$\mathscr{S}$PDEs without PDEs

(with M. Foondun)

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The stochastic heat equation

- The archetypal equation

$$\frac{\partial}{\partial t}u(t,x) = \kappa(\Delta u)(t,x) + \sigma(u(t,x))\dot{w}(t,x):$$

  - $\kappa > 0$ fixed
  - $\dot{w}$ = space-time white noise [$t \geq 0$, $x \in \mathbf{R}^d$]
  - σ globally Lipschitz [say!]
- Typically $d = 1$
- Important examples that arise:
  - $\sigma(u) = 1$ [heat flow with random external environment]
  - $\sigma(u) = \lambda u$ [Kinetically-growing random surfaces & the parabolic Anderson model]
  - $\sigma(u) = \lambda\sqrt{u}$ [The Brownian density process] ...
- Existence/uniqueness? Properties? ... [technical]

Discrete random walks

- Let $\{S_n\}_{n=0}^{\infty}$ be a random walk on $\mathbf{Z}^d$ with $S_0 := 0$
- Transition matrix P:

$$\begin{aligned} P_{x,y} &:= \mathrm{P}\big(S_{n+1} = y \,\big|\, S_n = x\big) \\ &= \mathrm{P}\big(S_{n+1} - S_n = y - x \,\big|\, S_n = x\big) \\ &= \mathrm{P}\big\{S_1 = y - x\big\}. \end{aligned}$$

- Generator L:

$$L_{x,y} := P_{x,y} - I_{x,y} \quad [I_{x,y} = \mathbf{1}_{\{y\}}(x)]$$

- Identify functions with vectors, so that

$$(Pf)(x) := \sum_{y \in \mathbf{Z}^d} P_{x,y}\, f(y), \quad [= \mathrm{E}\, f(S_1 + x)]$$

$$(Lf)(x) := \sum_{y \in \mathbf{Z}^d} L_{x,y}\, f(y) \quad [= \mathrm{E}\, f(S_1 + x) - f(x)] \ \ldots$$

Discrete random walks

- Note $(P^2)_{x,y} = P^2_{x,y}$ satisfies

$$\begin{aligned} P^2_{x,y} &= \sum_{z \in \mathbf{Z}^d} P_{x,z} P_{z,y} \\ &= \mathrm{P}\left(S_{n+2} = y \,\middle|\, S_n = x\right) \\ &= \mathrm{P}\left\{S_2 = y - x\right\} \end{aligned}$$

- Chapman–Kolmogorov
- More generally, $(P^k)_{x,y} = P^k_{x,y}$ satisfies

$$P^k_{x,y} = \mathrm{P}\left\{S_k = y - x\right\}$$

- Because vectors $\leftrightarrow$ functions,

$$(P^k f)(x) = \sum_{y \in \mathbf{Z}^d} P^k_{x,y} f(y) \quad [= \mathrm{E}\, f(S_k + x)]$$

An example

The discrete Laplacian

- $S =$ simple walk on $\mathbf{Z}^d$:
  - $P_{x,y} = (2d)^{-1}$ if $\|x-y\|_1 = 1$; else $P_{x,y} = 0$
  - $L_{x,y} = (2d)^{-1}$ if $\|x-y\|_1 = 1$; $L_{x,x} = -1$; else, $L_{x,y} = 0$
  - Therefore,

$$L = \frac{1}{2d} \times \Delta, \quad \text{where} \quad \Delta_{x,y} = \begin{cases} 1 & \text{if } \|x-y\|_1 = 1, \\ -2d & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

  - $\Delta :=$ the discrete Laplacian on $\mathbf{Z}^d$
- E.g., when $d = 1$:

$$\begin{aligned} (\Delta f)(x) &= \sum_{y \in \mathbf{Z}} \Delta_{x,y} f(y) = f(x-1) + f(x+1) - 2f(x) \\ &= \underbrace{\left[f(x+1) - f(x)\right]}_{f'(x)} - \underbrace{\left[f(x) - f(x-1)\right]}_{f'(x-1)} = f''(x-1) \end{aligned}$$

An example

The discrete Laplacian

- In general:

$$(\Delta f)(x) = \sum_{\substack{y\in\mathbf{Z}^d:\\ \|y-x\|_1=1}} \{f(y)-f(x)\}$$
$$= \sum_{k=1}^{d} \{f(x+\mathrm{e}_k)+f(x-\mathrm{e}_k)-2f(x)\}$$
$$\approx \sum_{k=1}^{d} \left(\frac{\partial^2 f}{\partial x_k^2}\right)(x)$$

- Good exercise: Prove
$(f\,,\Delta f) := \sum_{x\in\mathbf{Z}^d} f(x)(\Delta f)(x) \le 0 \quad \forall f\in \ell^2(\mathbf{Z}^d)$

More on the exercise

- The random-walk viewpoint is powerful; e.g., let us prove $(f, Lf) \le 0$ for all $f \in \ell^2(\mathbf{Z}^d)$
- $\leftrightarrow \ (f, Pf) \le (f, f)$
- But $|\text{lhs}| \le \|f\| \times \|Pf\|$ [Cauchy–Schwarz]
- And P is a contraction on $\ell^2(\mathbf{Z}^d)$, viz.:

$$\begin{aligned}\|Pf\|^2 &= \sum_{x \in \mathbf{Z}^d} (\mathrm{E}\, f(S_1 + x))^2 \le \sum_{x \in \mathbf{Z}^d} \mathrm{E}\left(|f(S_1 + x)|^2\right) \\ &= \mathrm{E} \sum_{x \in \mathbf{Z}^d} |f(S_1 + x)|^2 = \sum_{y \in \mathbf{Z}^d} |f(y)|^2 = \|f\|^2.\end{aligned}$$

Duhamel's principle

- Let's solve the discrete heat equation:

$$u_{n+1}(x) - u_n(x) = \underbrace{(Lu_n)(x)}_{(Pu_n)(x)-u_n(x)} + \sigma(u_n(x))\underbrace{F_n(x)}_{\text{external force}}$$

- If a solution $\exists$, then it is unique, and

$$\begin{aligned}
u_{n+1}(x) &= \sum_{y\in\mathbf{Z}^d} P_{x,y} u_n(y) + \sigma(u_n(x))F_n(x) \\
&= \sum_{y\in\mathbf{Z}^d} P_{x,y} \left(\sum_{z\in\mathbf{Z}^d} P_{y,z} u_{n-1}(z) + \sigma(u_{n-1}(y))F_{n-1}(y) \right) \\
&\quad + \sigma(u_n(x))F_n(x) \\
&= \sum_{z\in\mathbf{Z}^d} P^2_{x,z} u_{n-1}(z) + \sum_{y\in\mathbf{Z}^d} P_{x,y}\sigma(u_{n-1}(y))F_{n-1}(y) \\
&\quad + \sum_{y\in\mathbf{Z}^d} P^0_{x,y}\sigma(u_n(y))F_n(y) \quad [P^0 = I]
\end{aligned}$$

Duhamel's principle

- Continued $[u_{n+1}(x) = \sum_{y\in\mathbf{Z}^d} P_{x,y} u_n(y) + \sigma(u_n(x))F_n(x)]$:

$$\begin{aligned}
u_{n+1}(x) &= \sum_{z\in\mathbf{Z}^d} P^2_{x,z} u_{n-1}(z) + \sum_{k=0}^{1}\sum_{y\in\mathbf{Z}^d} P^k_{x,y}\sigma(u_{n-k}(y))F_{n-k}(y) \\
&= \sum_{z\in\mathbf{Z}^d} P^3_{x,z} u_{n-2}(z) + \sum_{k=0}^{2}\sum_{y\in\mathbf{Z}^d} P^k_{x,y}\sigma(u_{n-k}(y))F_{n-k}(y) \\
&= \sum_{z\in\mathbf{Z}^d} P^4_{x,z} u_{n-3}(z) + \sum_{k=0}^{3}\sum_{y\in\mathbf{Z}^d} P^k_{x,y}\sigma(u_{n-k}(y))F_{n-k}(y) \cdots \\
&= \sum_{z\in\mathbf{Z}^d} P^{n+1}_{x,z} u_0(z) + \sum_{k=0}^{n}\sum_{y\in\mathbf{Z}^d} P^k_{x,y}\sigma(u_{n-k}(y))F_{n-k}(y) \\
&= \left(P^{n+1}u_0\right)(x) + \sum_{k=0}^{n}\sum_{y\in\mathbf{Z}^d} P^{n-k}_{x,y}\sigma(u_k(y))F_k(y).
\end{aligned}$$

The discrete stochastic heat equation

- Let $\{F_n(x);\, n \geq 0,\, x \in \mathbf{Z}^d\}$ be iid with mean zero, variance one
  [discrete space-time white noise]
- Solve: $u_{n+1}(x) - u_n(x) = (Lu_n)(x) + \sigma(u_n(x))F_n(x)$
- The only possible solution:

$$u_{n+1}(x) \quad = \quad \left(P^{n+1}u_0\right)(x) + \sum_{k=0}^{n} \sum_{y \in \mathbf{Z}^d} P_{x,y}^{n-k} \sigma(u_k(y)) F_k(y).$$

- Solution $\exists$ iff sum converges

The discrete stochastic heat equation

- $[u_{n+1}(x) - u_n(x) = (Lu_n)(x) + \sigma(u_n(x))F_n(x)]$
- Suppose $|\sigma(u) - \sigma(v)| \leq C_\sigma |u - v|$ [discrete Lipschitz]
- For simplicity, assume $\sigma(0) = 0$ so that

$$|\sigma(u)| \leq C_\sigma |u| \qquad \text{for all } u \in \mathbf{Z}^d$$

- Suppose u_0 is bounded: $\sup_z |u_0(z)| := |u_0|_\infty < \infty$
- Theorem: $\exists!$ *solution & "sharp growth"*

The discrete stochastic heat equation

A semirigorous justification

- Recall the candidate solution:

$$
\begin{aligned}
u_{n+1}(x) &= \left(P^{n+1}u_0\right)(x) + \sum_{k=0}^{n}\sum_{y\in\mathbf{Z}^d} P^{n-k}_{x,y}\sigma(u_k(y))F_k(y) \\
\mathrm{E}\left(|u_{n+1}(x)|^2\right) &\le |u_0|_\infty^2 + \sum_{k=0}^{n}\sum_{y\in\mathbf{Z}^d}\left|P^{n-k}_{x,y}\right|^2 \mathrm{E}\left(\sigma^2(u_k(y))\right) \\
&\le |u_0|_\infty^2 + C_\sigma^2 \sum_{k=0}^{n}\sum_{y\in\mathbf{Z}^d}\left|P^{n-k}_{x,y}\right|^2 \mathrm{E}\left(|u_k(y)|^2\right)
\end{aligned}
$$

- Define, for all $\zeta\in(0\,,1)$,
  $\|u\|_\zeta^2 := \sup_{n\ge1}\sup_{x\in\mathbf{Z}^d}\zeta^n\mathrm{E}(|u_n(x)|^2)$:

$$
\mathrm{E}\left(|u_{n+1}(x)|^2\right) \le \text{const} + C_\sigma^2\|u\|_\zeta^2 \sum_{k=1}^{n}\sum_{y\in\mathbf{Z}^d}\left|P^{n-k}_{x,y}\right|^2\zeta^{-k}
$$

The discrete stochastic heat equation

A semirigorous justification

- $\|u\|_\zeta^2 := \sup_{n\geq 1}\sup_{x\in\mathbf{Z}^d} \zeta^n \mathrm{E}(|u_n(x)|^2)$
- $0<\zeta<1$:

$$\mathrm{E}\left(|u_{n+1}(x)|^2\right) \leq \text{const} + C_\sigma^2 \|u\|_\zeta^2 \sum_{k=1}^n \sum_{y\in\mathbf{Z}^d} \left|P_{x,y}^{n-k}\right|^2 \zeta^{-k}$$

$$= \text{const} + C_\sigma^2 \|u\|_\zeta^2\, \zeta^{-n} \sum_{j=1}^n \zeta^j \sum_{y\in\mathbf{Z}^d} \left|P_{x,y}^{j}\right|^2$$

$$= \text{const} + C_\sigma^2 \|u\|_\zeta^2\, \zeta^{-n} \sum_{j=1}^n \zeta^j \sum_{y\in\mathbf{Z}^d} \left|P_{0,y}^{j}\right|^2$$

$$\therefore\ \|u\|_\zeta^2 \leq \text{const} + \zeta C_\sigma^2 \|u\|_\zeta^2 \sum_{j=1}^\infty \zeta^j \sum_{y\in\mathbf{Z}^d} \left|P_{0,y}^{j}\right|^2$$

The discrete stochastic heat equation

A semirigorous justification

- $\mathrm{E}(|u_n(x)|^2) = O(\zeta^{-n})$ if $\zeta C_\sigma^2 \sum_{j=1}^\infty \zeta^j \sum_{y\in\mathbf{Z}^d} |P_{0,y}^j|^2 < 1$
- But

$$\sum_{y\in\mathbf{Z}^d} \left|P_{0,y}^j\right|^2 = \sum_{y\in\mathbf{Z}^d} \left|\mathrm{P}\{S_j = y\}\right|^2 = \sum_{y\in\mathbf{Z}^d} \mathrm{P}\left\{S_j = y = S_j'\right\}$$
$$= \mathrm{P}\left\{S_j - S_j' = 0\right\} = \bar{P}_{0,0}^j.$$

- Therefore,

$$\sum_{j=0}^\infty \zeta^j \sum_{y\in\mathbf{Z}^d} \left|P_{0,y}^j\right|^2 = \sum_{j=0}^\infty \zeta^j \bar{P}_{0,0}^j \qquad [\mathrm{E}(\text{local time of killed walk at 0})]$$

- Theorem: *The upper Liapounov L^2-exponent of u satisfies*

$$\limsup_{n\to\infty} \left\{\mathrm{E}\left(|u_n(x)|^2\right)\right\}^{1/n} \leq \left[\sup\left\{\zeta < 1 : \ \zeta \sum_{j=1}^\infty \zeta^j \bar{P}_{0,0}^j < \frac{1}{C_\sigma^2}\right\}\right]^{-1}.$$

The discrete stochastic heat equation

$$\limsup_{n\to\infty}\left\{\mathrm{E}\left(|u_n(x)|^2\right)\right\}^{1/n} \le \left[\sup\left\{\zeta\in(0\,,1):\ \zeta\sum_{j=1}^{\infty}\zeta^j\bar{P}^j_{0,0}<\frac{1}{C_\sigma^2}\right\}\right]^{-1}.$$

- If $|\sigma(z)|\geq c_\sigma|z|$, then the preceding can be reversed [sup $\leftrightarrow$ inf; "$< C_\sigma^{-2}$" $\leftrightarrow$ "$> c_\sigma^{-2}$"].
- In the continuous setting [$L=\text{const}\cdot\Delta$], when $\sigma(u)=\lambda u$:
  - Lieb & Liniger (1963); Kardar, Parisi, & Zhang (1986); Kardar (1987); Krug & Spohn (1991) ...
  - Molchanov (1991); Bertini & Cancrini (1994); Carmona & Molchanov (1994) ...
    [Key words: Anderson model; Anderson localization; intermittency; ...]
- In particular, for the linear stochastic heat equation [$\sigma\equiv\text{const}$], the solution grows subexponentially

The parabolic Anderson model in dimension $(1+1)$

- $u_{n+1}(x) - u_n(x) = \frac{1}{2}(\Delta u_n)(x) + \lambda u_n(x) F_n(x) \quad \forall n \geq 0, x \in \mathbf{Z}$
- ${}_1F_1$ = the confluent hypergeometric function of the 1st kind

Corollary

For all $x \in \mathbf{Z}$,

$$\limsup_{n\to\infty} \left\{ \mathrm{E}\left(|u_n(x)|^2 \right) \right\}^{1/n} = \left[\inf\left\{ \zeta \in (0\,,1) : {}_1F_1\left(\frac{1}{2}\,, 1\,, \zeta\right) > \frac{1}{\zeta\lambda^2} \right\} \right]^{-}$$

The parabolic Anderson model in dimension $(1+1)$

- Want $\sum_{j=1}^{\infty} \zeta^j \bar{P}^j_{0,0}$ $\qquad [\bar{P}^j_{0,0} = \mathrm{P}\{S_j - S'_j = 0\}]$
- Fourier analysis $\to$

$$\begin{aligned}\bar{P}^j_{0,0} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{E}\left(e^{i\xi(S_1 - S'_1)j}\right) \mathrm{d}\xi = \frac{1}{\pi}\int_0^{\pi}\left(\frac{1+\cos(2\xi)}{2}\right)^j \mathrm{d}\xi \\ &= \frac{2}{\pi}\int_0^{\pi/2} (\cos\xi)^{2j}\,\mathrm{d}\xi = \frac{(2j-1)!!}{(2j)!!}\end{aligned}$$

- $\bar{P}^{j+1}_{0,0} / \bar{P}^j_{0,0} = (j+\frac{1}{2})/(j+1) \ \to\ \sum_{j=1}^{\infty} \zeta^j \bar{P}^j_{0,0} = {}_1F_1(\frac{1}{2}, 1; \zeta)$