On the chaotic character of some parabolic SPDEs

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(joint with Daniel Conus, Mathew Joseph, and Shang-Yuan Shiu)

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(Zeldovich–Ruzmaikin–Sokoloff, 1990)

- Intermittency occurs when we multiply many roughly-independent r.v.'s ; e.g., $\xi_1, \xi_2, \ldots$ i.i.d. with $P\{\xi_1 = 2\} = P\{\xi_1 = 0\} = 1/2$

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- Is this degeneracy because of the many zeros? No

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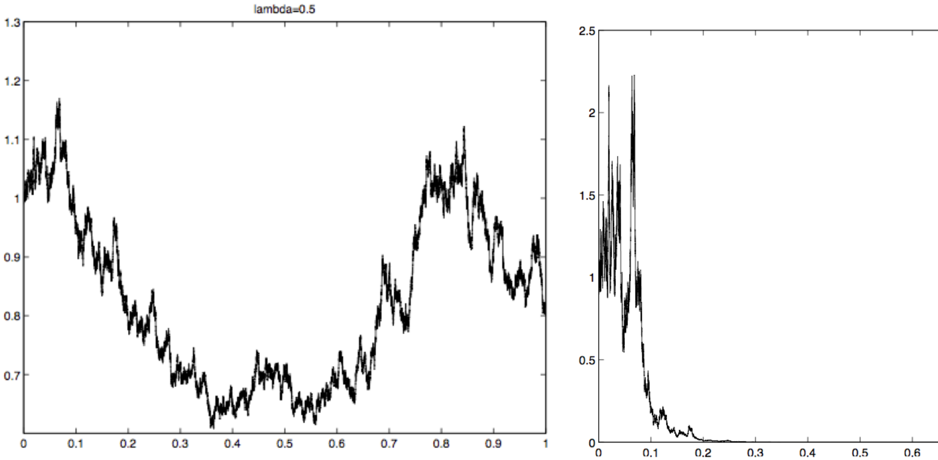
- The examples are "similar,"

$$e^{b_t - (t/2)} \approx \prod_j \left(1 - (\Delta b)_j - \frac{1}{2}(\Delta t)_j\right)$$

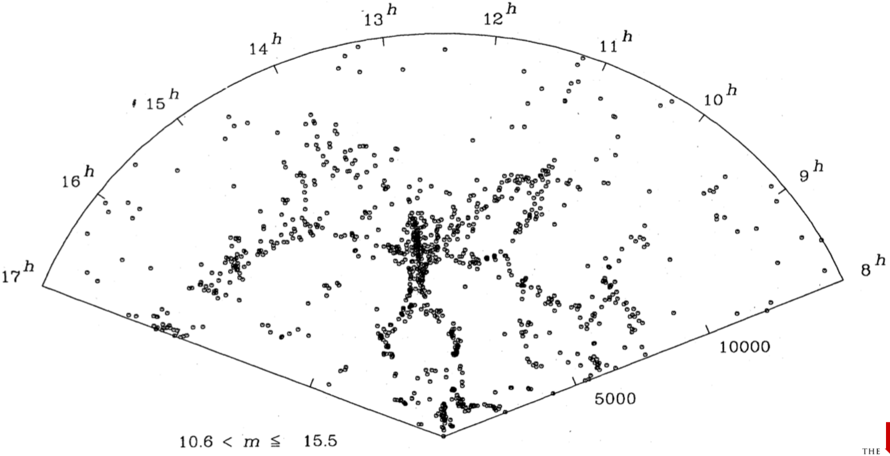
A simulation $[\dot{u}_t(x) = (\varkappa/2)u_t''(x) + \lambda u_t(x)\eta_t,\ u_0 \equiv 1]$

$u_t = \exp\{\lambda b_t - (\lambda t/2)\}$ with $\lambda = 0.5$ (left) and $\lambda = 5$ (right)



Intermittency in cosmology

S. F. Shandarin and Ya. B. Zeldovitch, *Rev. Modern Physics* **61***(2)* (1989) 185–220



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6. Either $0 < \inf \sigma \leq \sup \sigma < \infty$, or $\sigma(u) \propto u$ [random media].

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$\partial_t u = (\varkappa/2)\partial_{xx} u + \sigma(u)\eta$

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- Today: What happens before the onset of localization?

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- If $u_0 \in C^\alpha(\mathbf{R})$ for some $\alpha > \frac{1}{2}$ and has compact support, and if $\sigma(0) = 0$, then $\sup_{x\in\mathbf{R}} u_t(x) < \infty$ a.s. for all $t > 0$ (Foondun–Kh, 2010)

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- Today's goal: The solution can be sensitive to the choice of u_0 (we study cases where u_t is unbounded for all $t > 0$)

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- "KPZ fluctuation exponents" $(1/3\,, 2/3)$

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  - Similar results for Majda's passive–scalar model [stretched exponential tails, but on a non-log scale] by Bronski–McLaughlin (2000)

Colored noise

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- Now

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- $\exists$ KPZ version also (Medina–Hwa–Kardar–Zhang, 1989)

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- There are other variations as well

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- If $\lambda > 0$ and h is "nice," then

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- Are there in-between models? Yes.

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- $f = h * \tilde{h} \;\Leftrightarrow\; \alpha = 0$, and $f = \delta_0 \;\Leftrightarrow\; \alpha = 1 = \min(d\,, 2)$ [spectral analogies]

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- **Question:** (Ben Arous, Quastel, 2011) What if $u_0 = \delta_0$?
- **Theorem.** (Conus–Joseph–Kh–Shiu, 2011 [?]) Consider

$$\partial_t u_t(x) = \frac{\varkappa}{2} u_t''(x) + \sigma(u_t(x))\eta_t(x),$$

subject to $u_0 :=$ a finite Borel measure of bounded support, and $\sigma(0) = 0$. Then $\sup_x |u_t(x)| < \infty$ a.s. for all $t > 0$.