Additive Lévy Processes, I: Background and Motivation

Davar Khoshnevisan
University of Utah, USA
davar@math.utah.edu
http://www.math.utah.edu/~davar

Workshop on Probabilistic Analysis
Taipei, June 2006
Outline of Lecture 1:

- Definition of LPs.
- Examples of LPs.
- Definition of ALPs.
- Do ALPs arise naturally?
- An application of ALPs to Lévy processes.
- Additive Brownian Motion and the Brownian Sheet [time permitting].
Definition of LPs

Let $X := \{X(t)\}_{t \geq 0}$ be a stoch. process [i.e., a sequence of rv’s indexed by $\mathbb{R}_+]$. Suppose it takes values in $\mathbb{R}^d$; i.e., $X(t) \in \mathbb{R}^d$ with probab. one.

$X$ is a Lévy process if:

- For all $t, s \geq 0$, $\{X(t+s) - X(s)\}_{t \geq 0}$ is [totally] independent of $\{X(u)\}_{0 \leq u \leq s}$; [“indep. incs”]

- for all $t, s \geq 0$, $\{X(t+s) - X(s)\}_{t \geq 0}$ has the same [fi-di] distributions as $\{X(t)\}_{t \geq 0}$; [“stat. inc’s”]

- $X(0) = 0$ and $X$ is continuous in $L^0(\Omega, \mathcal{F}, P)$.

The distribution of the entire process $X$ depends on the distribution of $X(t)$ which we realize via the Lévy–Khintchine formula for

$$
\mathbb{E} e^{i\xi \cdot X(t)} = e^{-t \Psi(\xi)} \quad \forall \xi \in \mathbb{R}^d, t \geq 0.
$$

$\Psi :=$ Lévy exponent of $X$. 
A Connection to Semigroups

Define for all \( f : \mathbb{R}^d \to \mathbb{R}_+ \) [Borel meas.], \( x \in \mathbb{R}^d \), and \( t \geq 0 \),

\[
(T_t f)(x) := \mathbb{E}[f(x + X(t))].
\]

Let \( \hat{f}(\xi) := \int_{\mathbb{R}^d} e^{i\xi \cdot x} f(x) \, dx \) and note that if \( f, \hat{f} \in L^1(\mathbb{R}^d) \), then \( f(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i\xi \cdot x} \hat{f}(\xi) \, d\xi \). Thus,

\[
(T_t f)(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathbb{E}[e^{i\xi \cdot X(t)}] e^{i\xi \cdot x} \hat{f}(\xi) \, d\xi
\]

\[
= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-t\Psi(\xi)} e^{i\xi \cdot x} \hat{f}(\xi) \, d\xi.
\]

Thus, \( T_t \) is a convolution kernel with multiplier \( \hat{T}_t(\xi) = e^{-t\Psi(\xi)} \).

\[\{T_t\}_{t \geq 0} \text{ is a convolution semigroup.}\]
Example: Brownian Motion

Think of $X : \mathbb{R}^+ \to \mathbb{R}^d$ as a “random function.”

- (Bachelier, 1900; Einstein, 1905) $\Psi(\xi) = \|\xi\|^2$.
- (Wiener, 1910) $X$ is continuous a.s.
- (Paley, Wiener, Zygmund, 1933) $X$ is nowhere-differentiable a.s.
- (Taylor, 1952/53) The random image-set $X(\mathbb{R}^+)$ a.s. has Hausdorff dimension $\min(d, 2)$.
- The Hille–Yosida generator of $\{T_t\}_{t \geq 0}$ is $\Delta$ [distribution sense]; i.e., $T_t = e^{t\Delta}$. 
Recall on Hausdorff Dimension

Let \( s, \varepsilon > 0 \) be fixed; \( A \subset \mathbb{R}^d \) a set.

\[
\mathcal{H}_\varepsilon^s(A) := \inf \sum_{j=1}^{\infty} (2r_j)^s,
\]

where the infimum is taken over all balls \( B_1, B_2, \ldots \) of respective radii \( r_1, r_2, \ldots \in (0, \varepsilon) \) such that \( \bigcup_{j=1}^{\infty} B_j \supset A \). The \( s \)-dimensional Hausdorff measure of \( A \) is

\[
\mathcal{H}^s(A) := \lim_{\varepsilon \to 0^+} \mathcal{H}_\varepsilon^s(A).
\]

\( \mathcal{H}^s \) is an outer measure; measure on Borel sets.

\[
\dim_H A := \sup \{ s > 0 : \mathcal{H}^s(A) > 0 \}
\]

\[
= \inf \{ s > 0 : \mathcal{H}^s(A) < \infty \}.
\]

(Hausdorff, \( \leq 1927 \))
A Relation to [Bessel-] Riesz Capacities

\[ I_s(\mu) := \begin{cases} 
\int \int \|x - y\|^{-s} \mu(dx) \mu(dy), & \text{if } s > 0, \\
\int \int \log_+ \|x - y\|^{-1} \mu(dx) \mu(dy), & \text{if } s = 0, \\
1, & \text{if } s < 0. 
\end{cases} \]

\[ \text{Cap}_s(A) := \left[ \inf_{\mu \in \mathcal{P}(A)} I_s(\mu) \right]^{-1}, \quad [\inf \emptyset := \infty, \ 1/\infty := 0]. \]

**Theorem. [Frostman, 1935]** For all Borel sets \( A \subset \mathbb{R}^d \),

\[ \dim_H A = \sup \{ s > 0 : \text{Cap}_s(A) > 0 \} \]

\[ = \inf \{ s > 0 : \text{Cap}_s(A) = 0 \}. \]
Example: Isotropic Stable Processes

- (Lévy, 1937) \( \Psi(\xi) = \|\xi\|^\alpha; \alpha \in (0, 2] \).
- \( T_t = \exp(t\Delta^{\alpha/2}) \).
- (Lévy, 1937) When \( \alpha < 2 \), \( X \) is pure-jump a.s.
- (McKean, 1955) The random image-set \( X(\mathbb{R}_+) \) a.s. has Hausdorff dimension \( \min(d, \alpha) \).
- (Kakutani, 1944; Dvoretzky, Erdős, and Kakutani, 1950; McKean, 1955) For all Borel sets \( A \subset \mathbb{R}^d \),
  \[
P \{ X(\mathbb{R}_+) \cap A \neq \emptyset \} > 0 \leftrightarrow \text{Cap}_{d-\alpha}(A) > 0.
\]
- (Nevanlinna, 1936; Noshiro, 1948; Ninomiya, 1953) Connections to the Dirichlet problem for \( \Delta^{\alpha/2} \) with removable singularities.
Codimension and a Drawback

Recall that if $X_\alpha$ is iso. stable-$\alpha$ in $\mathbb{R}^d$ then

$$P \{X_\alpha(\mathbb{R}_+) \cap A \neq \emptyset\} > 0 \iff \text{Cap}_{d-\alpha}(A) > 0.$$ 

Also recall (Frostman, 1935) that

$$\inf \{ s \in (0, d) : \text{Cap}_{d-s}(A) > 0 \} + \dim_H A = d.$$ 

Thus,

**Proposition. [Taylor, 1966]** For all Borel sets $A \subset \mathbb{R}^d$ with $\dim_H A \geq d - 2$,

$$\inf \left\{ \alpha \in (0, 2) \left| P \{X_\alpha(\mathbb{R}_+) \cap A \neq \emptyset\} > 0 \right\} + \dim_H A = d.$$ 

What if $\dim_H A < d - 2$? An answer is given by Peres (1996; 1998), but this answer does not involve Lévy processes.
Additive Stable Processes

Let $X_1, \ldots, X_N$ denote independent iso. stable-$\alpha$ processes in $\mathbb{R}^d$. Define the $(N,d)$-random field

$$\mathcal{X}_{N,\alpha}(t) := X_1(t_1) + \cdots + X_N(t_N), \quad t = (t_1, \ldots, t_N) \in \mathbb{R}_+^N.$$ 

["additive stable process"]

**Theorem. [Hirsch–Song, 1995; Kh. 2002]** For all Borel sets $A \subset \mathbb{R}^d$,

$$P \{ \mathcal{X}_{N,\alpha}(\mathbb{R}_+^N) \cap A \neq \emptyset \} > 0 \iff \text{Cap}_{d-\alpha N}(A) > 0.$$ 

So now we can characterize $\dim_H A$ by seeing for which pairs $(N, \alpha)$ the range of $\mathcal{X}_{N,\alpha}$ can hit $A$. 
Definition of ALPs

- $X_1, \ldots, X_N = \text{independent Lévy in } \mathbb{R}^d$;
- ALP [additive Lévy process]:

$$\mathcal{X}(t) := X_1(t_1) + \cdots + X_N(t_N),$$

for $t := (t_1, \ldots, t_N) \in \mathbb{R}_+^N$.
- Law is characterized by

$$\mathbb{E} e^{i \xi \cdot \mathcal{X}(t)} = \exp \left( - \sum_{j=1}^N t_j \Psi_j(\xi) \right) = e^{-t \cdot \Psi(\xi)},$$

where $\Psi(\xi) := (\Psi_1(\xi), \ldots, \Psi_N(\xi))$ and $\Psi_j$ is the Lévy exponent of $X_j$:

$$\mathbb{E} e^{i \xi \cdot X_j(s)} = e^{-s \Psi_j(\xi)}.$$
Do ALPs Arise Naturally?

Yes. Here are 4 ways; there are others as well.

1. Double Points

2. Triple Points, etc.

3. Arithmetic properties [“Kahane’s Problem”]

4. Brownian sheet [time permitting]
Reason 1: Intersections of Paths
[""Double Points"""]

Let \( Y \) be a Lévy process in \( \mathbb{R}^d \). An old question:

\[
\mathcal{P} := P \{ \exists s \neq t : Y(s) = Y(t) \} > 0?
\]

- \( Y = \text{BM} \):
  \[ \mathcal{P} > 0 \text{ iff } d \leq 3 \]
  (Dvoretzky, Erdős, and Kakutani, 1950; Aizenmann, 1985; Peres, 1996; Kh. 2003).

- Dvoretzky, Erdős, Kakutani, and Taylor (1957); Hendricks (1973/74); Hawkes (1977, 1978); Hendricks (1979); Kahane (1983, 1985); Evans (1987); Tongring (1988); Rogers (1989); Le Gall, Rosen, and Shieh (1989); Fitzsimmons and Salisbury (1989); Ren (1990); Hirsch and Song (1995); Shieh (1998); Peres (1999); Kh. (2002).
Connection to Additive Lévy Processes

When is $\mathcal{P} := P \{ \exists s \neq t : Y(s) = Y(t) \} > 0$?

Let $Y_1$ and $Y_2$ be i.i.d. copies. The above is equivalent to:

When is $P \{ \exists s, t > 0 : Y_1(s) = Y_2(t) \} > 0$?

Consider the additive Lévy process

$\mathcal{Y}(t) := Y_1(t_1) - Y_2(t_2)$.

We wish to know

When does $\mathcal{Y}$ hit zero?
Reason 2: Variants ["Triple Points"]

When is \( P \{ \exists \text{ distinct } s, t, u : Y(s) = Y(t) = Y(u) \} > 0? \)

If \( Y = \text{BM} \) then the answer is "iff \( d \leq 2 \)" (Dvoretzky, Erdős, Kakutani, and Taylor, 1957).

Equivalently, if \( Y_1, Y_2, Y_3 \) are i.i.d. Lévy processes then we wish to know when

\[
P \{ \exists s, t, u > 0 : Y_1(s) = Y_2(t) = Y_3(u) \} > 0?
\]

Define \( Y(t) := \begin{bmatrix} Y_1(t_1) \\ 0_d \\ 0_d \end{bmatrix} + \begin{bmatrix} 0_d \\ Y_2(t_2) \\ 0_d \end{bmatrix} + \begin{bmatrix} 0_d \\ 0_d \\ Y_3(t_3) \end{bmatrix} \). Then we wish to know when \( Y \) hits the diagonal of \( \mathbb{R}^{3d} \); i.e., the collection of all points of the form \( x \otimes x \otimes x \) where \( x \in \mathbb{R}^d \).
Reason 3: Arithmetic Properties

- $B, B_1,$ and $B_2 = \text{indept. Br. motions in } \mathbb{R}^d$;

- $E, F \subset [0, \infty)$ compact, nonrandom, disjoint.

Then the following are equivalent (Kahane, 1985):

- $|B(E) \oplus B(F)| > 0$ with pos. probab.;

- $|B_1(E) \oplus B_2(F)| > 0$ with pos. probab. [does not require that $E \cap F = \emptyset$].

Kahane also provided a necessary as well as a sufficient condition, and asked for a precise condition on $E \times F$ to ensure $|B(E) \oplus B(F)| > 0$.

**Answer:** (Kh. 1999): $\text{Cap}_{d/2}(E \times F) > 0$. 
Define
\[ \mathcal{B}(t) := B_1(t_1) + B_2(t_2) \]
"Additive Brownian motion."

Then the following are equivalent:

- \(|B_1(E) \oplus B_2(F)| > 0\) with pos. probab.
- \(\mathcal{B}(E \times F)\) has positive Leb. meas. with pos. probab.

This problem is now very well understood for very general Lévy processes (Kh. and Xiao, 2005).
An Application of ALPs to LPs

Let $Y$ be an arbitrary Lévy process in $\mathbb{R}^d$ with Lévy exponent $\Psi_Y$: $\mathbb{E}[e^{i\xi \cdot Y(s)}] = \exp(-s\Psi_Y(\xi))$. Our immediate goal is to derive the following:

**Theorem. [Kh., Xiao, and Zhong, 2003]** With probability one, $\dim_H Y(\mathbb{R}_+)$ is equal to

$$\sup\left\{ s > 0 : \int_{\mathbb{R}^d} \Re \left( \frac{1}{1 + \Psi_Y(\xi)} \right) \frac{d\xi}{\|\xi\|^{d-s}} < \infty \right\}.$$  

This solved a relatively old problem (Taylor, 1952/53; McKean, 1955; Blumenthal and Getoor, 1960, 1961; Pruitt 1969; Fristedt, 1974).

⇒ If $\tilde{Y} :=$ symmetrization of $Y$ [i.e., $\tilde{Y}(t) = Y(t) - Y'(t)$ for an indep. copy $Y'$] then $\dim_H Y(\mathbb{R}_+) \geq \dim_H \tilde{Y}(\mathbb{R}_+)$ a.s. Completes the observation of Kesten (1969).
Let $X_1, \ldots, X_N$ be indep. Lévy processes in $\mathbb{R}^d$; $\Psi_j = $ Lévy exponent of $X_j$.

$$\mathcal{X}(t) := X_1(t_1) + \cdots + X_N(t_N).$$

Theorem. [Kh. and Xiao 2006+] Let $F \subset \mathbb{R}^d = \text{nonrandom Borel}$. Then, we have $|\mathcal{X}(\mathbb{R}^N_+ \oplus F)| > 0$ with pos. probab. iff $\exists \mu \in \mathcal{P}(F)$:

$$\int_{\mathbb{R}^d} \prod_{j=1}^N \Re \left( \frac{1}{1 + \Psi_j(\xi)} \right) |\hat{\mu}(\xi)|^2 d\xi < \infty.$$ 

This improves on Kh., Xiao, and Zhong (2003).

When $N = 1$ this is well known (Orey, 1967; Kesten, 1969; Port and Stone, 1971; Hawkes, 1979, 1986; \ldots). For $N > 1$ we need different ideas.
For instance, suppose $X_1, \ldots, X_N$ are i.i.d., isotropic $\alpha$-stable Lévy processes in $\mathbb{R}^d$. The "additive stable process" $\mathcal{X}_{N, \alpha}(t) := \sum_{j=1}^{N} X_j(t_j)$ has the property that for all non-random analytic sets $F \subset \mathbb{R}^d$ the following are equivalent:

1. $\Pr\{|\mathcal{X}_{N, \alpha}(\mathbb{R}^N) \oplus F| > 0\} > 0$;

2. $\exists \mu \in \mathcal{P}(F)$ such that
   \[
   \int_{\mathbb{R}^d} \frac{|\hat{\mu}(\xi)|^2}{1 + \|\xi\|^{\alpha N}} d\xi < \infty;
   \]

3. $\exists \mu \in \mathcal{P}(F)$ such that
   \[
   I_{d-\alpha N}(\mu) = \iint \frac{\mu(dx) \mu(dy)}{\|x - y\|^{d-\alpha N}} < \infty;
   \]

4. $\text{Cap}_{d-\alpha N}(F) > 0$.

We saw this before too.
I.e., if $X_1, \ldots, X_N$ are i.i.d. isotropic stable-$\alpha$ in $\mathbb{R}^d$ and $\mathcal{X}_{N,\alpha}(t) = X_1(t_1) + \cdots + X_N(t_N)$, then

$$P\left\{ \left| \mathcal{X}_{N,\alpha}(\mathbb{R}^+_N) \oplus F \right| > 0 \right\} > 0 \iff \text{Cap}_{d-\alpha N}(F) > 0.$$ 

Let $Y$ be an independent [ordinary] Lévy process in $\mathbb{R}^d$, and apply the preceding to $F := Y(\mathbb{R}_+)$, conditional on $Y$: As positive-probability events,

$$\left| \mathcal{X}_{N,\alpha}(\mathbb{R}^+_N) \oplus Y(\mathbb{R}_+) \right| > 0 \iff \text{Cap}_{d-\alpha N}(Y(\mathbb{R}_+)) > 0.$$ 

But $\mathcal{X}_{N,\alpha}(\mathbb{R}^+_N) \oplus Y(\mathbb{R}_+) = \mathcal{L}(\mathbb{R}^N_{+1})$, where $\mathcal{L}$ is the ALP $\mathcal{L}(t) := X_1(t_1) + \cdots + X_N(t_N) + Y(t_{N+1})$. We can apply the previous theorem with $F := \{0\}$ to find that

$$\left| \mathcal{L}(\mathbb{R}^N_{+1}) \right| > 0 \iff \int_{\mathbb{R}^d} \text{Re} \left( \frac{1}{1 + \Psi_Y(\xi)} \right) \frac{d\xi}{\|\xi\|^{\alpha N}} < \infty.$$ 

Compare the boxed equations to find $\dim_H Y(\mathbb{R}_+)$ (Frostman, 1935).
Additive BM and the Brownian Sheet

Let $B(t_1, t_2)$ denote two-parameter Brownian sheet in $\mathbb{R}^d$. I.e., $B$ is a centered Gaussian process and

$$\text{Cov}(B_i(s_1, s_2), B_j(t_1, t_2)) = \min(s_1, t_1) \min(s_2, t_2) \delta_{i,j}.$$ 

Locally, Brownian sheet looks like additive Brownian motion. Here is a precise statement near the point $t := (1, 1)$ (say!): For $\epsilon, \delta > 0$,

$$B(1 + \epsilon, 1 + \delta) = B(1, 1) + X_1(\epsilon) + X_2(\delta) + Y(\epsilon, \delta),$$

where:

- $X_1$ and $X_2$ are Brownian motions;
- $Y$ is a Brownian sheet;
- $B(1, 1), X_1, X_2, \text{ and } Y$ are totally independent.
When $\varepsilon, \delta \approx 0$,

$$B(1 + \varepsilon, 1 + \delta) - B(1, 1) \approx X_1(\varepsilon) + X_2(\delta).$$

One makes precise sense of this in a problem-dependent manner. Here is an application:

**Theorem. [Kh. and Shi, 1999]** If $F \subset \mathbb{R}^d$ is analytic, then

$$P\{B((0, \infty)^2) \cap F \neq \emptyset\} > 0 \text{ iff } \text{Cap}_{d-4}(F') > 0.$$ [Cap$_0$ := log. capacity; Cap$_{-r}$ := 1 for $r < 0$]

For instance, let $F := \{0\}$ to find that $B$ hits singletons iff $d \leq 3$ (Orey and Pruitt, 1973).

By Girsanov this theorem translates to a statement about SPDEs:
Let $\dot{W}$ denote $d$-dimensional white noise over $\mathbb{R}^2$. Consider the system of SPDEs of the wave type,

$$
\frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial^2 u(t,x)}{\partial x^2} = b(u(t,x)) + \Sigma \dot{W}(t,x).
$$

Here, $\Sigma$ is a non-singular, $d \times d$ matrix, and $b : \mathbb{R}^d \to \mathbb{R}^d$ is bounded and globally Lipschitz (say!). Note that $u : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}^d$. Then, for all nonrandom analytic sets $F \subset \mathbb{R}^d$,

$$
P \{ u((0,\infty) \times \mathbb{R}_+) \cap F \neq \emptyset \} > 0 \iff \text{Cap}_{d-4}(F) > 0.
$$

A remarkable recent theorem of Robert Dalang and Eulalia Nualart (2004) shows that the same result holds for the system of SPDEs,

$$
\frac{\partial^2 u(t,x)}{\partial t^2} = \frac{\partial^2 u(t,x)}{\partial x^2} + b(u(t,x)) + a(u(t,x)) \dot{W}(t,x),
$$

as long as $a$ and $b$ are Lipschitz and bounded (say!), and $a$ is strictly elliptic.