intermittency
(Zeldovich–Ruzmaikin–Sokoloff, 1990)

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Is this degeneracy because of the many zeros? No.
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Let $b$ denote 1-D Brownian motion and consider the exponential martingale $u_t := e^{\lambda b_t - (\lambda^2 t/2)}$.
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The examples are “similar,”

$$e^{b_t - (t/2)} \approx \prod_j \left( 1 - (\Delta b)_j - \frac{1}{2}(\Delta t)_j \right)$$
A simulation \[
\dot{u}_t(x) = (\pi/2)u''_t(x) + \lambda u_t(x)\eta_t, \quad u_0 \equiv 1
\]
\[u_t = \exp\{\lambda b_t - (\lambda t/2)\}\] with \(\lambda = 0.5\) (left) and \(\lambda = 5\) (right)
Intermittency in cosmology
The model (for today)

\[ \frac{\partial}{\partial t} u_t(x) = \kappa \frac{\partial^2}{\partial x^2} u_t(x) + \sigma(u_t(x))\eta_t(x), \]

where:

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where:

1. \( \kappa > 0; \)
2. \( \sigma : \mathbb{R} \rightarrow \mathbb{R} \) is Lipschitz continuous;
3. \( \eta \) is space-time white noise; i.e., a centered GGRF with

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6. Either \( 0 < \inf \sigma \leq \sup \sigma < \infty \), or \( \sigma(u) \propto u \) [random media].
Weak intermittency

\[ \partial_t u = (\kappa/2) \partial_{xx} u + \sigma(u) \eta \]


\[
0 < \limsup_{t \to \infty} \frac{1}{t} \log E \left( |u_t(x)|^k \right) < \infty \quad (k \geq 2, x \in \mathbb{R})
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- (weak) intermittency \([\text{Bertini–Cancrini, 1994; Carmona–Molchanov, 1994; Molchanov, 1991; Foondun–K., 2010; Zel’dovitch et al, 1985, 1988, 1990; \ldots}]:

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- Today: What happens before the onset of localization?
Optimal regularity
\[ \partial_t u = (\kappa/2) \partial_{xx} u + \sigma(u) \eta \]

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  [Lunardi, 1995, and older works by Pazy, Kato, ...]
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- If \( u_0 \in C^\alpha(\mathbb{R}) \) for some \( \alpha > \frac{1}{2} \) and has compact support, and if \( \sigma(0) = 0 \), then \( \sup_{x \in \mathbb{R}} u_t(x) < \infty \) a.s. for all \( t > 0 \) (Foondun–Kh, 2010)
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- Today’s goal: The solution can be sensitive to the choice of \( u_0 \) (we study cases where \( u_t \) is unbounded for all \( t > 0 \))
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Power of \( \kappa \) suggests the universality class of random walks in weak interactions with their random environment
Theorem (Conus–Joseph–Kh)
The parabolic Anderson model

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\[ u_t(x) \approx \exp \left\{ \text{const} \cdot \left( \log |x| / \sqrt{\kappa} \right)^{2/3} \right\} \]
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Power of \( \kappa \) suggests the universality class of random-matrix models (GUE)
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\[ u_t(x) \approx \exp \left\{ \text{const} \cdot \left( \log |x| / \sqrt{\kappa} \right)^{2/3} \right\} \]

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“KPZ fluctuation exponents” \( (1/3, 2/3) \)
Ideas used in proofs

- **Coupling.** If $x_1, \ldots, x_N$ are sufficiently far apart, then $u_t(x_1), \ldots, u_t(x_N)$ are “approximately independent”
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  - $\log P\{u_t(x) \geq \lambda\} \asymp -x^{1/2} \lambda^2$ if $\sigma$ bounded above and below
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  - $\log P\{u_t(x) \geq \lambda\} \asymp -x^{1/2}(\log \lambda)^{3/2}$ for parabolic Anderson model
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  - $\log P\{u_t(x) \geq \lambda\} \asymp -\kappa^{1/2} (\log \lambda)^{3/2}$ for parabolic Anderson model
- **Similar results for Majda’s passive–scalar model** [stretched exponential tails, but on a non-log scale] by Bronski–McLaughlin (2000)
Colored noise

\[ \dot{u}_t(x) = (\kappa/2)(\Delta u_t)(x) + \sigma(u_t(x))\eta_t(x) \quad (t > 0, \ x \in \mathbb{R}^d) \]

- Now

\[ \text{Cov}(\eta_t(x), \eta_s(y)) = \delta_0(s - t)f(x - y) \]

(Dalang, 1999; Hu–Nualart, 2009, …)
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Suppose \( f = h \ast \tilde{h} \) for some \( h \in L^2(\mathbb{R}^d) \), so \( \exists! \) solution \( \forall d \geq 1 \)
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- \( \exists \) KPZ version also (Medina–Hwa–Kardar–Zhang, 1989)
Theorem (Conus–Joseph–Kh–Shiu)

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If \( \lambda > 0 \) and \( h \) is "nice," then

\[
\limsup_{|x| \to \infty} \frac{\log u_t(x)}{\log |x|} \approx \frac{1}{2} \text{ a.s. for all } t > 0 \text{ and } \kappa \text{ small}
\]

There are other variations as well

"fluctuation exponent" \((0, 1/2)\)

Are there in-between models? Yes.
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\limsup_{|x| \to \infty} \frac{\log u_t(x)}{(\log |x|)^{1/2}} \lesssim 1 \quad \text{a.s. for all } t > 0 \text{ and } \kappa \text{ small}
\]
Theorem (Conus–Joseph–Kh–Shiu)
The parabolic Anderson model

\[ \dot{u} = (\kappa/2) \Delta u + \lambda u \eta \quad [\sigma(x) = \lambda x] \]

- If \( \lambda > 0 \) and \( h \) is “nice,” then

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“fluctuation exponent” \((0, 1/2)\)
Theorem (Conus–Joseph–Kh–Shiu)
The parabolic Anderson model

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- There are other variations as well
- “fluctuation exponent” \((0, \frac{1}{2})\)
- Are there in-between models? Yes.
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The solution \( \exists! \) when \( \alpha < \min(d, 2) \) [Dalang, 1999]
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If \( \lambda > 0 \), then

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“fluctuation exponent” \( (2\psi - 1, \psi) = (\alpha/(4-\alpha), 2/(4-\alpha)) \)

\( f = h \ast \tilde{h} \iff \alpha = 0 \), and \( f = \delta_0 \iff \alpha = 1 = \min(d, 2) \) [spectral analogies]
Initial point mass

- In all of the preceding, we assumed that

\[ 0 < \inf u_0 \leq \sup u_0 < \infty. \]
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**Question:** (Ben Arous, Quastel, 2011) What if \( u_0 = \delta_0 \)?
Initial point mass

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$$0 < \inf u_0 \leq \sup u_0 < \infty.$$ 

- **Question:** (Ben Arous, Quastel, 2011) What if $u_0 = \delta_0$?

- **Theorem.** (Conus–Joseph–Kh–Shiu, 2011 [?]) Consider

$$\partial_t u_t(x) = \frac{\kappa}{2} u''_t(x) + \sigma(u_t(x))\eta_t(x),$$

subject to $u_0 :=$ a finite Borel measure of bounded support, and $\sigma(0) = 0$. Then $\sup_x |u_t(x)| < \infty$ a.s. for all $t > 0$. 

D. Khoshnevisan (Univ of Utah)  SPDEs and chaos  September 22, 2011  15 / 15