The Favorite Point of a Poisson Process

Davar Khoshnevisan & Thomas M. Lewis

University of Utah and Furman University

Abstract. In this paper we consider the most visited site, X_t, of a Poisson process up to time t. Our point of departure from the literature on maximal spacings is our asymptotic analysis of where the maximal spacing occurs (i.e., the size of X_t) and not the size of the spacings.

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Suggested Short Title. Favorite Points

1. Introduction.

Let $(N_t,\, t \geq 0)$ denote a rate one Poisson process. For every $x \in \mathbb{R}^1_+$, let us define the occupation time of $\{x\}$ by time t as follows:

$$\xi_t(x) \stackrel{\Delta}{=} \int_0^t 1_{\{x\}}(N_u) du.$$

We have adopted the customary notation that $1_A(r)$ is one if $r \in A$ and zero otherwise. Evidently, $\xi_t(x) = 0$ almost surely if and only if $x \notin \mathbb{Z}^1_+$. With this in mind, we can now define the *most visited site* of N up to time t as

$$X_t \stackrel{\Delta}{=} \min\{k \geq 0 : \, \xi_t(k) \geq \xi_t(i) \, \text{for all } i \geq 0\}.$$

In this paper, we are interested in the growth properties of the process X.

Suppose, instead, N were replaced by a mean zero finite variance lattice random walk. Then defining X in the obvious way, asymptotic properties of this favorite point process were studied by Erdős and Révész (1984) and Bass and Griffin (1985). Erdős and Révész have shown that the limsup of the favorite point process is infinity and have determined the rate with which this occurs. The surprising results of Bass and Griffin demonstrate that this limsup is, in fact, a limit; consequently, the favorite point process is transient. In the present setting, it is not hard to see that almost surely

$$\lim_{t\to\infty} X_t \to \infty.$$

Indeed, elementary properties of N demonstrate that X_t is uniformly distributed on the the set $\{0, 1, \ldots, m\}$ conditional on the event $\{N_t = m\}$. Therefore, by the strong law of large numbers applied to N, X_t/t converges in law to the uniform measure on $[0, 1]$. From this alone, one can deduce that almost surely

$$\liminf_{t\to\infty} \frac{X_t}{t} = 0 \qquad \text{and} \qquad \limsup_{t\to\infty} \frac{X_t}{t} = 1.$$

This paper is an attempt to refine these statements. Unlike the recurrent case, where the liminf (viz., Bass and Griffin) is harder to derive than the limsup (viz., Erdős and Révész), in the present case it is quite the opposite.

Although sometimes in disguise, the favorite point process, X_t, appears quite naturally in the literature on maximal spacings. For instance, the fact that X_t/t is nearly uniformly distributed on [0,1] appears in several applications; see Slud (1978) and Pyke (1970,1980). For a thorough survey paper on this subject, see Pyke (1965).

We begin with a result on the lower envelope of X, which demonstrates the rate of escape of X_t as $t \to \infty$.

Theorem 1.1. *Suppose $\psi : [1,\infty) \mapsto \mathbb{R}^1_+$ is decreasing to zero as $t \to \infty$ and that $t \mapsto t\psi(t)$ is increasing. Then almost surely*

$$\liminf_{t\to\infty} \frac{X_t}{t\psi(t)} = \begin{cases} 0, & \text{if } J(\psi) = \infty \\ \infty, & \text{if } J(\psi) < \infty \end{cases},$$

where

$$J(\psi) \triangleq \int_1^\infty \psi(t)\frac{dt}{t}.$$

For example, one can take $\psi(t) = \big(\ln t\big)^{-a}$ to see that almost surely

$$\liminf_{t\to\infty} \frac{(\ln t)^a}{t} X_t = \begin{cases} 0, & \text{if } a \le 1 \\ \infty, & \text{if } a > 1 \end{cases}.$$

We have the following theorem on the upper envelope of X.

Theorem 1.2. *Almost surely*

$$\limsup_{t\to\infty} \frac{X_t - t}{\sqrt{2t\ln\ln t}} = 1.$$

It should be noted that this is not an ordinary law of the iterated logarithm: it is closer, in spirit, to the one–sided laws of the iterated logarithm described in Pruitt (1981). As we have already noted, X_t/t is asymptotically uniformly distributed on $[0,1]$. From this, it is evident that $\mathbb{E}(X_t) \sim t/2$ as $t \to \infty$. From this perspective, the centering in Theorem 1.2 is rather exotic. Moreover, unlike the case for the classical law of the iterated logarithm, $(X_t - t)/\sqrt{t}$ is not converging weakly to the normal law. Indeed, for each $a \ge 0$, our Lemma 3.1 implies

$$\mathbb{P}(X_t \ge t + a\sqrt{t}) \to 0 \qquad \text{as } t \to 0.$$

This shows that the event that X is near its upper envelope is very rare; so rare that proving Theorem 1.2 by the usual methods (i.e., obtaining sharp probability estimates and then using blocking arguments) is formidable, if not impossible. Instead we will take a different route: we will define a sequence of stopping times, (T_k), and show that (with probability one) infinitely often $N(T_k)$ is large in the sense of the law of the iterated logarithm and $N(T_k) = X(T_k)$ eventually. Thus, we show that X automatically inherits a law of the iterated logarithm from N.

We also have the following refinement of the upper half of Theorem 1.2:

Theorem 1.3. *Let $\psi : [1, \infty) \mapsto \mathbb{R}^1_+$ be increasing to infinity as $t \to \infty$. Suppose further that ψ satisfies*

$$\int_1^\infty \exp\big(- \psi^2(t)/2\big) \frac{dt}{t\psi(t)} < \infty.$$

Then almost surely $X_t \le t + \sqrt{t}\psi(t)$ for all t sufficiently large.

Consequently for any $p > 1$ and with probability one,

$$X_t \le t + \sqrt{t}\sqrt{2 \ln\ln t + p \ln\ln\ln t} \qquad \text{eventually,}$$

while, by the integral test of Feller (see Bai (1989)),

$$N_t \ge t + \sqrt{t}\sqrt{2 \ln\ln t + 3 \ln\ln\ln t} \qquad \text{infinitely often.}$$

Finally, we point out that by the strong law of large numbers for N and our Theorem 1.1, one easily obtains the following result on the size of the gap between N_t and X_t.

Corollary 1.4. *With probability one,*

$$\limsup_{t\to\infty} \frac{N_t - X_t}{t} = 1.$$

The corresponding liminf result is trivial, since by its very definition, $X_t = N_t$, infinitely often.

2. Proof of Theorem 1.1.

In this section, we will prove Theorem 1.1. Before doing so, we will prove some lemmas and propositions concerning the distribution of X_t (Lemma 2.1), the joint distribution of X_s and X_t (Proposition 2.3) and the rate of convergence of X_t/t to the uniform measure on $[0, 1]$ as $t \to \infty$ (Lemma 2.5).

We start with the following characterization of the joint distribution of the interarrival times of a conditioned Poisson process: Given $\{N_t = m\}$, the joint distribution of $(\xi_t(0), \ldots, \xi_t(m))$ is the same as that of $(Y_0/S, \ldots, Y_m/S)$, where $(Y_i, 0 \le i \le m)$ is a collection of independent exponentially distributed random variables with parameter one, and $S = t^{-1}(Y_0 + \ldots + Y_m)$ (see, e.g., Karlin and Taylor (1981), p. 105). As such, for $0 \le k \le m$, we have

$$\mathbb{P}(X_t = k | N_t = m) = \mathbb{P}(Y_k > Y_i \text{ for } 0 \le i \le m \text{ with } i \ne k) = \frac{1}{m+1}. \tag{2.1}$$

Thus, given $\{N_t = m\}$, X_t is uniformly distributed on the set $\{0, 1, \ldots, m\}$. From this we can readily calculate the distribution of X_t, which is what we do next.

Lemma 2.1. *For all integers* $k \geq 1$,

(1) $\mathbb{P}(X_t \geq k) = \mathbb{P}(N_t \geq k) - \frac{k}{t}\mathbb{P}(N_t \geq k+1)$.

(2) $\mathbb{P}(X_t < k) = \mathbb{P}(N_t < k) + \frac{k}{t}\mathbb{P}(N_t \geq k+1)$.

Proof. We will demonstrate (1): (2) follows from (1) by taking complements.

Since $X_t \leq N_t$, we have

$$\mathbb{P}(X_t \geq k) = \sum_{m=k}^{\infty} \mathbb{P}(X_t \geq k | N_t = m)\mathbb{P}(N_t = m)$$

By (2.1) we have

$$\mathbb{P}(X_t \geq k | N_t = m) = 1 - \frac{k}{m+1}.$$

Consequently,

$$\begin{aligned}\mathbb{P}(X_t \geq k) &= \mathbb{P}(N_t \geq k) - \sum_{m=k}^{\infty} \frac{k}{m+1}\mathbb{P}(N_t = m) \\ &= \mathbb{P}(N_t \geq k) - \frac{k}{t}\sum_{m=k}^{\infty} \frac{t^{m+1}}{(m+1)!}e^{-t} \\ &= \mathbb{P}(N_t \geq k) - \frac{k}{t}\mathbb{P}(N_t \geq k+1),\end{aligned}$$

which establishes (1). □

Next we turn to estimating the joint distribution of X_s and X_t. To do so, we will need additional notation: given $0 < s < t$, and a nonnegative integer k, let

$$\xi_{s,t}(k) \stackrel{\Delta}{=} \xi_t(k) - \xi_s(k) = \int_s^t 1_{\{k\}}(N_u)du.$$

The following lemma is an important calculation in the joint probability estimate:

Lemma 2.2. *Let* $0 < s < t$ *and let* $0 \leq j \leq m < k$ *be integers. Then*

$$\mathbb{P}(X_s \leq j,\, X_t = k,\, N_s = m) \leq \mathbb{P}(X_s \leq j,\, N_s = m) \cdot \mathbb{P}(X_{t-s} = k - m).$$

Proof. The events $\{N_t = q\}$, $q \geq k$, partition the event, $\{X_s \leq j,\, X_t = k,\, N_s = m\}$. Thus

$$\mathbb{P}(X_s \leq j,\, X_t = k,\, N_s = m) = \sum_{q=k}^{\infty} \mathbb{P}(A_q),$$

where

$$A_q = \{X_s \le j,\, N_s = m,\, X_t = k,\, N_t = q\}$$

Given $\{N_t = q\}$ and $\{X_t = k\}$, we observe that k is the minimum index for which

$$\xi_t(k) \ge \xi_t(i) \quad \text{for } 0 \le i \le q.$$

Since $N_s = m < k$, we note that $\xi_s(k) = 0$; thus, $\xi_t(k) = \xi_{s,t}(k)$. Now $\xi_{s,t}(i) \le \xi_t(i)$ for all $i \ge 0$; thus, k is the minimum index for which

$$\xi_{s,t}(k) \ge \xi_{s,t}(i) \quad \text{for } m \le i \le q.$$

Given $N_s = m$, and $m \le i \le q$, we have

$$\xi_{s,t}(i) = \int_s^t 1_{\{i-m\}}(N_u - N_s)du,$$

which is independent of the event $\{N_s = m,\, X_s \le j\}$ and, as a process in $i \in \{m, \ldots, q\}$, is distributed as $\{\xi_{t-s}(i-m),\, m \le i \le q\}$. Hence

$$\mathbb{P}(A_q) \le \mathbb{P}(X_s \le j,\, N_s = m) \cdot \mathbb{P}(X_{t-s} = k - m,\, N_{t-s} = q - m).$$

We obtain the desired result upon summing over q. □

Concerning this result, it should be noted that it was significant that $N_s < X_t$: this is what permitted us to identify the most visited site of the random walk between times s and t. If $N_s \ge X_t$, then all knowledge of the most visited site between times s and t is lost.

In our next result, we obtain an estimate for the joint distribution of X_s and X_t.

Proposition 2.3. *Let $0 < s < t$ and $0 < a < b$. Then*

$$\mathbb{P}(X_s \le a,\, X_t \le b) \le \mathbb{P}(X_t \le a) + \mathbb{P}(X_t = N_s) + \mathbb{P}(X_s \le a) \cdot \mathbb{P}(X_{t-s} \le b).$$

Proof. Without loss of generality, we may assume that a and b are integers. Certainly

$$\mathbb{P}(X_s \le a,\, X_t \le b) \le \mathbb{P}(X_t \le a) + \mathbb{P}(X_t = N_s) + \mathbb{P}(X_s \le a,\, a < X_t \le b,\, X_t \ne N_s).$$

However, if $X_s < X_t$, then there is a (random) time $t_0 \in (s,t]$ at which $X_{t_0} = N_{t_0}$. Therefore, $X_t \geq N_s$. Thus

$$\mathbb{P}(X_s \leq a,\, X_t \leq b) \leq \mathbb{P}(X_t \leq a) + \mathbb{P}(X_t = N_s) + \\ + \sum_{m=0}^{b-1} \mathbb{P}(X_s \leq a,\, a < X_t \leq b,\, N_s < X_t,\, N_s = m).$$

If $0 \leq m \leq a-1$, then, by Lemma 2.2, this summand can be estimated as follows:

$$\mathbb{P}(X_s \leq a,\, a < X_t \leq b,\, N_s = m) \leq \mathbb{P}(X_s \leq a,\, N_s = m) \cdot \mathbb{P}(X_{t-s} \leq b).$$

Likewise, if $a \leq m \leq b$, then, by Lemma 2.2, this summand can be estimated as follows:

$$\mathbb{P}(X_s \leq a,\, m < X_t \leq b,\, N_s = m) \leq \mathbb{P}(X_s \leq a,\, N_s = m) \cdot \mathbb{P}(X_{t-s} \leq b).$$

We obtain the desired result upon summing over m. □

Let

$$\lambda(x) = \begin{cases} x \ln x + 1 - x & \text{for } x > 0 \\ 1 & \text{for } x = 0 \end{cases}.$$

Standard exponential Chebyshev's inequality arguments can be used to show the following:

$$\text{(2.2}a\text{)} \qquad \mathbb{P}(N_t \geq tx) \leq e^{-t\lambda(x)} \quad \text{for } x > 1$$

$$\text{(2.2}b\text{)} \qquad \mathbb{P}(N_t \leq tx) \leq e^{-t\lambda(x)} \quad \text{for } 0 \leq x < 1.$$

From this it is easy to obtain the following estimates:

Lemma 2.4. *Let $(N_t,\, t \geq 0)$ be a rate one Poisson process. Then there exists a positive constant c such that*

(1) $\mathbb{P}(N_t \geq t + \alpha\sqrt{t}) \leq e^{-\alpha^2/4}$ *for* $0 \leq \alpha \leq \sqrt{t}/2$.

(2) $\mathbb{P}(N_t \geq t + \alpha\sqrt{t}) \leq e^{-c\alpha\sqrt{t}}$ *for* $\alpha \geq \sqrt{t}/2$.

(3) $\mathbb{P}(N_t \leq t - \alpha\sqrt{t}) \leq e^{-\alpha^2/2}$ *for* $\alpha \geq 0$.

Proof. Throughout let $x = 1 + \alpha/\sqrt{t}$. Consider (1): the case $\alpha = 0$ is trivial; thus, without loss of generality, assume that $0 < \alpha \leq \sqrt{t}/2$ and hence $1 < x \leq 3/2$. Since

$$\ln(x) \geq (x-1) - \frac{(x-1)^2}{2} \quad \text{for } x \geq 1,$$

we have $\lambda(x) \geq (x-1)^2/4 = \alpha^2/(4t)$; thus (1) follows from (2.2).

The condition: $\alpha \geq \sqrt{t}/2$ implies $x \geq 3/2$. Since $\lambda(x)$ is convex and $\lambda(1) = 0$, we obtain $\lambda(x) \geq \frac{2}{3}\lambda(3/2)(x-1)$ for $x \geq 3/2$. Letting $c = 2\lambda(3/2)/3$, we obtain (2) from (2.1).

To verify (3), observe that Taylor's theorem implies $\lambda(x) \geq (x-1)^2/2$ for $0 \leq x < 1$: (3) follows immediately from (2.2). □

Next we calculate the rate with which X_t/t converges to the uniform measure on $[0,1]$.

Lemma 2.5.

(1) Let $0 \leq \alpha \leq 1$, *then* $\lim_{t\to\infty} |\mathbb{P}(X_t \leq \alpha t) - \alpha| = 0$.

(2) Let $0 < \alpha^* < 1$, *then, for all* t *sufficiently large,*

$$\sup_{0\leq\alpha\leq\alpha^*} |\mathbb{P}(X_t \leq \alpha t) - \alpha| \leq \frac{2}{t}.$$

Remark 2.5.1. Lemma 2.5(1) shows that X_t/t converges in law to the uniform measure on $[0,1]$. Lemma 2.5(2) gives a uniform rate for this convergence for α bounded away from 1. With more care, one can show that $|\mathbb{P}(X_t \leq \alpha t) - \alpha|$ is uniformly bounded by $6t^{-1/3}(\ln t)^{1/3}$ for $0 \leq \alpha \leq 1$ and t sufficiently large. Since we will make no use of this latter fact, we will not prove it here.

Proof. By Lemma 2.2, we have

$$\begin{aligned}\mathbb{P}(X_t \leq \alpha t) &= \mathbb{P}(X_t \leq \lfloor \alpha t \rfloor) \\ &= \mathbb{P}(N_t \leq \lfloor \alpha t \rfloor) + \frac{\lfloor \alpha t \rfloor + 1}{t}\mathbb{P}(N_t \geq \lfloor \alpha t \rfloor + 2).\end{aligned}$$

Thus

$$|\mathbb{P}(X_t \leq \alpha t) - \alpha| \leq \frac{1}{t} + (1-\alpha)\mathbb{P}(N_t \leq \alpha t).$$

To verify (1), observe that the case $\alpha = 1$ is trivial. If $0 \leq \alpha < 1$, then (2.2) shows that $\mathbb{P}(N_t \leq \alpha t) \leq e^{-t\lambda(\alpha)}$ (with $\lambda(\alpha) > 0$), which tends to zero as $t \to \infty$: this demonstrates (1).

To establish (2), let α^* be as given. If $0 \leq \alpha \leq \alpha^*$, then, for all t sufficiently large,

$$\mathbb{P}(N_t \leq \alpha t) \leq \mathbb{P}(N_t \leq \alpha^* t) \leq e^{-t\lambda(\alpha^*)} \leq \frac{1}{t},$$

which demonstrates (2) and hence the lemma. □

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let ψ be as in the statement of Theorem 1.1. In fact, we will show that $J(\psi) < \infty$ implies $\mathbb{P}\big(X_t \le t\psi(t),\ \text{i.o.}\big) = 0$ whereas $J(\psi) = \infty$ implies $\mathbb{P}\big(X_t \le t\psi(t),\ \text{i.o.}\big) = 1$. This suffices to prove the theorem. Indeed, suppose $J(\psi) < \infty$. Then for any $K > 0$, $J(K\psi) = KJ(\psi) < \infty$. Hence, we have shown that for all $K > 0$

$$\liminf_{t\to\infty} \frac{X_t}{t\psi(t)} \ge K.$$

Letting $K \to \infty$, we see that $J(\psi) < \infty$ implies that the above liminf is infinity. Likewise, if $J(\psi) = \infty$, so is $J(\varepsilon\psi) = \varepsilon J(\psi)$, for any $\varepsilon > 0$. Hence we have shown that

$$\liminf_{t\to\infty} \frac{X_t}{t\psi(t)} \le \varepsilon.$$

The theorem is proved upon letting $\varepsilon \to 0$.

For $j \ge 1$, let $t_j \triangleq 2^j$ and

$$A_j \triangleq \{X_{t_j} \le t_j\psi(t_j)\}.$$

By Lemma 2.5, for j sufficiently large, we have

$$|\mathbb{P}(A_j) - \psi(t_j)| \le \frac{1}{2^{j-1}}.$$

This, together with the inequalities

$$\frac{1}{2}\psi(t_{j+1}) \le \int_{t_j}^{t_{j+1}} \frac{\psi(t)}{t} dt \le \psi(t_j),$$

demonstrates that $\sum_j \mathbb{P}(A_j)$, $\sum_j \psi(t_j)$ and $J(\psi)$ converge or diverge simultaneously.

Suppose that $\sum_j \psi(t_j)$ converges. Let $\eta(t) = 2\psi(2t)$ and let $B_j = \{X_{t_j} \le t_j\eta(t_j)\}$. Then the convergence of $\sum_j \eta(t_j)$ implies the convergence of $\sum_j \mathbb{P}(B_j)$. By the Borel-Cantelli lemma, it follows that X_{t_j} eventually exceeds $t_j\eta(t_j)$ with probability one. As a consequence, if $t_j \le t \le t_{j+1}$, then eventually

$$X_t \ge X_{t_j} > t_j\eta(t_j) = t_{j+1}\psi(t_{j+1}) \ge t\psi(t) \quad \text{a.s.}$$

(where we have used the fact that $t \mapsto t\psi(t)$ is increasing to obtain the last inequality). Thus the convergence of $J(\psi)$ implies $\mathbb{P}(X_t \le t\psi(t), \text{i.o.}) = 0$.

If, however, $J(\psi)$ diverges, then by the above considerations, $\sum_j \mathbb{P}(A_j) = \infty$. Consequently to show $\mathbb{P}(A_j, \text{ i.o.}) = 1$, by the Kochen-Stone lemma (see Kochen and Stone (1964)) it suffices to demonstrate that

$$\mathbb{P}(A_j \cap A_k) \le \mathbb{P}(A_j)\mathbb{P}(A_k) + R(j,k) \quad \text{for } 1 \le j < k \text{ with} \tag{2.3a}$$

$$\sum_{1 \le j < k \le n} R(j,k) = o\left(\Sigma_n^2\right) \qquad \text{as } n \to \infty, \tag{2.3b}$$

where

$$\Sigma_n \triangleq \sum_{j=1}^{n} \mathbb{P}(A_j). \tag{2.3c}$$

By Proposition 2.3,

$$\mathbb{P}(A_j \cap A_k) \le A(j,k) + B(j,k) + C(j,k),$$

where

$$\begin{aligned} A(j,k) &\triangleq \mathbb{P}(X_{t_k} \le t_j \psi(t_j)) \\ B(j,k) &\triangleq \mathbb{P}(N_{t_j} = X_{t_k}) \\ C(j,k) &\triangleq \mathbb{P}(A_j) \cdot \mathbb{P}(X_{t_k - t_j} \le t_k \psi(t_k)). \end{aligned}$$

By Lemma 2.5,

$$A(j,k) \le 2^{j-k}\psi(t_j) + 2^{1-k}.$$

Therefore as $n \to \infty$,

$$\sum_{1 \le j < k \le n} A(j,k) = o\left(\Sigma_n^2\right).$$

The term $B(j,k)$ is quite small: with high probability N_{t_j} is close to t_j (deviations from t_j can be measured by Lemma 2.4). However, with small probability, X_{t_k} will assume a value in a neighborhood of t_j (being, more or less, distributed uniformly in $[0, t_k]$). Precisely we have $B(j,k) \le B_1(j,k) + B_2(j,k)$, where

$$\begin{aligned} B_1(j,k) &= \mathbb{P}\left(N_{t_j} \le t_j - 2\sqrt{t_j \ln t_k}\right) + \mathbb{P}\left(N_{t_j} \ge t_j + 2\sqrt{t_j \ln t_k}\right) \\ B_2(j,k) &= \mathbb{P}\left(t_j - 2\sqrt{t_j \ln t_k} \le X_{t_k} \le t_j + 2\sqrt{t_j \ln t_k}\right). \end{aligned}$$

By Lemmas 2.4 and 2.5, we obtain the following:

$$\begin{aligned} B_1(j,k) &\le \frac{1}{4^k} + \frac{1}{2^k} + \exp\left(-c\sqrt{t_j \ln t_k}\right) \\ B_2(j,k) &\le \frac{2^{j/2}k + 1}{2^{k-2}}, \end{aligned}$$

which demonstrates that $\sum_{1\le j<k\le n} B(j,k)$ is bounded as $n\to\infty$.

Finally, for j sufficiently large, we have

$$\mathbb{P}(X_{t_k-t_j}\le t_k\psi(t_k))\le \mathbb{P}(A_k)+\frac{3}{2^{k-j-1}}.$$

Thus, eventually

$$C(j,k)\le \mathbb{P}(A_j)\cdot\mathbb{P}(A_k)+\mathbb{P}(A_j)\frac{3}{2^{k-j-1}}.$$

Since $\sum_{1\le j<k\le n}\mathbb{P}(A_j)2^{j-k+1}=o(\Sigma_n^2)$ as $n\to\infty$, this implies (2.3), which proves the Theorem in question. □

3. Proof of Theorem 1.3.

We shall begin this section with a lemma which will lead to the proof of Theorem 1.3.

Lemma 3.1. *Let $1\le\alpha\le\sqrt{t}$. Then*

$$\mathbb{P}(X_t\ge t+\alpha\sqrt{t})=\frac{\alpha}{\sqrt{2\pi t}}\int_\alpha^\infty x^{-2}e^{-x^2/2}dx \quad + \quad O\left(\frac{\alpha^2}{t}\right).$$

Proof. Let $k\stackrel{\Delta}{=}\lceil t+\alpha\sqrt{t}\,\rceil$ and observe that

$$\begin{aligned}\mathbb{P}(X_t\ge t+\alpha\sqrt{t})&=\mathbb{P}(X_t\ge k)\\ &=\mathbb{P}(N_t=k)+\left(1-\frac{k}{t}\right)\mathbb{P}(N_t\ge k+1),\end{aligned}$$

by an application of Lemma 2.1(2). We will estimate the first term on the right hand side directly and the second term by the classical Berry-Esseen theorem.

Let $\delta\stackrel{\Delta}{=}k-t$. By the Stirlings formula,

$$k!=(k/e)^k\sqrt{2\pi k}\cdot(1+\varepsilon(k)),$$

where $k\varepsilon(k)\to 1/12$ as $k\to\infty$ (see, e.g., Artin (1964), p. 24). Thus

$$\mathbb{P}(N_t=k)=\frac{1}{\sqrt{2\pi k}}(k/t)^{-k}e^{\delta}\cdot(1+O(1/t)). \tag{3.1}$$

Recall the following inequality: for $x\ge 0$,

$$\exp\left(-kx+kx^2/2-kx^3/3\right)\le(1+x)^{-k}\le\exp\left(-kx+kx^2/2\right).$$

Therefore,

$$(k/t)^{-k} = \exp\left(-\delta - \frac{\delta^2}{2t}\right) \cdot (1 + O(\delta/t)).$$

Also, we have

$$\frac{1}{\sqrt{k}} = \frac{1}{\sqrt{t}} \cdot (1 + O(\delta/t)).$$

Inserting this into (3.1), we obtain the following:

$$\mathbb{P}(N_t = k) = \frac{1}{\sqrt{2\pi t}} \exp\left(\frac{-\delta^2}{2t}\right) \cdot (1 + O(\delta/t)).$$

However, $\delta = \lceil t + \alpha\sqrt{t}\,\rceil - t$. Thus

$$\exp\left(\frac{-\delta^2}{2t}\right) = \exp\left(\frac{-\alpha^2}{2}\right) \cdot (1 + O(\alpha/\sqrt{t})).$$

Consequently

$$\mathbb{P}(N_t = k) = \frac{1}{\sqrt{2\pi t}} \exp\left(\frac{-\alpha^2}{2}\right) \cdot (1 + O(\alpha/\sqrt{t})). \tag{3.2}$$

Let

$$\Phi(b) \triangleq \frac{1}{\sqrt{2\pi}} \int_b^\infty e^{-x^2/2} dx.$$

By the classical Berry-Esseen theorem (and some basic estimates), we have,

$$\begin{aligned} \mathbb{P}(N_t \geq k+1) &= \Phi\big((k+1-t)/\sqrt{t}\big) + O(1/\sqrt{t}) \\ &= \Phi(\alpha) + O(\alpha/\sqrt{t}). \end{aligned}$$

Moreover,

$$\frac{t-k}{t} = \frac{-\alpha}{\sqrt{t}} + O(1/t).$$

Hence

$$\frac{t-k}{t} \cdot \mathbb{P}(N_t \geq k+1) = \frac{-\alpha}{\sqrt{t}} \Phi(\alpha) + O(\alpha^2/t).$$

Finally, an integration by parts reveals:

$$\Phi(\alpha) = \frac{1}{\sqrt{2\pi}\alpha} e^{-\alpha^2/2} - \frac{1}{\sqrt{2\pi}} \int_\alpha^\infty x^{-2} e^{-x^2/2} dx.$$

Combining this with (3.2), we obtain

$$\mathbb{P}(X_t \geq t + \alpha\sqrt{t}) = \frac{\alpha}{\sqrt{2\pi t}} \int_\alpha^\infty x^{-2} e^{-x^2/2} dx + O(\alpha^2/t),$$

as was to be shown. □

Now we are prepared to prove Theorem 1.3.

Proof of Theorem 1.3. By Theorem 1.2 (proved in the next section) and a classical argument of Erdős (1942), we can assume that

$$(1.3)\sqrt{\ln\ln t} < \psi(t) < (1.5)\sqrt{\ln\ln t} \tag{3.3}$$

For all $t \geq 1$, let

$$\psi_0(t) \triangleq \max\left\{1, \psi(t) - \frac{7}{\psi(t)}\right\}$$

Observe that $t \mapsto \psi_0(t)$ is nondecreasing and $\psi_0(t) \sim \psi(t)$ as $t \to \infty$. From this and (3.3), it follows that

$$\sqrt{\ln\ln t} < \psi_0(t) < 2\sqrt{\ln\ln t} \tag{3.4}$$

for all t sufficiently large. Moreover, for all t sufficiently large,

$$\exp\left(-\frac{\psi_0^2(t)}{2}\right) \leq C\exp\left(-\frac{\psi^2(t)}{2}\right).$$

for some positive constant C. Consequently,

$$\int_1^\infty \exp\left(-\frac{\psi_0^2(t)}{2}\right)\frac{dt}{t\psi_0(t)} < \infty. \tag{3.5}$$

Finally, for future reference, we point out that, for all t sufficiently large,

$$\psi_0(t) + \frac{6}{\psi_0(t)} \leq \psi(t). \tag{3.6}$$

For each integer $m \geq 20$, let

$$t_m \triangleq \frac{m^2}{\ln\ln(m)}$$

By the mean value theorem,

$$t_{m+1} - t_m \sim \frac{2m}{\ln\ln(m)} \tag{3.7}$$

as $m \to \infty$. Observe that, for all m sufficiently large,

$$\begin{aligned}\int_{t_{m-1}}^{t_m} \exp\left(-\frac{\psi_0^2(t)}{2}\right)\frac{dt}{t\psi_0(t)} &\geq \exp\left(-\frac{\psi_0^2(t_m)}{2}\right)\frac{(t_m - t_{m-1})}{t_m\psi_0(t_m)} \\ &\geq \frac{1}{m\psi_0(t_m)}\exp\left(-\frac{\psi_0^2(t_m)}{2}\right)\end{aligned} \tag{3.8}$$

where we have used the fact that the integrand is nonincreasing and (3.7) to obtain the first and second inequalities, respectively.

By L'Hôpital's rule, as $a \to \infty$,

$$\int_a^\infty \exp(-x^2/2)x^{-2}dx \sim a^{-3}\exp(-a^2/2).$$

Therefore, by Lemma 3.1, there exist positive constants C_1, C_2, C_3 and C_4 such that

$$\begin{aligned}\mathbb{P}(X_{t_{m+1}} \geq t_{m+1} + \sqrt{t_{m+1}}\psi_0(t_m)) &\leq C_1 \frac{\exp(-\psi^2(t_m)/2)}{\sqrt{t_{m+1}}\psi_0^2(t_m)} + C_2 \frac{\psi_0^2(t_m)}{t_m} \\ &\leq C_3 \frac{\exp(-\psi_0^2(t_m)/2)}{m\psi_0(t_m)} + C_4 \frac{(\ln\ln(m))^2}{m^2},\end{aligned}$$

where we have used (3.4) and (3.7) to obtain the second inequality. By (3.5) and (3.8), it follows that

$$\sum_{m=1}^\infty \mathbb{P}(X_{t_{m+1}} \geq t_{m+1} + \sqrt{t_{m+1}}\psi_0(t_m)) < \infty.$$

Consequently, by the easy half of the Borel–Cantelli lemma,

$$X_{t_{m+1}} \leq t_{m+1} + \sqrt{t_{m+1}}\psi_0(t_m)$$

on a set of full measure and for all m sufficiently large. However, by some algebra,

$$t_{m+1} + \sqrt{t_{m+1}}\psi_0(t_m) = t_m + \sqrt{t_m}\big(\psi_0(t_m) + r_m\big) \tag{3.9}$$

where

$$\begin{aligned} r_m &\triangleq \frac{t_{m+1} - t_m}{\sqrt{t_m}} + \frac{\sqrt{t_{m+1}} - \sqrt{t_m}}{\sqrt{t_m}\psi_0(t_m)} \\ &= \frac{t_{m+1} - t_m}{\sqrt{t_m}}(1 + o(1)),\end{aligned}$$

By (3.4) and (3.7), we have, for all m sufficiently large,

$$\frac{t_{m+1} - t_m}{\sqrt{t_m}} < \frac{5}{\psi_0(t_m)}.$$

Consequently, $r_m < 6/\psi_0(t_m)$ for all m large enough. By inserting this into (3.9) and recalling (3.6), we may conclude that with probability one

$$X_{t_{m+1}} \leq t_m + \sqrt{t_m}\psi(t_m)$$

for all m sufficiently large. Finally, given $t_m < t \le t_{m+1}$, with probability one and for t sufficiently large, we obtain:

$$\begin{aligned} X_t &\le X_{t_{m+1}} \\ &\le t_m + \sqrt{t_m}\psi(t_m) \\ &\le t + \sqrt{t}\psi(t), \end{aligned}$$

which is what we wished to show. □

4. Proof of Theorem 1.2.

As usual, the proof of such a theorem is divided up into upper bound and a lower bound arguments. Since $X_t \le N_t$, the upper bound for the limsup, i.e.,

$$\limsup_{t\to\infty} \frac{X_t - t}{\sqrt{2t\ln\ln t}} \le 1, \qquad \text{a.s.}$$

is a trivial consequence of the law of the iterated logarithm for N. It therefore remains to prove the lower bound, i.e.,

$$\limsup_{t\to\infty} \frac{X_t - t}{\sqrt{2t\ln\ln t}} \ge 1, \qquad \text{a.s.} \tag{4.1}$$

It is not too difficult to see that standard proofs for obtaining such lower bounds fail. In a standard proof of the law of the iterated logarithm for, say, Brownian motion, one shows that the lower bound is achieved along geometric subsequences: these subsequences are sparse enough that the dependence amongst the samples is negligible. Let (t_n) denote an increasing sequence of real numbers and, for $n \ge 1$, let

$$E_n \triangleq \{X_{t_n} \ge t_n + \sqrt{2t_n \ln\ln(t_n)}\}.$$

Lemma 3.1 shows that $\sum_n \mathbb{P}(E_n) < \infty$ whenever $\sum_n t_n^{-1/2} < \infty$, which indicates that the events in question are rare — so rare that any attempt at using the independence half of the Borel–Cantelli lemma is bound to fail. Instead we shall prove (4.1) along a random subsequence, (T_k). For this subsequence, (T_k), we will demonstrate that infinitely often $N(T_k)$ is large (in the sense of the LIL) and eventually $N(T_k) = X(T_k)$.

With this strategy in mind, let us start with some definitions. Given $L \in \mathbb{R}^1_+$, define

$$\tau(L) \triangleq \inf\{t \ge L : N_t - N_{t-L} = 0\}.$$

One of the main results of this section is Lemma 4.3, in which we obtain an exponential lower bound for the random variable

$$\frac{N_{\tau(L)} - \tau(L)}{\sqrt{\tau(L)}}.$$

In preparation for this result, we will need some preliminary lemmas.

We begin by a process of "decimation". More precisely, define for all $k \geq 1$,

$$\mathbb{D}_k \stackrel{\Delta}{=} \{ j2^{-k} : \ j \in \mathbb{Z} \}.$$

Define stopping times,

$$\tau_k(L) \stackrel{\Delta}{=} \inf\{t \in \mathbb{D}_k \cap [L, \infty) : \ N_t - N_{t-L} = 0\}.$$

Of course, $\tau(L) \leq \tau_k(L)$ and $\tau_{k+1}(L) \leq \tau_k(L)$, for all $k \geq 0$. In fact, we have the following strong convergence result:

Lemma 4.1. *For each $L > 0$, with probability one,*

$$\lim_{k \to \infty} \tau_k(L) = \tau(L).$$

Proof. Let $\varepsilon > 0$ be fixed. Choose k large enough that $2^{-k} < \varepsilon$. Then, by the strong Marov property, we obtain:

$$\mathbb{P}\big(|\tau(L) - \tau_k(L)| > \varepsilon\big) \leq \mathbb{P}\big(N_{2^{-k}} > 0\big).$$

Since $\mathbb{P}\big(N_{2^{-k}} > 0\big) \sim 2^{-k}$, it follows that

$$\sum_{k=1}^{\infty} \mathbb{P}\big(|\tau(L) - \tau_k(L)| > \varepsilon\big) < \infty.$$

The proof is completed by an application of the easy half of the Borel–Cantelli lemma and letting $\varepsilon \downarrow 0$ along a countable sequence. $\square$

Our next lemma is a modification of a moderate deviation inequality, which we state without proof (see, e.g., Feller (1971), p. 552).

Lemma 4.2. *Given $A \in (0,1)$ there exist $x_0 = x_0(A) > 0$, $\eta = \eta(A) > 0$ and $\rho = \rho(A) > 0$, such that for every $L > 0$, $x \in [x_0, \rho t^{1/6}]$ and $t \geq \eta L^2$,*

$$\mathbb{P}\big(N_{t-L} \geq t + \sqrt{t}x\big) \geq \frac{1}{2} \exp\bigg(-\frac{x^2}{2A}\bigg).$$

Our next lemma shows that such an inequality can be obtained for the stopped Poisson process, $N_{\tau(L)}$.

Lemma 4.3. *Fix $A \in (0,1)$ and $L > 0$. For x_0, η and ρ as given by Lemma 4.2, let $\gamma \triangleq \big(\eta L^2\big) \vee \big(x/\rho\big)^6$. Then for all such $L > 0$ and all $x \geq x_0$,*

$$\mathbb{P}\big(N_{\tau(L)} \geq \tau(L) + \sqrt{\tau(L)}x\big) \geq \frac{1}{2}\mathbb{P}\big(\tau(L) \geq \gamma\big) \exp\bigg(-\frac{x^2}{2A}\bigg).$$

Proof. We begin by recalling some facts about associated random variables. Following Esary, Proschan and Walkup (1967), we say that a collection of random variables, $Y_1, \cdots, Y_n$ is *associated* if

$$\text{Cov}\,\big(f(Y_1, \cdots, Y_n), g(Y_1, \cdots, Y_n)\big) \geq 0,$$

where f and g are any two measurable coordinatewise nondecreasing functions mapping $\mathbb{R}^n$ into $\mathbb{R}$ (provided the covariance is defined). In Esary, Proschan and Walkup (1967), it is shown that independent random variables are associated. Thus, by direct application of the definition, nonnegative linear combinations of a collection of independent random variables are associated. Consequently, a finite collection of (possibly overlapping) increments of a Poisson process is associated. Whenever the collection $Y, Y_1, \cdots, Y_n$ is associated, by choosing f and g to be appropriate indicator functions, we obtain

$$\mathbb{P}\big(Y \geq a, Y_1 \geq a_1, \cdots, Y_n \geq a_n\big) \geq \mathbb{P}\big(Y \geq a\big)\mathbb{P}\big(Y_1 \geq a_1, \cdots, Y_n \geq a_n\big). \tag{4.2}$$

First, we will establish the desired inequality for $\tau_k(L)$. Then we will take limits as $k \to \infty$. With this in mind, let $t \in \mathbb{D}_k$ with $t > \gamma \vee L$. Define the measurable event

$$\begin{aligned} F_t \triangleq & \big\{N_s - N_{s-L} \geq 1 \text{ if } s \in \mathbb{D}_k \cap [L, t-L)\big\} \\ & \cap \big\{N_{t-L} - N_{s-L} \geq 1 \text{ if } s \in \mathbb{D}_k \cap [t-L, t)\big\}. \\ = & \{\tau_k(L) \geq t\} \end{aligned}$$

We note that,

$$\{\tau_k(L) = t\} = F_t \cap \{N_t - N_{t-L} = 0\}.$$

Furthermore, on the event $\{\tau_k(L) = t\}$, we have $N_t = N_{t-L}$. Finally, the events $\{N_{t-L} \geq t + \sqrt{t}x\}$ and F_t are independent of $\{N_t - N_{t-L} = 0\}$. Consequently

$$\mathbb{P}\big(N_t \geq t + \sqrt{t}x \mid \tau_k(L) = t\big) = \mathbb{P}\big(N_{t-L} \geq t + \sqrt{t}x \mid F_t\big).$$

However, from (4.2) it follows that,

$$\mathbb{P}\big(N_{t-L} \geq t + \sqrt{t}x \mid F_t\big) \geq \mathbb{P}\big(N_{t-L} \geq t + \sqrt{t}x\big).$$

From Lemma 4.2, by our choice of γ, this last probability is bounded below by $(1/2) \exp\big(-x^2/(2A)\big)$. Thus, by the law of total probability, we arrive at the following estimate:

$$\mathbb{P}\big(N_{\tau_k(L)} \geq \tau_k(L) + \sqrt{\tau_k(L)}\ x\big) \geq \frac{1}{2}\mathbb{P}\big(\tau_k(L) \geq \gamma\big) \exp\left(-\frac{x^2}{2A}\right). \tag{4.3}$$

By Lemma 3, $\tau_k(L) \downarrow \tau(L)$, a.s. as $k \to \infty$. Since the Poisson process is right continuous, the proof of this lemma is completed upon taking limits in (4.3) as $k \to \infty$. □

In the next lemma, we give the momment generating function of $\tau(L)$.

Lemma 4.4.

(1) For all $\lambda > -1$,

$$\mathbb{E} \exp\big(-\lambda\tau(L)\big) = \frac{(\lambda+1)e^{-\lambda L}}{\lambda e^L + e^{-\lambda L}}.$$

(2) $\mathbb{E}\tau(L) = e^L$.

(3) There exists some $C > 0$ (not depending on L), such that for all $a \geq 1$,

$$\mathbb{P}\big(\tau(L) \leq a\big) \leq Cae^{-L}.$$

Proof. The proof of (1) is based on a technique for the study of patterns connected with repeated trials as developed in Ch.XIII, §7 and §8 of Feller (1971). For simplicity, let $\tau \triangleq \tau(L)$ and for $t \geq L$ define

$$E_t \triangleq \{\omega : \ N_t - N_{t-L} = 0\}.$$

By the law of total probability,

$$\mathbb{P}(E_t) = \int_L^t \mathbb{P}(E_t \mid \tau = s)\mathbb{P}(\tau \in ds).$$

If $L \leq s < t - L$, by independence,

$$\mathbb{P}(E_t \mid \tau = s) = \mathbb{P}(E_t) = \mathbb{P}(N_L = 0) = e^{-L}.$$

Otherwise, we obtain

$$\begin{aligned}\mathbb{P}(E_t \mid \tau = s) &= \mathbb{P}(N_t - N_s = 0, N_s - N_{t-L} = 0 \mid \tau = s)\\ &= e^{-(t-s)}.\end{aligned}$$

Since $\mathbb{P}(E_t) = \exp(-L)$, we arrive at the formula,

$$1 = \mathbb{P}(\tau \leq t - L) + \int_{t-L}^t \exp\big(L + (t-s)\big)\mathbb{P}(\tau \in ds). \tag{4.4}$$

Define a function, g, by

$$g(u) \triangleq \begin{cases} \exp(L-u), & \text{if } 0 \leq u < L \\ 1, & \text{if } u \geq L \\ 0, & \text{if } u < 0 \end{cases}.$$

By (4.4), for all $t \geq L$, we have

$$\int_{-\infty}^{\infty} g(t-s)\mathbb{P}(\tau \in ds) = 1. \tag{4.5}$$

On the other hand, if $t < L$, we have

$$\int_{-\infty}^{\infty} g(t-s)\mathbb{P}(\tau \in ds) = 0. \tag{4.6}$$

Let $H(t) = 1_{[L,\infty)}(t)$ be the indicator function of $[L,\infty)$. Then we have shown in (4.5) and (4.6) that H is a certain convolution. In particular,

$$H(t) = \int_{-\infty}^{\infty} g(t-s)\mathbb{P}(\tau \in ds).$$

Therefore

$$\widetilde{H}(\lambda) = \widetilde{g}(\lambda)\mathbb{E}\exp\big(-\lambda\tau\big),$$

where for any positive function, F, $\widetilde{F}$ denotes the Laplace transform: $\int e^{-\lambda t} F(t)dt$. Since for all $\lambda > -1$,

$$\widetilde{H}(\lambda) = \lambda^{-1} \exp\big(-\lambda L\big),$$
$$\widetilde{g}(\lambda) = \frac{\lambda e^{L} + e^{-\lambda L}}{(\lambda+1)\lambda},$$

we obtain the Laplace transform of τ by solving. Equation (2) follows by differentiating this transform and setting λ to zero. To prove (3), we use Chebyhev's inequality, viz.

$$\begin{aligned}\mathbb{P}(\tau \le a) &= \mathbb{P}\big(\exp(1-\tau/a) \ge 1\big)\\ &\le e\frac{(1+a^{-1})e^{-L/a}}{a^{-1}e^{L} + e^{-L/a}}\\ &\le 2ea\exp(-L).\end{aligned}$$

Taking $C \triangleq 2e$, we arrive at (3). In the above, we have used the inequalities: $0 \le \exp(-L/a) \le 1$ and $a+1 \le 2a$ for $a \ge 1$.

Remark 4.4.1. A by–product of Lemma 4.4(1) is that as $L \to \infty$, $e^{-L}\tau(L)$ converges weakly to a mean one exponential distribution. We shall have no need for this fact.

Let $t_1 \triangleq 1$ and for $k \ge 2$, define $t_k \triangleq t_{k-1} + 4\ln k$. We point out the elementary fact that as $k \to \infty$, $t_k \sim 4k\ln k$. Next we define a family of stopping times, T_k: let

$$T_1 \triangleq \Delta T_1 \triangleq \inf\big\{t \ge t_1 :\ N_t - N_{t-t_1} = 0\big\}.$$

For $k \ge 2$, let

$$T_k \triangleq \inf\big\{t \ge T_{k-1} + t_k :\ N_t - N_{t-t_k} = 0\big\} \quad \text{and} \quad \Delta T_k \triangleq T_k - T_{k-1}.$$

By the strong Markov property, $\{\Delta T_k; k \ge 1\}$ is an independent sequence of random variables with ΔT_k distributed as $\tau(t_k)$. Since $T_k = \sum_{j=1}^{k} \Delta T_j$, by Lemma 4.4(2)

$$\mathbb{E}T_k = \sum_{j=1}^{k} \exp(t_j),$$

from which it is easy to verify that $\mathbb{E}T_k \sim \exp(t_k)$ as $k \to \infty$.

Lemma 4.5. *With probability one,*

(1) $\lim_{k\to\infty}\big(T_{k-1}/T_k\big)=0$*;*

(2) $\lim_{k\to\infty}\big(\ln\ln T_k/\ln\ln \mathbb{E}T_k\big)=1$*;*

(3) $X_{T_k}=N_{T_k}$*, eventually.*

Proof. Let $a_k \stackrel{\Delta}{=} k^2\exp\big(t_{k-1}\big)$ and let $\varepsilon>0$ be fixed. Then

$$\mathbb{P}\big(T_{k-1}\geq \varepsilon T_k\big)\leq \mathbb{P}\big(T_{k-1}\geq \varepsilon a_k\big)+\mathbb{P}\big(T_k\leq a_k\big).$$

By Chebychev's inequality and the remarks preceding the statement of the lemma,

$$\mathbb{P}\big(T_{k-1}\geq \varepsilon a_k\big)\leq \frac{\mathbb{E}T_{k-1}}{\varepsilon a_k}\sim\frac{\exp(t_{k-1})}{\varepsilon a_k}.$$

Thus, there exists some $C_1>0$ such that

$$\mathbb{P}\big(T_{k-1}\geq \varepsilon a_k\big)\leq C_1 k^{-2}.$$

Let us observe that $a_k^{-1}\exp(t_k)=k^2$. Therefore, by Lemma 4.4(3),

$$\mathbb{P}\big(T_k\leq a_k\big)\leq \mathbb{P}\big(\Delta T_k\leq a_k\big)\leq Ck^{-2}.$$

This shows that $\sum_k \mathbb{P}(T_{k-1}\geq \varepsilon T_k)<\infty$. Thus (1) follows from the easy half of the Borel–Cantelli lemma.

To establish (2), we note that $\ln\ln\mathbb{E}T_k\sim\ln t_k$, as $k\to\infty$. Let $\varepsilon\in(0,1)$ be fixed. By Lemma 4.4(3), we obtain:

$$\begin{aligned}\mathbb{P}\big(\ln\ln T_k\geq(1+\varepsilon)\ln\ln t_k\big)&=\mathbb{P}\big(T_k\geq\exp(t_k^{1+\varepsilon})\big)\\&\leq \mathbb{E}T_k\exp\big(-t_k^{1+\varepsilon}\big)\\&\leq 2\exp\big(t_k-t_k^{1+\varepsilon}\big),\end{aligned}\tag{4.7}$$

for all k sufficiently large. Likewise, since $\Delta T_k\leq T_k$,

$$\mathbb{P}\big(\ln\ln T_k\leq(1-\varepsilon)\ln t_k\big)\leq C\exp\big(t_k^{1-\varepsilon}-t_k\big).\tag{4.8}$$

Since the right hand side of (4.7) and (4.8) are both summable in k, (2) follows from the easy half of the Borel–Cantelli lemma.

It remains to prove (3). First let us note that $X(T_k) = N(T_j)$ for some $1 \le j \le k$. Thus

$$P(X(T_k) \neq N(T_k)) = \sum_{j=1}^{k-1} P(X(T_k) = N(T_j)).$$

If $X(T_k) = N(T_j)$, then this would imply $N((t_k - t_j) + T_j) - N(T_j) = 0$. Thus, by the strong Markov property,

$$P(X(T_k) = N(T_j)) \le \exp\big(- (t_k - t_j)\big).$$

Consequently,

$$P(X(T_k) \neq N(T_k)) \le \exp\big(- (t_k - t_{k-1})\big) \sum_{j=1}^{k-1} \exp\big((t_j - t_{k-1})\big).$$

But $t_k - t_{k-1} = 4\ln(k)$ and each summand is dominated by 1, so we arrive at the estimate:

$$P(X(T_k) \neq N(T_k)) \le \frac{1}{k^3},$$

which certainly sums in k. The proof is finished by an application of the easy-half of the Borel-Cantelli lemma. □

We are now ready to prove Theorem 1.2. Let $p \in (0,1)$ be fixed. For every $k \ge 1$, define the event

$$A_k \stackrel{\Delta}{=} \big\{\omega : \ N_{T_k} - N_{T_{k-1}} - \Delta T_k \ge p\sqrt{2\Delta T_k \ln k}\big\}.$$

We will show that $\mathbb{P}\big(A_k, \text{ i.o.}\big) = 1$. By the strong Markov property, (A_k) are independent. Consequently, it is sufficient to prove that $\sum_k \mathbb{P}(A_k) = \infty$.

For each $k \ge 1$, let $S_k \stackrel{\Delta}{=} \tau(t_k)$. By the strong Markov property,

$$\mathbb{P}(A_k) = \mathbb{P}\big(N_{S_k} - S_k \ge p\sqrt{2S_k \ln k}\big).$$

Let $A \stackrel{\Delta}{=} p^2$ and $\gamma_k \stackrel{\Delta}{=} \big(\eta t_k^2\big) \vee \big(p\sqrt{2\ln k}/\rho\big)^6$. By Lemma 4.3, we have

$$\mathbb{P}(A_k) \ge \mathbb{P}(S_k \ge \gamma_k)/2k.$$

It is easy to check that $\lim_k \mathbb{P}(S_k \ge \gamma_k) = 1$. Therefore, we have shown that $\sum_l \mathbb{P}(A_k) = \infty$ and hence that $\mathbb{P}(A_k, \text{ i.o.}) = 1$. It follows that

$$\limsup_{k\to\infty} \frac{N_{T_k} - N_{T_{k-1}} - \Delta T_k}{\sqrt{2\Delta T_k \ln k}} \ge 1, \qquad \text{a.s.}$$

By Lemma 4.5, with probability one, as $k \to \infty$,

$$\sqrt{2\Delta T_k \ln k} \sim \sqrt{2T_k \ln\ln T_k},$$

and

$$\frac{T_{k-1} \ln\ln T_{k-1}}{T_k \ln\ln T_k} \to 0.$$

Therefore, by the ordinary law of the iterated logarithm for N,

$$\lim_{k\to\infty} \frac{N_{T_{k-1}} - T_{k-1}}{\sqrt{2T_k \ln\ln T_k}} = 0, \qquad \text{a.s.}$$

Hence, we have shown the following:

$$\limsup_{k\to\infty} \frac{N_{T_k} - T_k}{\sqrt{2T_k \ln\ln T_k}} \geq 1, \qquad \text{a.s.}$$

Finally, by Lemma 4.5(3), we are done. □

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D. Khoshnevisan
Department of Mathematics
University of Utah
Salt Lake City, UT. 84112
davar@math.utah.edu

T.M. Lewis
Department of Mathematics
Furman University
Greenville SC. 29613
Lewis_Tom/furman@furman.edu