Exceptional Times and Invariance for Dynamical Random Walks

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$\nu :=$ a probability distribution on $(-\infty, \infty)$

Assume: Mean$(\nu) = 0$ and SD$(\nu) = 1$

ν-Random Walk: $X_1(0), X_2(0), \ldots \overset{\text{i.i.d.}}{\sim} \nu$

$S_n(0) := X_1(0) + \cdots + X_n(0)$

Adding Dynamics: (Benjamini, Häggström, Peres, and Steif, *Ann. Prob.*, 2003)

Want $t \mapsto \{S_n(t)\}_{n=1}^{\infty}$ to be stationary, strong Markov in $(-\infty, \infty)^{\mathbf{R}_+}$, and with invariant measure ν.

Thus, in particular, for any $t \geq 0$, $\{S_1(t), S_2(t), \ldots\} \overset{(d)}{=} \{S_1(0), S_2(0), \ldots\}$; evolve in stationarity.

One Interesting Solution:

❶ For every index $i \geq 1$ run an indep't rate-one Poisson process

❷ Every time the PP jumps replace $X_i(t-)$ by an indep't copy [$X_i(0-) := X_i(0)$]

❸ All PP's are independent of all X's

❹ Finally define ($n \geq 1$, $t \geq 0$)

$$S_n(t) := X_1(t) + \cdots + X_n(t)$$

As t varies, $\{S_1(t), S_2(t), \ldots\}$ forms an infinite family of interacting random walks; interactions are "local."

An a.s.-property that holds for $S_1(t), S_2(t), \ldots$ simultaneously for all $t \geq 0$ is said to be "dynamically stable." Else, it is "dynamically sensitive."

Theorem 1 (Benjamini et al) ❶ *If ν has only one moment ($:= 0$), then SLLN is dyn. stable; i.e.,*

$$\mathrm{P}\left\{ \lim_{n\to\infty} \frac{S_n(t)}{n} = 0 \text{ for all } t \geq 0 \right\} = 1.$$

❷ *If Mean$(\nu) = 0$ and SD$(\nu) = 1$, then the LIL is dyn. stable; i.e.,*

$$\mathrm{P}\left\{ \limsup_{n\to\infty} \frac{S_n(t)}{\sqrt{2n \ln\ln n}} = 1 \text{ for all } t \geq 0 \right\} = 1.$$

Define dyn. walks in $\mathbf{Z}^d$ analogously. Then,

Theorem 2 (Benjamini et al, 2003) *If ν defines the simple walk on $\mathbf{Z}^d$ then "Transience" is dyn. stable iff $d \geq 5$.*

[Cf. Pólya: $\{S_1(0), S_2(0), S_3(0), \ldots\}$ is transient iff $d \geq 3$.]

Theorem 3 (Benjamini et al, 2003) *If ν lives on a finite subset of $\mathbf{Z}$ and Mean$(\nu) = 0$ then "Recurrence" is dyn. stable.*

Benjamini et al (2003) conjectured that there probably exists a connection to the OU process in Wiener space. Answer: "Yes." Without having to define the latter process:

Invariance: Suppose Mean$(v) = 0$ and SD$(v) = 1$.

Theorem 4 (Kh., Levin, Méndez, 2004) *As $n \to \infty$,*

$$\left\{ \frac{S_{[ns]}(t)}{\sqrt{n}} \right\}_{s,t \in [0,1]} \xrightarrow{\mathscr{D}([0,1]^2)} \{U_s(t)\}_{s,t \in [0,1]}$$

where U is a centered, Gaussian process with

$$\mathrm{E}\left[U_s(t)U_u(v)\right] = \min(s,u) \times e^{-|t-v|}.$$

For a related result for a related model see Rusakov (*Teor. Veroyatnost. i Primenen*, 1989).

An "explicit construction" of U: Set $U_s(t) = e^{-t}\beta(s, e^{2t})$ where β denotes the Brownian sheet:

$$\mathrm{E}\left[\beta(s,t)\beta(u,v)\right] = \min(s,u) \times \min(t,v).$$

Dyn. Instability of the LIL: Suppose $\nu = N(0,1)$. Set $\Phi = N(0,1)$-cdf, and $\overline{\Phi} = 1 - \Phi$.

Theorem 5 (Kh., Levin, Méndez, *Ann. Prob.*, 2004+) *The integral-test refinement to the LIL is dyn. unstable. In fact, for $H \uparrow$,*

$$S_n(t) > H(n)\sqrt{n} \ {}^{\forall}t \geq 0 \text{ i.o. iff}$$

$$\int_1^\infty H^4(t) \frac{\overline{\Phi}(H(t))\, dt}{t} < \infty.$$

Cf. Erdős: For $t \geq 0$ fixed, $S_n(t) > H(n)\sqrt{n}$ i.o. iff

$$\int_1^\infty H^2(t) \frac{\overline{\Phi}(H(t))\, dt}{t} < \infty.$$

Question: How big is the set of exceptional times t?

A Multifractal Analysis: Set $\nu = N(0,1)$. If $H \uparrow$ then

$$\Lambda_H := \left\{ t \geq 0 : \ S_n(t) > H(n)\sqrt{n} \text{ i.o.} \right\}.$$

Theorem 6 (Kh., Levin, Méndez, 2004) *A.s.:*

$$\dim_{\mathscr{H}} \Lambda_H = \min\left(1\ ,\ \frac{4-\delta(H)}{2}\right), \textit{ where}$$

$$\delta(H) := \sup\left\{ \zeta > 0 : \ \int_1^\infty H^\zeta(t) \frac{\overline{\Phi}(t)\,dt}{t} < \infty \right\}.$$

($\dim_{\mathscr{H}} A < 0$ means $A = \varnothing$.) The proof rests on several calculations, one of which is interesting in the present context:

Moderate Deviations: Let $\nu = N(0,1)$. For any fixed compact set $E \subset [0,1]$ consider $\mathrm{K}_E(\varepsilon)$ to be the *Kolmogorov ε-entropy* of E; i.e., the maximum n for which $\exists\ x_1, \ldots, x_n \in E$ such that $\min_{1 \le i \ne j \le n} |x_i - x_j| > \varepsilon$.

Theorem 7 (Kh., Levin, Méndez, 2004) *Suppose $z_n \uparrow \infty$ while $z_n = o(n^{1/4})$. Then, there exists $c > 1$ such that for all compact $E \subseteq [0,1]$ and all $n \ge 1$,*

$$c^{-1} \le \frac{\mathrm{P}\left\{\sup_{t \in E} S_n(t) \ge z_n \sqrt{n}\right\}}{\mathrm{K}_E(1/z_n^2)\overline{\Phi}(z_n)} \le c.$$

Corollary 8 (Kh., Levin, Méndez, 2004) *Suppose Z is the OU process; i.e., it solves $dZ = -Z\,dt + \sqrt{2}\,dW$. Then, there exists $c > 1$ such that for all compact $E \subseteq [0,1]$ and all $\lambda \ge 1$,*

$$c^{-1} \le \frac{\mathrm{P}\left\{\sup_{t \in E} Z(t) \ge \lambda\right\}}{\mathrm{K}_E(1/\lambda^2)\overline{\Phi}(\lambda)} \le c.$$

Other Implications Exist: For instance, for all compact, non-random $E \subseteq [0,1]$,

$$\sup_{t \in E} \limsup_{n \to \infty} \frac{(S_n(t))^2 - 2n \ln\ln n}{n \ln\ln\ln n} = 3 + 2 \dim_{\mathscr{P}} E,$$

where $\dim_{\mathscr{P}}$ denotes "packing dimension." When $E = \{0\}$ (any singleton will do) $\dim_{\mathscr{P}} E = 0$, and we obtain a classical result of Kolmogorov. On the other hand, $\dim_{\mathscr{P}}[0,1] = 1$, and this yields an earlier results of the authors (*Ann. Prob.*, 2004+).

A Stability Result: If ν denotes a distribution on $\mathbf{Z}$ that has finite support, then a theorem of Benjamini et al (2003) asserts that all $S_n(t)$'s are recurrent simultaneously. This holds for more general walks: Suppose Mean$(\nu) = 0$ and SD$(\nu) = 1$. Also assume that ν has $(2+\varepsilon)$ finite moments for some $\varepsilon > 0$. Then,

Theorem 9 (Kh., Levin, Méndez, 2004) *A.s.:*

$$\sum_{n=1}^{\infty} \mathbf{1}_{\{S_n(t)=0\}} = \infty \quad \forall t \geq 0.$$

- Not a "standard" extension
- We do not know what happens when $\varepsilon = 0$
- Requires a new "gambler's ruin" result of indep't interest:

Gambler's Ruin: Henceforth, $\{x_i\}_{i=1}^{\infty}$ are i.i.d. $\mathbf{Z}$-valued, and define a random walk $s_n := x_1 + \cdots + x_n$. We assume that $\mathrm{E}[x_1] = 0$ and $\mathrm{Var}(x_1) = \sigma^2 < \infty$. Consider the first–passage times,

$$T(z) := \inf\{n \geq 1 : \; s_n = z\}.$$

Gambler's ruin problem (Pascal, Fermat, $\cdots$) asks for an evaluation of $\mathrm{P}\{T(z) \leq T(0)\}$. If x's are nice, then use martingales. In general, this idea does not seem to work.

Theorem 10 (Kh., Levin, Méndez, 2004) *If G denotes the additive subgroup of $\mathbf{Z}$ generated by the possible values of $\{s_n\}_{n=1}^{\infty}$ then there exists $c = c(\sigma^2, G) > 1$ such that for all $z \in G$,*

$$\frac{c^{-1}}{1+|z|} \leq \mathrm{P}\{T(z) \leq T(0)\} \leq \frac{c}{1+|z|}.$$

An Outline: First prove that $\mathrm{P}\{T(0)>n\} \asymp n^{-1/2}$. [Half is easy: $\mathrm{P}\{T(0)>n\} \geq \mathrm{P}\{\mathscr{T}>n\}$ where $\mathscr{T}$ denotes the first time s_n enters $(-\infty, 0)$. Then appeal to Feller's Tauberian estimates.]

Then go one more step and prove that $\mathrm{P}_z\{T(0)>n\} \asymp |z|/\sqrt{n}$ (lower bound OK if $|z| = O(\sqrt{n})$; upper bound generic.) Once again, half is easy: $\mathrm{P}_z\{T(0)>n\} \geq \mathrm{P}_z\{\mathscr{T}>n\}$, which is greater than $c|z|/\sqrt{n}$ (Pemantle and Peres, 1995).

One more easy half-proof: By the strong Markov property,

$$\mathrm{P}\{T(0)>n\} \geq \mathrm{P}\{T(z) \leq T(0)\} \times \mathrm{P}_z\{T(0)>n\}.$$

Assemble the preceding 2 estimates to obtain an upper bound for $\mathrm{P}\{T(0)<T(z)\}$.