Slices of the Brownian Sheet

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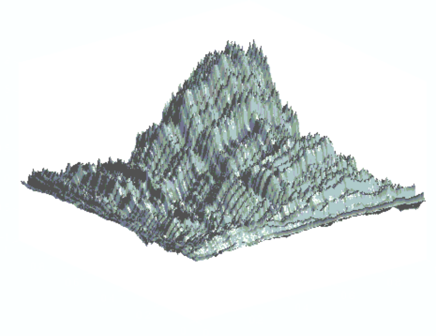
- $t \mapsto e^{-t}B(\bullet\,,e^{2t})$ Malliavin's process (Ornstein–Uhlenbeck proc. on $C[0\,,1]$; inv meas = Wiener) [Paul-André Meyer]

[Brownian sheet]



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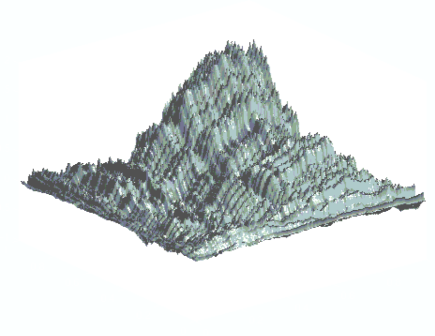

[Ron Pyke and Paul-André Meyer, 1971 Oberwolfach]

$X_{i,j}$ i.i.d., $EX_{i,j} = 0$, Var$X_{i,j} = 1$.

$$S_n(s\,,t) := \frac{1}{n} \sum_{1 \le i \le ns} \sum_{1 \le j \le nt} X_{i,j} \qquad 0 \le s,t \le 1.$$

Theorem (Pyke)

$S_n \Rightarrow$ *Brownian sheet on* $[0\,,1]^2$.

Coin Tosser's View ($d = 1$ say)

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"$\Rightarrow$" means weak conv in space $D([0\,,1]\,,\, D[0\,,1]) \simeq D([0\,,1]^2)$ (Bickel and Wichura)

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Exceptional Slices

Example (Fukushima, 1984)

Set $d = 1$. At $t = 1$, the slice along $s \in [1\,,2]$ does not hit 0 $[B(s\,,1) \neq 0]$, but there can $\exists s \in [1\,,2]$ such that $B(s\,,1) = 0$.
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Theorem (Penrose, 1990)

Set $d = 1$. Every slice along $s \in [1\,,2]$ hits 0 at some time $t \in [1\,,2]$ $[B(s\,,t) = 0]$ with pos. probab.

Exceptional Slices

...Still more interesting when $d > 1$.

Theorem (Orey and Pruitt, 1973; Fukushima, 1984)

$\exists (s\,,t) \in [1\,,2]^2 \ \ni \ B(s\,,t) = 0$ *iff* $d \leq 3$.

Exceptional Slices

Theorem (2006)

$F \subset [1\,,2]$ *compact, non-random.*

$$P\left\{{}^{\exists}s \in F:\ B(s\,,t) = 0 \text{ for some } t \in [1\,,2]\right\} > 0 \ \Leftrightarrow\ \mathscr{C}_{(d-2)/2}(F) > 0.$$

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Example

Set $F := W^{-1}(\{0\}) \cap [1\,,2]$, where W is an independent 1-D BM. Then,

$$P\left\{{}^{\exists}s \in W^{-1}(\{0\}):\ B(s\,,t) = 0 \text{ for some } t \in [1\,,2]\right\} > 0 \ \Leftrightarrow\ d = 1, 2.$$

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Corollary (Penrose, 1990)

When $d \in \{2, 3\}$,

$$\dim_{\rm H} \{s \in [1,2] : \; B(s,t) = 0 \textit{ for some } t \in [1,2]\} = 2 - \frac{d}{2}.$$

$\dim_{\rm H} < 0 \Rightarrow \varnothing$.

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Marstrand (1954)

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When $d \in \{2,3\}$, outside one null set:

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Open Pbm: $d \in \{2,3\}$: $\exists s \ni B^{-1}(\{0\}) \cap (\{s\} \times (0,\infty))$ is uncountable.

Proof of Main Theorem

Goal:

Theorem (2006)

$F \subset [1\,,2]$ compact, non-random.

$$P\left\{{}^{\exists}s \in F:\ B(s\,,t)=0 \text{ for some } t \in [1\,,2]\right\} > 0 \ \Leftrightarrow\ \mathscr{C}_{(d-2)/2}(F) > 0.$$

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Conjecture of Kahane (1983–1985??)

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$$\mathscr{C}_s(F \times [0\,,1)) > 0 \quad \text{iff} \quad \mathscr{C}_{s-1}(F) > 0.$$

[A much more general fact holds.]

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- $[0\,,1) \simeq \mathbf{T}$.
- $\exists! \mu \in \mathscr{P}(F) \ni \mathscr{C}_{s-1}(F) = 1/I_{s-1}(\mu)$. Also with $\mathscr{C}_s \cdots$

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- $\mathrm{e}_{F \times \mathbf{T}} = \mathrm{e}_{F \times \mathbf{T}} \circ \tau_a^{-1}$ if $a' = 0$. (equil meas is unique).

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$$\mathrm{e}_{F\times\mathbf{T}}(A \times B) \;=\; P\{X' \in A, X'' \in B\}$$

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$\Rightarrow \mathscr{C}_s(F \times \mathbf{T}) \leq \mathscr{C}_{s-1}(F)$. The other bound is easy.

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Self-Intersecting Slices

Theorem (2006)

Fix $F \subset [1\,,2]$ *compact nonrandom.*

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Corollary (Mountford, 1990)

$$\dim_{\mathrm{H}} \{s:\ \textit{slice along s is self-intersecting}\} = 1 \wedge \left(3 - \frac{d}{2}\right)_+.$$

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Theorem (Dvoretzky and Erdős, 1951)

For all $s > 0$,

$$P\{R(s) = d - 2\} = 1.$$

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Central Step:

$$P\left\{ \inf_{s\in F} \inf_{t\in[1,2]} |B(s,t)| \leq \epsilon \right\} \asymp \epsilon^{d-2} K_F(\epsilon^2) \wedge 1,$$

where $K_F :=$ Kolmogorov capacitance of F (i.e., max points $\geq \epsilon$ apart in F)

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1. $F = \cup_{n=1}^{\infty} F_n$; and
2. For all $n \geq 1$,

$$\int_1^\infty \left[\frac{K_{F_n}(1/\psi(x))}{\psi^{(d-2)/2}(x)} \wedge 1\right] \frac{dx}{x} < \infty.$$

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$F \subset [1,2]$ compact nonrandom and $\psi : (0,\infty) \to (0,\infty)$, meas non-decreasing:

Theorem (2006)

$$\inf_{s \in F} \liminf_{t \to \infty} \left(\frac{\psi(t)}{t} \right)^{1/2} |B(s,t)| = \begin{cases} 0, & \textit{if } (F,\psi) \notin \mathrm{FIN}_{loc}, \\ \infty, & \textit{otherwise}. \end{cases}$$