SEMINAR ON STOCHASTIC PROCESSES, 2000

RANDOM WALKS ON GROUPS, 40 YEARS AFTER HARRY KESTEN'S THESIS: CLASSICAL AND EXOTIC BEHAVIORS

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The Ph.D. thesis of Harry Kesten (1958-AMS-Transaction) investigates random walks on finitely generated groups. It emphasizes the problem of understanding how the behavior of a random walk relates to the algebraic structure of the group. For a random walk governed by a given probability measure p on a finitely generated group G, let $\phi(n)$ be the the probability of return to the starting point after $2n$ steps. What are the possible behaviors of $\phi(n)$ as n tends to infinity? How does this behavior relates to the structure of G?

One of the celebrated results of Kesten's thesis is that, assuming that p is symmetric and that its support generates G, $\phi(n)$ decays exponentially fast if and only if G is non-amenable. In his last paper on the subject (Fifth Berkeley Symposium 1966), Kesten collects some open questions focussing around the recurrence/transience dichotomy. From this emerged what became to be known as Kesten's conjecture: The only recurrent groups are the finite extentions of $\{e\}$, $\mathbb{Z}$ and $\mathbb{Z}^2$.

Although important results were obtained in the sixties and seventies concerning special cases and related problems on Lie groups, little progress were made during this period concerning the most natural questions arising from Kesten's thesis and his Fifth Berkeley Symposium paper. The simple reason is that finitely generated groups can be extremely complicated objects.

In the eighties, it emerged from the work of N. Varopoulos, that the simple heuristic idea that $\phi(n)$ should be small if the group is large can be made very precise without a deep understanding of the algebraic structure of the group involved. Let $V(n)$ be the number of elements of a group G that can be obtained as product of at most n generators, for some fixed generating set. One of Varopoulos' results is the following: if $V(n)$ growth faster than n^d then $\phi(n)$ deacys faster than $n^{-d/2}$. Using Gromov's result describing the algebraic structure of groups having polynomial growth, Varopoulos settled by the affirmative the conjecture of Kesten: The only recurrent groups are, indeed, the finite extentions of $\{e\}$, $\mathbb{Z}$ and $\mathbb{Z}^2$.

For amenable discrete subgroups of Lie groups, Varopoulos and his student Alexopoulos obtained the following complete result describing the possible behaviors of $\phi(n)$. Either $V(n) \approx n^d$ for some integer d, in which case $\phi(n) \approx n^{-d/2}$, or $V(n)$ grows exponentially, in which case $\phi(n) \approx \exp(-n^{1/3})$. Because of this, the three behaviors $\phi(n) \approx \exp(-n)$, $\phi(n) \approx (-n^{1/3})$, $\phi(n) \approx n^{-d/2}$ for some integer d, can be coined the classical behaviors of simple random walk.

In recent work with Christophe Pittet, we show that exotic behaviors (i.e., different from the classiacl behaviors above) can occur even in the class of solvable groups. For instance, there exists a solvable (hence amenable) group such that $\phi(n)$ decays faster than any of the functions $\exp(-n^\alpha)$, $\alpha \in (0, 1)$.