AN INTRODUCTION TO PARABOLIC SPDES

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ABSTRACT. This is a synopsis of my lectures notes on stochastic partial differential equations for the lecture series, "Lectures in Probability & Stochastic Processes, XI," held at ISI-Delhi from November 25–29, 2016. The main topics are: Existence and regularity of solutions; asymptotic analysis; the strong Markov property; and topics on comparison and positivity principles.

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Contents

1. Remarks on the heat equation

Consider the solution $u = u(t\,,x)$ to the following boundary-value problem:

$$\begin{cases} \dot{u} = u'' & \text{on } (0\,,\infty)\times[0\,,1], \text{ subject to} \\ \text{I.C.:} \quad u(0) = u_0 & \text{on } [0\,,1], \text{ and} \\ \text{B.C.:} \quad u(t\,,0) = u(t\,,1) = 0 & \forall t > 0, \end{cases} \tag{1.1}$$

where $u_0 \in L^2[0\,,1]$ is fixed.

We can solve (1.1) by using Fourier series as follows: First let

$$\psi_n(x) = \sqrt{2}\,\sin(n\pi x) \qquad \forall x \in [0\,,1],\ n \geqslant 1.$$

Then, write any $L^2[0\,,1]$-solution u to (1.1) in Fourier sine series as

$$u(t\,,x) = \sum_{n=1}^{\infty} u_n(t)\psi_n(x) \quad \text{where} \quad u_n(t) = (u(t)\,,\psi_n) := \int_0^1 u(t\,,x)\psi_n(x)\,\mathrm{d}x.$$

Now,

$$\begin{aligned} \dot{u}_n(t) &= (\dot{u}(t)\,,\psi_n) = (u''(t)\,,\psi_n) \\ &= (u(t)\,,\psi_n'') && \text{(integration by parts)} \\ &= -n^2\pi^2\,(u(t)\,,\psi_n) = -n^2\pi^2 u_n(t). \end{aligned}$$

This is an ODE for $u_n(t)$. Solve it in order to see that

$$u_n(t) = u_n(0)\mathrm{e}^{-n^2\pi^2 t} = (u_0\,,\psi_n)\mathrm{e}^{-n^2\pi^2 t}.$$

Therefore, in particular,

$$u(t\,,x) = \sum_{n=1}^{\infty} (u_0\,,\psi_n)\mathrm{e}^{-n^2\pi^2 t}\psi_n(x) \qquad \text{in } L^2[0\,,1]. \tag{1.2}$$

It is easy to rigorize this procedure in order to deduce

Theorem 1.1. *If $u_0 \in L^2[0\,,1]$, then* (1.2) *is the unique solution to* (1.1) *in* $L^2[0\,,1]$.

Theorem 1.1 has a well-known connection to probability that deserves special mention. Define for all $t > 0$ and $x, y \in [0\,,1]$,

$$p_t(x\,,y) := \sum_{n=1}^{\infty} \psi_n(x)\psi_n(y)\mathrm{e}^{-n^2\pi^2 t} \qquad \text{(the “heat kernel”).} \tag{1.3}$$

For every fixed $x \in [0\,,1]$, the function $u(t\,,y) := p_t(x\,,y)$ weakly solves (1.1) with $u_0 = \delta_x$. Therefore, $p_t(x\,,y) > 0$ $\forall t > 0, (x\,,y) \in (0\,,1)^2$, by the maximum principle. (Consult, for example, Evans [18].)

For all $\varphi \in L^2[0\,,1]$ define

$$(P_t\varphi)(x) := \int_0^1 p_t(x\,,y)\varphi(y)\,\mathrm{d}y \qquad \forall t > 0 \text{ and } x, y \in [0\,,1], \text{ and} \qquad P_0\varphi := \varphi.$$

It is easy to see that

$$(P_t\varphi)(x) = \sum_{n=1}^{\infty} \psi_n(x)(\varphi\,,\psi_n)\mathrm{e}^{-n^2\pi^2 t}.$$

Thanks to (1.2), $u(t\,,x) := (P_t u_0)(x)$ is the unique solution to (1.1). Moreover, $\{P_t\}_{t\geqslant 0}$ is a semigroup; that is, $P_{t+s} = P_t P_s$ for all $s, t \geqslant 0$. In fact, $\{P_t\}_{t\geqslant 0}$ is the semigroup associated to Brownian motion killed upon leaving $[0\,,1]$. Here is a way to state this last assertion a little more carefully: Let B denote a 1-dimensional Brownian motion and

$$\tau := \inf\{t > 0 : \; B_{2t} = 0 \text{ or } 1\}.$$

Then, it can be shown that

$$(P_t\varphi)(x) = \mathrm{E}_x\left[\varphi(B_{2t});\, \tau > t\right]. \tag{1.4}$$

See Bass's book [3] on one-dimensional diffusions, for example. By letting φ approximate δ_y we obtain the following:

Theorem 1.2. *The mapping $(t\,,x\,,y) \mapsto p_t(x\,,y)$ is the transition function for a Brownian motion B – run at twice the standard speed – to go from x to y in t time units before B leaves* $[0\,,1]$.

2. Remarks on white noise

2.1. **Warmup: White noise on $[0\,,1]$ via Brownian motion.** Let $\{X_i\}_{i=1}^\infty$ be i.i.d. $\mathrm{N}(0\,,1)$ random variables, and define for all $\varphi \in L^2[0\,,1]$ and $N \geqslant 1$,

$$B'_N(\varphi) := \sum_{n=1}^N X_n(\varphi\,,\psi_n).$$

Then, B'_N is a mean-zero Gaussian random field (GRF), indexed by $L^2[0\,,1]$, and

$$\mathrm{Cov}[B'_N(f)\,,B'_N(g)] = \sum_{n=1}^N (f\,,\psi_n)(g\,,\psi_n) \xrightarrow{(N\to\infty)} (f\,,g) \qquad \forall f,g \in L^2[0\,,1].$$

It follows easily from this that:

- $B'(\varphi) := \lim_{N\to\infty} B'_N(\varphi)$ exists in $L^2(\Omega)$ for every $\varphi \in L^2[0\,,1]$;
- $\varphi \mapsto B'(\varphi)$ is a linear mapping from $L^2[0\,,1]$ to $L^2(\Omega)$;
- B' is a mean-zero GRF with $\mathrm{Cov}[B'(f)\,,B'(g)] = (f\,,g)$; and as such
- $B' : L^2[0\,,1] \mapsto L^2(\Omega)$ is a linear isometry [the "isonormal process"].
- Let $\xi(A) := B'(\mathbf{1}_A)$ for all Borel sets $A \subset [0\,,1]$ to see that ξ is an $L^2(\Omega)$-valued measure ["white noise"]. We may, and often do, identify ξ with B' in this way.

Some notations:

- $B'(\varphi) :=$ the "Wiener integral of φ."
- $B'(\varphi) := \int_0^1 B'(x)\varphi(x)\,\mathrm{d}x := \int_0^1 \varphi\,\mathrm{d}W := \int_0^1 \varphi\,\mathrm{d}\xi := \int_0^1 \varphi(x)\,\xi(\mathrm{d}x)$.
- $\int_A \varphi\,\mathrm{d}B := \int_A \varphi(x)\,B(\mathrm{d}x) := \int_A \varphi(x)\,\xi(\mathrm{d}x) := \int_A \varphi\,\mathrm{d}\xi := B'(\varphi\mathbf{1}_A)$.
- If $B'(x)$ made sense as a real-valued stochastic process, then it would have to be $B'(\delta_x)$. This is purely formal since $\delta_x \not\in L^2[0\,,1]$. In any case, formally speaking, $B'(x) = \sum_{n=1}^\infty X_n\psi_n(x)$, though the series does not converge pointwise in $L^2(\Omega)$.
- Since $\mathrm{Cov}[B'(f)\,,B'(g)] = (f\,,g)$, we formally replace f by δ_x and g by δ_y to obtain the formal statement that $\{B'(x)\}_{x\in[0,1]}$ defines a mean-zero Gaussian process with

  $$\mathrm{Cov}[B'(x)\,,B'(y)] = \delta_0(x-y).$$

  Since $\xi(A) = \int_A B'(x)\,\mathrm{d}x$, we formally identify measures with their Radon–Nykodym densities and say that $\{B'(x)\}_{x\in[0,1]}$ is "white noise on $[0\,,1]$," as well.

Define B to be the "cdf" of B'. That is, let

$$B(x) := \int_0^x B'(y)\,\mathrm{d}y = B'(\mathbf{1}_{[0,x]}) \qquad \forall x \in [0\,,1]. \tag{2.1}$$

In terms of the ψ_n's,

$$B(x) = \sum_{n=1}^{\infty} X_n(\mathbf{1}_{[0,x]}\,,\psi_n) = \sum_{n=1}^{\infty} X_n \int_0^x \psi_n(y)\,\mathrm{d}y, \tag{2.2}$$

where $\{X_n\}_{n=1}^{\infty}$ are i.i.d. $\mathrm{N}(0\,,1)$s (Wiener's construction). It is easy to see from this that B is a mean-zero Gaussian process with

$$\mathrm{Cov}[B(x)\,,B(y)] = (\mathbf{1}_{[0,x]}\,,\mathbf{1}_{[0,y]}) = x \wedge y.$$

That is, B = Brownian motion. It follows from (2.1) that

$$B'(x) = \frac{\mathrm{d}}{\mathrm{d}x}B(x) \quad \text{in the sense of distributions.}$$

A little more precisely put, for all $\varphi \in C_0^1[0\,,1]$,

$$\begin{aligned}\int_0^1 \varphi'(x)B(x)\,\mathrm{d}x &= \int_0^1 \varphi'(x)B'(\mathbf{1}_{[0,x]})\,\mathrm{d}x = B'\left(\int_0^1 \varphi'(x)\mathbf{1}_{[0,x]}(\cdot)\,\mathrm{d}x\right)\\ &= B'\left(-\varphi(\cdot)\right) = -B'(\varphi) = -\int_0^1 B'(x)\varphi(x)\,\mathrm{d}x,\end{aligned}$$

thanks to the linearity of the linear operator B'.

Remark 2.1. Though $B(x+\varepsilon) - B(x)$ scales pointwise as $\sqrt{\varepsilon}$, it scales weakly as ε: $\forall \varphi \in C_0^1[0\,,1]$,

$$\begin{aligned}\int_0^1 \varphi(x)\,\frac{B(x+\varepsilon)-B(x)}{\varepsilon}\,\mathrm{d}x &= -\int_0^1 \frac{\varphi(x)-\varphi(x-\varepsilon)}{\varepsilon}\,B(x)\,\mathrm{d}x\\ &\xrightarrow{(\varepsilon\to 0)} -\int_0^1 \varphi'(x)B(x)\,\mathrm{d}x = B'(\varphi) \quad \text{a.s.},\end{aligned}$$

where we have extended every function f on $[0\,,1]$ to a function on $\mathbb{R}$ by defining f to be zero outside $[0\,,1]$.

2.2. **Space-time white noise via Brownian sheet.** Recall from the previous section that if $X_1, X_2, \ldots$ are i.i.d. $\mathrm{N}(0\,,1)$s, then the Gaussian process $B(x) = \sum_{n=1}^{\infty} X_n \int_0^x \psi_n(y)\,\mathrm{d}y$ is a Brownian motion. If we now replace the i.i.d. $\mathrm{N}(0\,,1)$s by i.i.d. Brownian motions, all now indexed by $\mathbb{R}_+$ instead of $[0\,,1]$, then we obtain a "Brownian sheet." More precisely, let $B_1, B_2, \ldots$ denote i.i.d. Brownian motions, each indexed by $\mathbb{R}_+$, and define the following infinite-dimensional version of (2.2):

$$W(t\,,x) := \sum_{n=1}^{\infty} B_n(t)\int_0^x \psi_n(y)\,\mathrm{d}y \qquad \forall t \geqslant 0, x \in [0\,,1].$$

Elementary computations show that W is a [continuous] mean-zero Gaussian process with

$$\mathrm{Cov}[W(t\,,x)\,,W(s\,,y)] = (s \wedge t)(x \wedge y).$$

Space-time "white noise" $\dot{W}$ can now be defined as follows, in analogy with white noise on $[0\,,1]$:

$$\dot{W}(t\,,x) := \frac{\partial^2}{\partial t\,\partial x} W(t\,,x) \quad \text{in the sense of distributions.}$$

It might have been more clear to write $\dot{W}'$ for this mixed weak derivative, but we are simply following the current standards and notations of the subject.

Formally, $\dot{W}$ is a mean-zero Gaussian random field with

$$\mathrm{Cov}[\dot{W}(t\,,x)\,,\dot{W}(s\,,y)] = \delta_0(t-s)\delta_0(x-y).$$

Corresponding to white noise, we also have the formally-defined space-time Wiener integral,

$$\dot{W}(\varphi) := \int \varphi\,\mathrm{d}W := \int_{\mathbb{R}_+\times[0,1]} \varphi(t\,,x)\dot{W}(t\,,x)\,\mathrm{d}t\,\mathrm{d}x := \sum_{n=1}^{\infty} \int_0^{\infty} \mathrm{d}t \int_0^1 \mathrm{d}x\ \varphi(t\,,x)\psi_n(x)B_n'(t),$$

whose precise meaning is

$$\dot{W}(\varphi) := \sum_{n=1}^{\infty} \int_0^{\infty} (\varphi(t)\,,\psi_n)\,B_n(\mathrm{d}t) \qquad \forall \varphi \in L^2(\mathbb{R}_+ \times [0\,,1]). \tag{2.3}$$

Each integral inside the sum is a Wiener integral, as was defined in the last section, and the sum converges in $L^2(\Omega)$, as is not hard to verify directly. It is easy to verify the following computation:

$$\mathrm{Cov}[\dot{W}(f)\,,\dot{W}(g)] = \int_0^{\infty} \mathrm{d}t \int_0^1 \mathrm{d}x\ f(t\,,x)g(t\,,x) := (f\,,g) \qquad \forall f,g \in L^2(\mathbb{R}_+ \times [0\,,1]).$$

Thus, it follows that the space-time white noise $\dot{W}$ can be viewed as a linear isometry from $L^2(\mathbb{R}_+ \times [0\,,1])$ to $L^2(\Omega)$.

3. **The Walsh integral**

If B is Brownian motion and $\varphi \in L^2(\mathbb{R}_+)$ is non random, then the Wiener integral $\int_0^\infty \varphi\,\mathrm{d}B$ is basically defined as it was earlier for Brownian motion on $[0\,,1]$; I will leave the requisite adjustments to you. Recall that Itô introduced a generalization of the Wiener integral—this is nowadays called the Itô integral—which included many random functions φ. The "Walsh integral" is the natural extension of Itô's integral against white noise/Brownian motion to a stochastic integral against space-time white noise/Brownian sheet. Next is an outline of the construction, and the first properties, of the Walsh integral.

Let $\Phi = \Phi(t\,,x)$ be a "predictable random field." That is:

(1) $\Phi(t) \in L^2[0\,,1]$ for [almost] every $t \geqslant 0$;
(2) $\mathrm{E}\sum_{n=1}^{\infty}\int_0^{\infty}(\Phi(t)\,,\psi_n)^2\,\mathrm{d}t < \infty$; and
(3) $t \mapsto (\Phi(t)\,,\psi_n)$ is predictable with respect to the filtration generated by the infinite-dimensional Brownian motion $(B_1\,,B_2\,,\ldots)$.

For such a random field Φ, we may define the "Walsh integral,"

$$\dot{W}(\Phi) := \sum_{n=1}^{\infty} \int_0^{\infty} (\Phi(t)\,,\psi_n)\,B_n(\mathrm{d}t),$$

where every integral is an ordinary Itô integral. We might also use alternative notation such as

$$\dot{W}(\Phi) := \int_{\mathbb{R}_+\times[0,1]} \Phi\,\mathrm{d}W := \int_{\mathbb{R}_+\times[0,1]} \Phi(t\,,x)\dot{W}(t\,,x)\,\mathrm{d}t\,\mathrm{d}x,$$

and analogous notation for definite Walsh integrals: For all Borel sets $T \subset \mathbb{R}_+$ and $S \subset [0\,,1]$,

$$\int_{T\times S} \Phi(t\,,x)\dot{W}(t\,,x)\,\mathrm{d}t\,\mathrm{d}x := \int_{T\times S} \Phi\,\mathrm{d}W := \dot{W}(\Phi\mathbf{1}_{T\times S}),$$

etc. Now that things are set up correctly, one can readily deduce the following from well-known properties of the Itô integral.

Theorem 3.1 (Pardoux, Krylov–Rozovskĭi, Walsh, …)**.** *For every predictable space-time random field* Φ*,*

$$M_t := \int_{[0,t]\times[0,1]} \Phi(s\,,y)\,\dot{W}(s\,,y)\,\mathrm{d}s\,\mathrm{d}y \qquad (t \geqslant 0)$$

defines a centered, continuous $L^2(\Omega)$*-martingale with quadratic variation,*

$$\langle M\rangle_t = \int_0^t \mathrm{d}s \int_0^1 \mathrm{d}y\;|\Phi(s\,,y)|^2 := \int_0^t \|\Phi(s)\|^2_{L^2[0,1]}\,\mathrm{d}s \qquad \forall t \geqslant 0.$$

4. A stochastic heat equation on $[0\,,1]$

Let $b, \sigma : \mathbb{R} \to \mathbb{R}$ be Lipschitz-continuous, non-random functions, and let $\dot{W}$ denote space-time white noise, as before. We now wish to consider the random solution $u = u(t\,,x)$ to the following "stochastic heat equation":

$$\begin{cases} \dot{u} = u'' + b(u) + \sigma(u)\dot{W} \qquad \text{on } (0\,,\infty)\times[0\,,1], \text{ subject to} \\ u(0) = u_0 \qquad \text{on } [0\,,1], \text{ and} \\ u(t\,,0) = u(t\,,1) = 0 \qquad \forall t > 0. \end{cases} \tag{4.1}$$

For the sake of simplicity, we will restrict attention to Hölder continuous and non-random initial functions $u_0 : [0\,,1] \to \mathbb{R}$.

Recall the heat semigroup $\{P_t\}_{t\geqslant 0}$, as well as the fact that $u(t\,,x) = (P_t u_0)(x)$ when $b \equiv \sigma \equiv 0$. For more general choices of b and σ we formally apply Duhamel's principle (or variation of constants) to perturb the nonlinear stochastic PDE (or SPDE) off the case $b \equiv \sigma \equiv 0$. Any integral that involves $\dot{W}$ will be interpretted as a Walsh integral. The end result is the following rigorous interpretation of (4.1): We wish to find a predictable random field u that solves the stochastic integral equation,

$$\begin{aligned} u(t\,,x) = (P_t u_0)(x) &+ \int_{[0,t]\times[0,1]} p_{t-s}(x\,,y) b(u(s\,,y))\,\mathrm{d}s\,\mathrm{d}y \\ &+ \int_{[0,t]\times[0,1]} p_{t-s}(x\,,y)\sigma(u(s\,,y))\dot{W}(s\,,y)\,\mathrm{d}s\,\mathrm{d}y. \end{aligned} \tag{4.2}$$

The first integral on the right is a Lebesgue integral, whereas the second is supposed to be a Walsh integral. See Walsh's St.-Flour notes [30] for the details of the argument that leads to this—socalled "mild"—formulation of the SPDE (4.1). It is possible to appeal to a "stochastic Fubini theorem" – see Walsh [30] – in order to prove that any mild solution (4.2) to (4.1) is in fact a solution in the usual [weak] sense. That is, it is possible to show that if u solves (4.2) then for all smooth and non-random $\phi : \mathbb{R}_+ \times [0\,,1]$ that satisfy $\phi(t\,,0) = \phi(t\,,1) = 0$ for all $t \geqslant 0$,

$$\begin{aligned}
&\int_0^1 \phi(0\,,x)u_0(x)\,\mathrm{d}x - \int_{\mathbb{R}_+\times[0,1]} \dot\phi(t\,,x)u(t\,,x)\,\mathrm{d}t\,\mathrm{d}x \\
&= \int_{\mathbb{R}_+\times[0,1]} \phi''(t\,,x)u(t\,,x)\,\mathrm{d}t\,\mathrm{d}x + \int_{\mathbb{R}_+\times[0,1]} \phi(t\,,x)\sigma(u(t\,,x))\,\dot W(t\,,x)\,\mathrm{d}t\,\mathrm{d}x.
\end{aligned} \tag{4.3}$$

Let us consider the simplest case where $\sigma \equiv b \equiv 0$. Then, the stochastic heat equation (4.1) reduces to the heat equation (1.1). It is known that even in this simple setting, there can be infinitely-many pointwise solutions to variations of (1.1) (consider the Tychonoff example [18] for instance). However, there is a unique mild solution to (1.1); that is, $u(t\,,x) = (P_t u_0)(x)$. In other words, by insisting on finding only mild solutions, we can ensure uniqueness – and also a measure of additional regularity – in such settings. In loose terms, mild solutions to stochastic PDEs are in some sense the only "physical" ones.

4.1. **Existence and uniqueness.** The main result of this section is the following basic existence/uniqueness theorem, that marks the beginning of any, and every, meaningful conversation about the SPDE (4.1).

Theorem 4.1 (Pardoux, Krylov–Rozovskiĭ, Walsh, ...). *There exists a unique predictable random field $u = \{u(t\,,x);\, t \geqslant 0,\, x \in [0\,,1]\}$ such that:*

(1) *$u(t\,,x)$ solves (4.2) a.s. for every $(t\,,x) \in (0\,,\infty) \times [0\,,1]$; and*
(2) *$\sup_{x\in[0,1]} \sup_{t\in[0,T]} \mathrm{E}(|u(t\,,x)|^2) < \infty$ for every real number $T > 0$.*

A careful proof requires some effort mainly because of the many implicit measurability requirements that predictable random fields have. Therefore, we will merely outline the part of the proof that does not discuss measurability issues.

Ideas of proof. In order to simplify the exposition, let us consider only the case that $b \equiv 0$. It is possible to extend the following general argument so that it covers the case $b \not\equiv 0$ as well.

The bulk of the proof is a fixed-point argument. Let $u_0(t\,,x) := u_0(x)$, and then iteratively define

$$u_{n+1}(t\,,x) := (P_t u_0)(x) + \int_{[0,t]\times[0,1]} p_{t-s}(x\,,y)\sigma(u_n(s\,,y))\,\dot W(s\,,y)\,\mathrm{d}s\,\mathrm{d}y, \tag{4.4}$$

for every $(t\,,x) \in (0\,,\infty) \times [0\,,1]$ and $n \in \mathbb{N}$. Assuming that the stochastic integral is inductively well defined, we might recall that, for every fixed $T > 0$,

$$M_t := \int_{[0,t]\times[0,1]} p_{T-s}(x\,,y)\sigma(u_n(s\,,y))\,\dot W(s\,,y)\,\mathrm{d}s\,\mathrm{d}y \qquad (0 < t \leqslant T)$$

is a continuous, centered, $L^2(\Omega)$-martingale $[M_0 := 0]$ with quadratic variation

$$\langle M\rangle_t = \int_0^t \mathrm{d}s \int_0^1 \mathrm{d}y\ |p_{T-s}(x\,,y)|^2\sigma^2(u_n(s\,,y)) \qquad \forall t \in (0\,,T].$$

Therefore, $\mathrm{E}(M_t) = 0$ and $\mathrm{E}(M_t^2) = \mathrm{E}[\langle M\rangle_t]$ for all $t \in [0\,,T]$. We specialize this with $T := t$ in order to see that

$$\mathrm{E}\left(|u_{n+1}(t\,,x)|^2\right) = |(P_t u_0)(x)|^2 + \mathrm{E}\int_0^t \mathrm{d}s \int_0^1 \mathrm{d}y\ |p_{t-s}(x\,,y)|^2\sigma^2(u_n(s\,,y)).$$

Since u_0 is bounded, the Brownian-motion representation (1.4) of $P_t u_0$ shows that

$$|(P_t u_0)(x)| = |\mathrm{E}_x\left(u_0(B_{2t});\, t < \tau\right)| \leqslant \sup_{y\in[0,1]} |u_0(y)| := K.$$

And by the Lipschitz continuity of σ, there exists a finite constant L such that

$$|\sigma(z)| \leqslant |\sigma(z) - \sigma(0)| + |\sigma(0)| \leqslant L(|z| + 1) \qquad \forall z \in \mathbb{R}.$$

We may combine these bounds in order to see that

$$\mathrm{E}\left(|u_{n+1}(t\,,x)|^2\right) \leqslant K^2 + L^2 \int_0^t \mathrm{d}s \int_0^1 \mathrm{d}y\ |p_{t-s}(x\,,y)|^2 \mathrm{E}\left(|u_n(s\,,y)|^2\right). \tag{4.5}$$

Let $\beta > 0$ be a fixed constant whose value will be determined shortly, and define for all $k \geqslant 0$,

$$m_k := \sup_{x\in[0,1]} \sup_{t\geqslant 0} \mathrm{e}^{-\beta t} \mathrm{E}\left(|u_k(t\,,x)|^2\right).$$

Then, we can multiply both sides of (4.5) by $\exp(-\beta t)$ and optimize over $(t\,,x)$ in order to obtain the recursive inequality,

$$\begin{aligned} m_{n+1} &\leqslant K^2 + L^2 m_n \sup_{x\in[0,1]} \sup_{t\geqslant 0} \int_0^t \mathrm{d}s \int_0^1 \mathrm{d}y\ \mathrm{e}^{-\beta(t-s)} |p_{t-s}(x\,,y)|^2 \\ &= K^2 + L^2 m_n \int_0^\infty \mathrm{e}^{-\beta s}\,\mathrm{d}s \int_0^1 \mathrm{d}y\ |p_s(x\,,y)|^2. \end{aligned}$$

Since $p_s(x\,,y)$ is the transition density of Brownian motion, run at twice its speed and killed when it leaves $[0\,,1]$, it is at most the transition density of free Brownian motion run at twice the standard speed, viz.,

$$p_s(x\,,y) \leqslant \Gamma(s\,,x-y),$$

where Γ denotes the "free-space heat kernel,"

$$\Gamma(s\,,a) = \frac{1}{\sqrt{4\pi s}} \exp\left(-\frac{a^2}{4s}\right) \qquad \forall a \in \mathbb{R},\ s > 0. \tag{4.6}$$

Thus, a direct computation yields

$$\begin{aligned} m_{n+1} &\leqslant K^2 + L^2 m_n \sup_{x\in[0,1]} \int_0^\infty \mathrm{e}^{-\beta s} \|\Gamma(s)\|_{L^2(\mathbb{R})}^2\,\mathrm{d}s \\ &= K^2 + C m_n \int_0^\infty \frac{\mathrm{e}^{-\beta s}}{\sqrt{s}}\,\mathrm{d}s, \end{aligned}$$

where C is an uninteresting, but fixed, finite constant. Another simple evaluation yields a finite constant C_1 such that

$$m_{n+1} \leqslant K^2 + \frac{C_1 m_n}{\sqrt{\beta}} \qquad \forall n \geqslant 0, \tag{4.7}$$

and C_1 is independent of β (though m_n and m_{n+1} depend on β). Now we choose and fix $\beta := 4C_1^2$ in order to find that, for this choice of β, $m_{n+1} \leqslant K^2 + \frac{1}{2} m_n$, uniformly for all $n \geqslant 0$. From this we can conclude that

$$m_{n+1} \leqslant K^2 + \tfrac{1}{2}\left(K^2 + \tfrac{1}{2} m_{n-1}\right) \leqslant \cdots \leqslant K^2\left(1 + \tfrac{1}{2} + \tfrac{1}{4} + \cdots\right) + 2^{-n} m_0 = 2K^2 + 2^{-n} m_0,$$

uniformly for all $n \geqslant 0$, provided that $u_1, \ldots, u_n$ are predictable random fields. In particular, the definition of the sequence $\{m_n\}_{n=0}^{\infty}$ yields

$$\sup_{x\in[0,1]} \sup_{t\in[0,T]} \mathrm{E}\left(|u_n(t\,,x)|^2\right) \leqslant \left(2K^2 + m_0\right) \mathrm{e}^{\beta T} < \infty, \tag{4.8}$$

uniformly for all $n \geqslant 0$ for which $u_1, \ldots, u_n$ are predictable random fields. It can be shown that (4.8) and the very definition (4.4) of the sequence $\{u_n\}_{n=0}^{\infty}$ together ensure this predictability. Hence, (4.8) holds uniformly for all $n \geqslant 0$, and unconditionally.

Next, define

$$d_k := \sup_{x\in[0,1]} \sup_{t\geqslant 0} \mathrm{e}^{-\beta t} \mathrm{E}\left(|u_{k+1}(t\,,x) - u_k(t\,,x)|^2\right) \qquad \forall k \geqslant 0,$$

for the same value of $\beta > 0$ as before. A computation very similar to the one that led to (4.7) yields

$$d_{n+1} \leqslant \frac{C_1 d_n}{\sqrt{\beta}} = \tfrac{1}{2} d_n \qquad \forall n \geqslant 0.$$

Thus, $d_n \to 0$ exponentially fast as $n \to \infty$; in particular,

$$\sup_{x\in[0,1]} \sup_{t\in[0,T]} \sum_{k=0}^{\infty} \mathrm{E}\left(|u_{n+1}(t\,,x) - u_n(t\,,x)|^2\right) \leqslant \mathrm{e}^{\beta T} \sum_{k=0}^{\infty} d_k < \infty.$$

This proves that $\{u_n(t\,,x)\}_{n=0}^{\infty}$ is a Cauchy sequence in $L^2(\Omega)$ for every fixed space-time point $(t\,,x) \in (0\,,\infty) \times [0\,,1]$. Let $u(t\,,x) := \lim_{n\to\infty} u_n(t\,,x)$ in $L^2(\Omega)$. One can then recycle the preceding arguments to show that

$$\begin{aligned} \lim_{n\to\infty} \int_{[0,t]\times[0,1]} & p_{t-s}(x\,,y)\sigma(u_n(s\,,y))\,\dot{W}(s\,,y)\,\mathrm{d}s\,\mathrm{d}y \\ &= \int_{[0,t]\times[0,1]} p_{t-s}(x\,,y)\sigma(u(s\,,y))\,\dot{W}(s\,,y)\,\mathrm{d}s\,\mathrm{d}y \qquad \text{in } L^2(\Omega). \end{aligned}$$

We have argued that both sides of (4.4) converge in $L^2(\Omega)$ to similar quantities that involve u in place of u_n and u_{n+1} everywhere. This completes the proof of existence in the case that $b \equiv 0$.

In order to verify uniqueness suppose (4.1)—still with $b \equiv 0$—has two predictable random field solutions u and v. Then,

$$u(t\,,x) - v(t\,,x) = \int_{[0,t]\times[0,1]} p_{t-s}(x\,,y)\left\{\sigma(u(s\,,y)) - \sigma(v(s\,,y))\right\} \dot{W}(s\,,y)\,\mathrm{d}s\,\mathrm{d}y.$$

This, and another martingale computation together yield a finite universal constant A such that, for all $(t\,,x) \in (0\,,\infty) \times [0\,,1]$,

$$\begin{aligned} \mathrm{E}\left(|u(t\,,x) - v(t\,,x)|^2\right) &= \int_0^t \mathrm{d}s \int_0^1 \mathrm{d}y\ |p_{t-s}(x\,,y)|^2\, \mathrm{E}\left(|\sigma(u(s\,,y)) - \sigma(v(s\,,y))|^2\right) \\ &\leqslant A \int_0^t \mathrm{d}s \int_0^1 \mathrm{d}y\ |p_{t-s}(x\,,y)|^2\, \mathrm{E}\left(|u(s\,,y) - v(s\,,y)|^2\right), \end{aligned} \tag{4.9}$$

thanks to the Lipschitz continuity of σ. Define

$$D := \sup_{t\geqslant 0} \sup_{x\in[0,1]} \mathrm{e}^{-\alpha t} \mathrm{E}\left(|u(t\,,x) - v(t\,,x)|^2\right),$$

where $\alpha > 0$ is a constant that will be described shortly. We multiply both sides of (4.9) by $\exp\{-\alpha t\}$ and optimize over $(t\,,x) \in (0\,,\infty) \times [0\,,1]$ in order to see that

$$D \leqslant AD \sup_{x\in[0,1]} \sup_{t\geqslant 0} \int_0^t \mathrm{d}s \int_0^1 \mathrm{d}y\ \mathrm{e}^{-\alpha(t-s)} |p_{t-s}(x\,,y)|^2 \leqslant AD \int_0^\infty \mathrm{e}^{-\alpha s} \|\Gamma(s)\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s = \frac{\tilde{A}D}{\sqrt{\alpha}},$$

after a direct computation, where $\tilde{A}$ is a finite constant that does not depend on α. We now choose $\alpha := \tilde{A}^2/4$ in order to see that, for this choice of α, $D \leqslant D/2$ and hence $D = 0$. This proves that $u(t\,,x) = v(t\,,x)$ a.s. for all $(t\,,x) \in (0\,,\infty) \times [0\,,1]$, and completes the proof of the uniqueness of the solution. □

4.2. **Higher moments.** The proof of part 2 of Theorem 4.1 yields a stronger result than asserted. In fact, we have shown that there exist real number $\alpha, \beta > 0$ such that

$$\mathrm{E}\left(|u(t\,,x)|^2\right) \leqslant \alpha \mathrm{e}^{\beta t} \qquad \forall t \geqslant 0,\ x \in [0\,,1].$$

Among other things, the proof hinged on the fact that $\mathrm{E}(M_t^2) = \mathrm{E}\langle M\rangle_t$ for all $t \geqslant 0$, whenever $\{M_t\}_{t\geqslant 0}$ is a continuous, $L^2(\Omega)$-martingale. Foondun and Khoshnevisan [20] have produced moment bounds for higher moments of a closely-related object to $u(t\,,x)$. Their arguments imply the following.

Theorem 4.2. *There exist real numbers $A, B > 0$ such that*

$$\mathrm{E}\left(|u(t\,,x)|^k\right) \leqslant A^k k^{k/2} \mathrm{e}^{Bk^3 t} \qquad \forall t \geqslant 0, x \in [0\,,1], k \geqslant 2;$$

the variable k need not be integral.

See also Conus, Joseph, and Khoshnevisan [12]. It is possible to prove that the constant k^3 is sharp; see Borodin and Corwin [5], Chen [10, 11], Conus, Joseph, and Khoshnevisan [12], and Conus, Joseph, Khoshnevisan and Shiu [14], and Chapter 6 of Khoshnevisan [25] for similar results. I will omit the lengthy proof. Suffice it to say that one adapts the proof of the moment estimates of Theorem 4.1, but use the following form of the Burkholder–Davis–Gundy inequality (hereforth, BDG):

$$C_k := \sup \frac{\|M_t\|^2_{L^k(\Omega)}}{\|\langle M\rangle_t\|_{L^{k/2}(\Omega)}} < \infty \quad \forall k \geqslant 2,$$

where the supremum is taken over all $t > 0$ and all continuous, $L^2(\Omega)$-martingales $\{M_t\}_{t\geqslant 0}$ that do not vanish a.s. The original BDG inequality (see [6]) contained an explicit but suboptimal upper bound for C_k. But it was good enough to show that $C_k < \infty$. Davis [15] was able to compute C_k, and Carlen and Kree [7] showed that: (i) $C_k \leqslant 4k$ for all $k \geqslant 2$; and (ii) $C_k/k \to 4$ as $k \to \infty$. To summarize, one of the key ingredients of Theorem 4.2 is the following form of the BDG inequality that is valid for all continuous, $L^2(\Omega)$-martingales $\{M_t\}_{t\geqslant 0}$:

$$\mathrm{E}(M_t^k) \leqslant (4k)^{k/2} \mathrm{E}\left[\langle M\rangle_t^{k/2}\right] \qquad \forall t \geqslant 0,\ k \geqslant 2. \tag{4.10}$$

4.3. **Regularity.** The goal of this course is to analyze the solution to the SPDE (4.1). The existence and uniqueness of that solution is guaranteed by Theorem 4.1. The following regularity result is a prefatory foray into the analysis of the solution to (4.1).

Theorem 4.3. *If $u_0(0) = u_0(1) = 0$ and u_0 is α-Hölder continuous for some $\alpha \in (0\,,1]$, then u has a continuous modification. In fact, that modification a.s. satisfies the following: For all real numbers $T > 0$, $\mu \in (0\,,\alpha \wedge (1/2))$, and $\rho \in (0\,,(\alpha/2) \wedge (1/4))$, there exists an a.s.-finite random variable V_T such that, with probability one,*

$$|u(t\,,x) - u(s\,,y)| \leqslant V_T\left(|x-y|^{\mu} + |t-s|^{\rho}\right) \quad \forall x,y \in [0\,,1],\ s,t \in [0\,,T].$$

Thanks to a suitable form of the Kolmogorov continuity theorem (see, for example, [25]), Theorem 4.3 follows readily from the following two quantitative bounds: For every $k \geqslant 2$ and $T > 0$ there exists $C_{k,T}$ such that

$$\begin{aligned} &\mathrm{E}\left(|u(t\,,x) - u(t\,,y)|^k\right) \leqslant C_{k,T}|x-y|^{k\{\alpha\wedge(1/2)\}}, \qquad \text{and} \\ &\mathrm{E}\left(|u(t\,,x) - u(s\,,x)|^k\right) \leqslant C_{k,T}|t-s|^{k\{(\alpha/2)\wedge(1/4)\}}, \end{aligned} \tag{4.11}$$

simultaneously for all $x,y \in [0\,,1]$, $s,t \in [0\,,T]$, and $k \geqslant 2$.

We will sketch the proof of a weaker form of the first bound in (4.11). After that, we will say a few things about the second bound as well. To simplify the exposition, we restrict attention to the case that $b \equiv 0$ throughout. The details of the derivation of (4.11) will soon appear in Dalang , Khoshnevisan, and Zhang [**?**].

First of all, it follows from Minkowski's inequality and the mild formulation (4.2) of the solution that

$$\begin{aligned} &\|u(t\,,x) - u(t\,,y)\|_{L^k(\Omega)} \\ &\leqslant |(P_tu_0)(x) - (P_tu_0)(y)| + \left\| \int_{[0,t]\times[0,1]} \{p_{t-s}(x\,,w) - p_{t-s}(y\,,w)\}\,\sigma(u(s\,,w))\dot{W}(s\,,w)\,\mathrm{d}s\,\mathrm{d}w \right\|_{L^k(\Omega)}. \end{aligned}$$

We estimate the first term next. It might be helpful to recall that $(t\,,x) \mapsto (P_tu_0)(x)$ solves the non-random heat equation (1.1). Thus, the following is a spatial modulus-of-continuity estimate for the solution to the heat equation (1.1).

Lemma 4.4. *If u_0 is Hölder continuous, then there exist $\alpha, L > 0$ such that*

$$|(P_tu_0)(x) - (P_tu_0)(y)| \leqslant L|x-y|^{\alpha} \qquad \forall t \geqslant 0,\ x,y \in [0\,,1].$$

Proof. Thanks to (1.3),

$$\begin{aligned} |(P_tu_0)(x) - (P_tu_0)(y)|^2 &\leqslant \left[\int_0^1 |p_t(x\,,z) - p_t(y\,,z)|\,|u_0(z)|\,\mathrm{d}z\right]^2 \\ &\leqslant \|u_0\|^2_{L^2[0,1]} \int_0^1 [p_t(x\,,z) - p_t(y\,,z)]^2\,\mathrm{d}z \\ &= 2\|u_0\|^2_{L^2[0,1]} \sum_{n=1}^{\infty} |\sin(n\pi x) - \sin(n\pi y)|^2 \mathrm{e}^{-2n^2\pi^2 t}. \end{aligned}$$

Because $|\sin(n\pi x)-\sin(n\pi y)|\leqslant n\pi|x-y|$, it follows that there exists $C=C(\|u_0\|_{L^2[0,1]})>0$ such that

$$|(P_tu_0)(x)-(P_tu_0)(y)|^2\leqslant C|x-y|^2\sum_{n=1}^\infty n^2\mathrm{e}^{-2n^2\pi^2t}\leqslant C|x-y|^2\int_0^\infty w^2\mathrm{e}^{-2w^2\pi^2t}\,\mathrm{d}w$$
$$=\frac{C'|x-y|^2}{t^{3/2}},$$

for a constant C' that depends only on $\|u_0\|_{L^2[0,1]}$. In particular, if $t\geqslant|x-y|$, then $|(P_tu_0)(x)-(P_tu_0)(y)|\leqslant c|x-y|^{1/4}$ with $c=\sqrt{C'}$.

Now consider the case that $0<t<|x-y|$. In this case, we proceed differently as follows:

$$\begin{aligned}|(P_tu_0)(x)-(P_tu_0)(y)|&\leqslant|(P_tu_0)(x)-u_0(x)|+|(P_tu_0)(y)-u_0(y)|+|u_0(x)-u_0(y)|\\&\leqslant|(P_tu_0)(x)-u_0(x)|+|(P_tu_0)(y)-u_0(y)|+c_1|x-y|^a,\end{aligned}$$

where $c_1>0$ and $a>0$ account for the Hölder continuity of u_0, and do not depend on $(t\,,x\,,y)$. Thanks to (1.4) and the facts that: (i) $u_0(B_{2\tau})=0$ a.s. on $\{\tau<\infty\}$; and (ii) $0<t<|x-y|$,

$$\begin{aligned}|(P_tu_0)(x)-\mathrm{E}_x[u_0(B_{2t})]|&\leqslant\mathrm{E}_x\left(|u_0(B_{2t})|;\tau\leqslant t\right)\\&\leqslant c_1\mathrm{E}_x\left(|B_{2t}-B_{2\tau}|^a;\ \tau\leqslant t\right)\\&\leqslant c_1\mathrm{E}_x\left(\sup_{s\in[0,t]}|B_{2t}-B_{2s}|^a\right)\leqslant c_2t^{2/a}\leqslant c_2|x-y|^{2a},\end{aligned}$$

where c_2 does not depend on $(t\,,x)$, thanks to a standard modulus-of-continuity estimate. The very same estimate holds if we replace x by y in the left-most quantity. These estimates together yield the desired result. □

Now we prove the first inequality in (4.11).

Proof of (4.11)*: First part.* Lemma 4.4 and a suitable application of the BDG inequality (4.10) together yield

$$\begin{aligned}&\|u(t\,,x)-u(t\,,y)\|_{L^k(\Omega)}\\&\qquad\leqslant L|x-y|^\alpha+\sqrt{\int_0^t\mathrm{d}s\int_0^1\mathrm{d}w\ \left\{p_{t-s}(x\,,w)-p_{t-s}(y\,,w)\right\}^2\|\sigma(u(s\,,y))\|_{L^k(\Omega)}^2},\end{aligned}$$

where L denotes the Lipschitz constant of u_0. Let K denote the Lipschitz constant of σ. Since $|\sigma(z)|\leqslant|\sigma(0)|+K|z|$ for all $z\in\mathbb{R}$, Theorem 4.2 implies that $\|\sigma(u(s\,,y))\|_{L^k(\Omega)}$ is bounded uniformly in $y\in[0\,,1]$ and $t\in[0\,,T]$. Therefore,

$$\|u(t\,,x)-u(t\,,y)\|_{L^k(\Omega)}\leqslant L|x-y|^\alpha+c_{k,T}\sqrt{\int_0^t\mathrm{d}s\int_0^1\mathrm{d}w\ \left\{p_{t-s}(x\,,w)-p_{t-s}(y\,,w)\right\}^2}.$$

Apply Parseval's identity – see also (1.3) – to compute

$$\begin{aligned}\int_0^1 \{p_{t-s}(x\,,w) - p_{t-s}(y\,,w)\}^2 \, \mathrm{d}w &= \sum_{n=1}^\infty [\psi_n(x) - \psi_n(y)]^2 \, \mathrm{e}^{-2n^2\pi^2(t-s)} \\ &= 2\sum_{n=1}^\infty [\sin(n\pi x) - \sin(n\pi y)]^2 \, \mathrm{e}^{-2n^2\pi^2(t-s)} \\ &\leqslant c\sum_{n=1}^\infty \left(1 \wedge n^2|x-y|^2\right) \mathrm{e}^{-2n^2\pi^2(t-s)},\end{aligned}$$

since $|\sin a - \sin b| \leqslant 2(1 \wedge |a-b|)$ for all $a, b \in \mathbb{R}$. Therefore,

$$\int_0^t \mathrm{d}s \int_0^1 \mathrm{d}w \;\; \{p_{t-s}(x\,,w) - p_{t-s}(y\,,w)\}^2 \leqslant c' \sum_{n=1}^\infty \frac{1 \wedge n^2|x-y|^2}{n^2}.$$

Split the sum according to whether or not $n \geqslant 1/|x-y|$ in order to see that

$$\int_0^t \mathrm{d}s \int_0^1 \mathrm{d}w \;\; \{p_{t-s}(x\,,w) - p_{t-s}(y\,,w)\}^2 \leqslant c''|x-y|.$$

This yields the inequality,

$$\|u(t\,,x) - u(t\,,y)\|_{L^k(\Omega)} \leqslant L|x-y|^\alpha + c'_{k,T}\sqrt{|x-y|} \leqslant c''_{k,T}|x-y|^{\alpha\wedge(1/2)},$$

valid uniformly for all $t \in [0\,,T]$ and $x, y \in [0\,,1]$. □

We will not spend too much time on the second bound in (4.11) since its proof is similar to that of the first part. The following lemma is a point of departure, and replaces the role of Lemma 4.4 in the preceding proof of the first part of (4.11). Also, it is worthy of record that this is where the additional conditions $u_0(0) = u_0(1) = 0$ enter the proof.

Lemma 4.5. *If $u_0(0) = u_0(1) = 0$ and u_0 is Hölder continuous with index $\geqslant \alpha$ for some $\alpha \in (0\,,1]$, then there exists a real number $K > 0$ such that*

$$|(P_t u_0)(x) - (P_s u_0)(x)| \leqslant K|t-s|^{\alpha/2} \qquad \forall s, t \geqslant 0,\; x, y \in [0\,,1].$$

Proof. Recall that for all $y \in [0\,,1]$ and $s > 0$, $(P_s u_0)(y) = \mathrm{E}_y[u_0(B_{2s})\,;\tau > s]$ where B denotes Brownian motion and τ denotes the first exit time of $[0\,,1]$ by $t \mapsto B_{2t}$. Then,

$$\begin{aligned}(P_{t+\varepsilon}u_0)(x) - (P_t u_0)(x) &= \mathrm{E}_x\left[u_0(B_{2(t+\varepsilon)})\mathbf{1}_{\{\tau > t+\varepsilon\}} - u_0(B_{2t})\mathbf{1}_{\{\tau > t\}}\right] \\ &= \mathrm{E}_x\left[u_0(B_{2(t+\varepsilon)}) - u_0(B_{2t}); \tau > t+\varepsilon\right] - \mathrm{E}_x\left[u_0(B_{2t});\, t < \tau \leqslant t+\varepsilon\right].\end{aligned}$$

Thus, it follows from the triangle inequality that

$$|(P_{t+\varepsilon}u_0)(x) - (P_t u_0)(x)| \leqslant \mathrm{E}\left(|u_0(B_{2(t+\varepsilon)}) - u_0(B_{2t})|\right) + \mathrm{E}_x\left(|u_0(B_{2t})|;\; t < \tau \leqslant t+\varepsilon\right).$$

Since u_0 is Hölder continuous, we can find a real number L such that

$$\mathrm{E}\left(|u_0(B_{2(t+\varepsilon)}) - u_0(B_{2t})|\right) \leqslant L\mathrm{E}\left(|B_{2(t+\varepsilon)} - B_{2t}|^\alpha\right) \leqslant L\varepsilon^{\alpha/2} \quad \forall t, \varepsilon \geqslant 0. \tag{4.12}$$

Because u_0 vanishes on $\{0\,,1\}$ and $B_{2\tau}\in\{0\,,1\}$ a.s.,

$$\begin{aligned}
\mathrm{E}_x\left(|u_0(B_{2t})|;\ t<\tau\leqslant t+\varepsilon\right) &= \mathrm{E}_x\left(|u_0(B_{2t})-u_0(B_{2\tau})|;\ t<\tau\leqslant t+\varepsilon\right)\\
&\leqslant L\mathrm{E}_x\left(|B_{2t}-B_{2\tau}|^\alpha;\ t<\tau\leqslant t+\varepsilon\right)\\
&\leqslant L\mathrm{E}\left(\sup_{r\in[0,\varepsilon]}|B_{2(t+r)}-B_{2t}|^\alpha\right)=L\mathrm{E}\left(\sup_{r\in[0,\varepsilon]}|B_{2r}|^\alpha\right),
\end{aligned}$$

which is equal to a constant real number times $\varepsilon^{\alpha/2}$ by Brownian scaling. This fact and (4.12) together imply the lemma. □

5. **Asymptotic analysis in the linear case**

Consider an idealized rod of unit length, and identify the rod with the interval $[0\,,1]$ in the usual way. Assume that, initially, heat density in the rod is given by the function $u_0:[0\,,1]\to\mathbb{R}$, and suppose that the endpoints of the rod are continually cooled at zero temperature. Then, the heat density $u(t)$ at time t satisfies the heat equation (1.1). As we saw in the first lecture, this heat density is explicitly given by

$$u(t\,,x)=(P_tu_0)(x)=\sum_{n=1}^\infty(u_0\,,\psi_n)\mathrm{e}^{-n^2\pi^2t}\psi_n(x).$$

This particular representation of the solution lets us see immediately that

$$\sup_{x\in[0,1]}|u(t\,,x)|\sim 2\mathrm{e}^{-\pi^2t}\left|\int_0^1 u_0(y)\sin(\pi y)\,\mathrm{d}y\right|\qquad\text{as } t\to\infty. \tag{5.1}$$

In other words, heat dissipates uniformly, and exponentially rapidly, in the rod at ever-larger times.

In this section we ask, and in part answer, similar dissipation questions about the stochastic heat equation (4.1) in the relatively simple case that $b\equiv 0$ and the function σ is a non-zero constant (still denoted by σ). We also will touch on small-time asymptotics.

5.1. **Large-time asymptotics.** First let us assume, in addition, that $u_0\equiv 0$. Thus, we look for a random field $Z=Z(t\,,x)$ such that

$$\left[\begin{array}{ll}\dot Z=Z''+\sigma\dot W & \text{on } (0\,,\infty)\times[0\,,1],\text{ subject to}\\ Z(0)\equiv 0 & \text{on } [0\,,1].\end{array}\right. \tag{5.2}$$

According to Theorem 4.1, the solution to (4.1) is unique as well as continuous. Moreover, the solution can be written as follows (see (4.2)):

$$Z(t\,,x)=\sigma\int_{[0,t]\times[0,1]}p_{t-s}(x\,,y)\dot W(s\,,y)\,\mathrm{d}s\,\mathrm{d}y. \tag{5.3}$$

In particular, the construction of the Wiener integral ensures that the random field Z is a mean-zero Gaussian random field with covariance,

$$\begin{aligned}
\mathrm{Cov}[Z(t\,,x)\,,Z(s\,,y)] &= \sigma^2\int_0^{s\wedge t}\mathrm{d}r\int_0^1\mathrm{d}w\ p_{t-r}(x\,,w)p_{s-r}(y\,,w)\\
&=\sigma^2\int_0^{t\wedge s}p_{t+s-2r}(x\,,y)\,\mathrm{d}r,
\end{aligned} \tag{5.4}$$

thanks to the symmetry and the semigroup property of the heat semigroup.

It is natural to think of the solution u to a parabolic PDE (random or otherwise) as a function $x \mapsto u(t\,,x)$ that evolves with time t. With this view in mind, we can first notice that, owing to (5.4),

$$\begin{aligned}\mathrm{Cov}[Z(t\,,x)\,,Z(t\,,y)] &= \sigma^2\int_0^t p_{2t-2r}(x\,,y)\,\mathrm{d}r = \sigma^2 t\int_0^1 p_{2t(1-s)}(x\,,y)\,\mathrm{d}s\\ &= \sigma^2 t\int_0^1 p_{2ts}(x\,,y)\,\mathrm{d}s = \sigma^2 t\sum_{n=1}^\infty\int_0^1 \psi_n(x)\psi_n(y)\mathrm{e}^{-2tsn^2\pi^2}\,\mathrm{d}s\\ &= \frac{\sigma^2}{2\pi^2}\sum_{n=1}^\infty \psi_n(x)\psi_n(y)\cdot\frac{1-\mathrm{e}^{-2tn^2\pi^2}}{n^2}.\end{aligned} \tag{5.5}$$

Therefore,

$$\mathrm{Cov}[Z(t\,,x)\,,Z(t\,,y)] \xrightarrow{(t\to\infty)} \frac{\sigma^2}{2}\sum_{n=1}^\infty \frac{\psi_n(x)\psi_n(y)}{n^2\pi^2}. \tag{5.6}$$

An inspection of the proof of (4.11) shows that for every $k \geqslant 2$ there exists $C_k > 0$ such that

$$\mathrm{E}\left(|Z(t\,,x)-Z(t\,,y)|^k\right) \leqslant C_k|x-y|^{k/2},$$

simultaneously for every $t \geqslant 0$ and $x, y \in [0\,,1]$. An appeal to the Kolmogorov continuity theorem then shows that the $C[0\,,1]$-valued process $\{Z(t)\}_{t\geqslant 0}$ is tight. It follows almost immediately from this and (5.6) that

$$Z(t) \xrightarrow{C[0,1]} \eta \quad \text{as } t\to\infty,$$

where $\eta := \{\eta(x)\}_{x\in[0,1]}$ is a centered Gaussian process, with continuous trajectories, whose covariance is

$$\mathrm{Cov}[\eta(x)\,,\eta(y)] = \frac{\sigma^2}{2}\sum_{n=1}^\infty \frac{\psi_n(x)\psi_n(y)}{n^2\pi^2} \qquad \forall x, y \in [0\,,1].$$

It is easy to use Fourier analysis to simplify this infinite sum. Let $c_0 := 1$ and $c_n(a) := \sqrt{2}\cos(n\pi a)$ for $n \geqslant 1$ and $a \in [0\,,1]$. Then, $\{c_n\}_{n=0}^\infty$ is a complete, orthonormal system in $L^2[0\,,1]$, and

$$(\mathbf{1}_{[0,z]}\,,c_n) = \sqrt{2}\int_0^z \cos(n\pi a)\,\mathrm{d}a = \frac{\sqrt{2}\sin(n\pi z)}{n\pi} = \frac{\psi_n(z)}{n\pi} \qquad \forall z\in[0\,,1],\ n \geqslant 1.$$

Therefore,

$$\begin{aligned}\sum_{n=1}^\infty \frac{\psi_n(x)\psi_n(y)}{n^2\pi^2} = \sum_{n=1}^\infty (\mathbf{1}_{[0,x]}\,,c_n)(\mathbf{1}_{[0,y]}\,,c_n) &= \sum_{n=0}^\infty (\mathbf{1}_{[0,x]}\,,c_n)(\mathbf{1}_{[0,y]}\,,c_n) - xy\\ &= (\mathbf{1}_{[0,x]}\,,\mathbf{1}_{[0,y]}) - xy = (x\wedge y) - xy.\end{aligned} \tag{5.7}$$

Let B denote a standard Brownian motion, and note that the "standard Brownian bridge,"

$$B_0(x) := B(x) - xB(1) \qquad (0 \leqslant x \leqslant 1)$$

is a mean-zero Gaussian process with covariance $\mathrm{Cov}[B_0(x)\,,B_0(y)] = (x\wedge y) - xy$. The preceding calculation proves that

$$\mathrm{Cov}[\eta(x)\,,\eta(y)] = \frac{\sigma^2}{2}\mathrm{Cov}[B_0(x)\,,B_0(y)] \qquad \forall x, y\in[0\,,1],$$

which in turn implies that η has the same finite-dimensional distributions as $(\sigma/\sqrt{2})B_0$. We have established the following.

Proposition 5.1 (Funaki [21]). *$Z(t) \xrightarrow{C[0,1]} \dfrac{\sigma}{\sqrt{2}} B_0$ as $t \downarrow 0$.*

In particular, we can see from an appeal to the continuous mapping theorem that, as $t \downarrow 0$, $\sup_{x\in[0,1]} |Z(t\,,x)|$ converges in distribution to $\sup_{x\in[0,1]} |\eta(x)| = (\sigma/\sqrt{2}) \sup_{x\in[0,1]} |B_0(x)|$, and hence is non zero almost surely. In other words, the solution to the the heat equation (5.2) does not experience dissipation at all. [Rather, heat content reaches equilibrium at large times.]

As part of a more general statement, we will soon see that the $C[0\,,1]$-valued stochastic process $t \mapsto Z(t)$ is Markov. This fact and Proposition 5.1 together imply that the law of the process $(\sigma/\sqrt{2})B_0$ – viewed as a probability measure on $C[0\,,1]$ – is the unique invariant measure of the $C[0\,,1]$-valued Markov process $\{Z(t)\}_{t\geqslant 0}$.

More generally still, if we consider (5.2), but with initial date u_0 that is non random and continuous (say), then the solution to our SPDE becomes $Z(t\,,x) + (P_tu_0)(x)$; see (4.2). Since $(P_tu_0)(x) \to 0$ [see the first paragraph of this section], it follows that $u(t)$ converges weakly in $C[0\,,1]$ to $(\sigma/\sqrt{2})B_0$, as does $Z(t)$.

5.2. **Small-time asymptotics.** One can also use (5.5) to describe the small-time behavior of $t \mapsto Z(t)$. Indeed, by (5.5),[1]

$$\mathrm{Cov}[Z(t\,,x)\,,Z(t\,,y)] = \sigma^2 \sum_{n=1}^\infty \sin(n\pi x)\sin(n\pi y)\cdot \frac{1-\mathrm{e}^{-2tn^2\pi^2}}{n^2\pi^2}.$$

It follows readily from this fact, and a Riemann-sum approximation by an integral, that

$$\mathrm{Cov}\left[Z\left(t\,,x\sqrt{t}\right)\,,Z\left(t\,,y\sqrt{t}\right)\right] \sim \sigma^2\sqrt{t}\int_0^\infty \sin(x\pi r)\sin(y\pi r)\cdot\frac{1-\mathrm{e}^{-2r^2\pi^2}}{r^2\pi^2}\,\mathrm{d}r,$$

as $t \downarrow 0$. In particular, we can deduce from the above that the finite-dimensional distributions of $x \mapsto t^{-1/4}Z(t\,,x\sqrt{t})$ converge, as $t \downarrow 0$, to the finite-dimensional distributions of $\{\sigma\zeta(x)\}_{x\in[0,1]}$ where ζ is a mean-zero Gaussian process with

$$\mathrm{Cov}[\zeta(x)\,,\zeta(y)] = \int_0^\infty \sin(x\pi r)\sin(y\pi r)\cdot\frac{1-\mathrm{e}^{-2r^2\pi^2}}{r^2\pi^2}\,\mathrm{d}r.$$

One can construct the limiting object $\zeta(x)$ as the following Wiener integral process:

$$\zeta(x) := \sigma\int_0^\infty \frac{\sin(x\pi r)}{\pi r}\sqrt{1-\mathrm{e}^{-2r^2\pi^2}}\,B(\mathrm{d}r) \qquad (0<x<1),$$

where B denotes Brownian motion; equivalently, B' denotes white noise on $\mathbb{R}_+$. This and a tightness argument, which we omit, together imply the following.

[1]The integral is absolutely convergent because

$$0 \leqslant \frac{1-\exp\{-2r^2\pi^2\}}{r^2} \leqslant 2\pi^2 \wedge \frac{1}{r^2} \qquad \forall r>0.$$

Proposition 5.2. *As $t \downarrow 0$,*

$$\left\{ \frac{Z\left(t\,, x\sqrt{t}\right)}{t^{1/4}} \right\}_{x\in[0,1]} \xrightarrow{C[0,1]} \{\sigma\zeta(x)\}_{x\in[0,1]}\,.$$

The next subsection contains a nontrivial generalization of Proposition 5.2.

5.3. **Small-time asymptotics in the non-linear setting.** We now make the transition from the constant-coefficient, Gaussian, case to the general non-Gaussian case by presenting the general nonlinear form of Proposition 5.2.

Theorem 5.3. *Let u denote the solution to* (4.1)*, where u_0 is Hölder continuous with index $\alpha \in (0\,,1]$, and vanishes on $\{0\,,1\}$. Then, as $t \downarrow 0$,*

$$\left\{ \frac{u\left(t\,, x\sqrt{t}\right)}{t^{1/4}} \right\}_{x\in[0,1]} \xrightarrow{C[0,1]} \{\sigma(u_0(0))\zeta(x)\}_{x\in[0,1]}\,.$$

Proof of the convergence of the finite-dimensional distributions. We will skip the proof of tightness and prove only that

$$\left\{ \frac{u\left(t\,, x\sqrt{t}\right)}{t^{1/4}} \right\}_{x\in[0,1]} \xrightarrow{\text{f.d.d.}} \{\sigma(u_0(0))\zeta(x)\}_{x\in[0,1]}\,.$$

where "$\xrightarrow{\text{f.d.d.}}$" denotes the convergence of the finite-dimensional distributions.

In light of Proposition 5.2, it suffices to prove that, uniformly over all $x \in [0\,,1]$,

$$\mathrm{E}\left(|u(t\,,x) - \sigma(u_0(x))Z(t\,,x)|^2\right) = O(t^{\alpha+(1/2)}) \qquad \text{as } t \downarrow 0, \tag{5.8}$$

where Z solves (5.2) with $\sigma \equiv 1$. For then we can write

$$\begin{aligned}
&\mathrm{E}\left(\left|u\left(t\,,x\sqrt{t}\right) - \sigma(u_0(0))Z\left(t\,,x\sqrt{t}\right)\right|^2\right)\\
&\leqslant 2\mathrm{E}\left(|u\left(t\,,x\sqrt{t}\right) - \sigma\left(u_0\left(x\sqrt{t}\right)\right)Z\left(t\,,x\sqrt{t}\right)|^2\right)\\
&\qquad\qquad + 2\left|\sigma\left(u_0\left(x\sqrt{t}\right)\right) - \sigma(u_0(0))\right|^2 \mathrm{E}\left(\left|Z\left(t\,,x\sqrt{t}\right)\right|^2\right)\\
&= O(t^{\alpha+(1/2)}) \qquad (t \downarrow 0),
\end{aligned}$$

owing to the Lipschitz continuity of σ, the α-Hölder continuioty of u_0, (5.8), and the fact that every $L^p(\Omega)$-norm of $Z(t\,,x\sqrt{t})$ behaves as $t^{1/4}$, essentially by Proposition 5.2.

Choose and fix some $x \in [0\,,1]$, and recall from (4.2) that for all $t > 0$,

$$u(t\,,x) = (P_t u_0)(x) + \int_0^t \mathrm{d}s \int_0^1 \mathrm{d}y\; p_{t-s}(x\,,y)b(u(s\,,y)) + I(t\,,x),$$

where

$$I(t\,,x) := \int_{[0,t]\times[0,1]} p_{t-s}(x\,,y)\sigma(u(s\,,y))\dot{W}(s\,,y)\,\mathrm{d}s\,\mathrm{d}y.$$

By Minkowski's inequality,

$$\begin{aligned}\left\|\int_0^t \mathrm{d}s \int_0^1 \mathrm{d}y\ p_{t-s}(x\,,y)b(u(s\,,y))\right\|_{L^2(\Omega)} &\leqslant \int_0^t \mathrm{d}s \int_0^1 \mathrm{d}y\ p_{t-s}(x\,,y)\,\|b(u(s\,,y))\|_{L^2(\Omega)}\\ &\leqslant A\int_0^t \mathrm{d}s\int_0^1 \mathrm{d}y\ p_{t-s}(x\,,y),\end{aligned}$$

Since $\int_0^1 p_v(x\,,y)\,\mathrm{d}y = \mathrm{P}_x\{\tau > v\} \leqslant 1$, it follows that

$$\left\|\int_0^t \mathrm{d}s \int_0^1 \mathrm{d}y\ p_{t-s}(x\,,y)b(u(s\,,y))\right\|_{L^2(\Omega)} = O(t) = o(t^{1/4}) \quad \text{as } t\downarrow 0.$$

This estimate shows that the integral that contains b does not contribute to the limit theorem. As a result, it remains to assume without loss of generality that $b\equiv 0$.

According to (5.1), we also have $|(P_tu_0)(x)| = o(t^{1/4})$ as $t\downarrow 0$. Therefore, it remains to prove that

$$\sup_{x\in[0,1]} \mathrm{E}\left(|I(t\,,x) - (P_tu_0)(x) - I_1(t\,,x)|^2\right) = O(t^{\alpha+(1/2)}) \qquad \text{as } t\downarrow 0, \tag{5.9}$$

where

$$I_1(t\,,x) := \sigma(u_0(x))Z(t\,,x) = \int_{[0,t]\times[0,1]} p_{t-s}(x\,,y)\sigma(u_0(x))\dot{W}(s\,,y)\,\mathrm{d}s\,\mathrm{d}y.$$

Define

$$I_2(t\,,x) := \int_{[0,t]\times[0,1]} p_{t-s}(x\,,y)\sigma(u_0(y))\dot{W}(s\,,y)\,\mathrm{d}s\,\mathrm{d}y.$$

Since we are assuming now that $b\equiv 0$,

$$\begin{aligned}\mathrm{E}\left(|I(t\,,x) - (P_tu_0)(x) - I_2(t\,,x)|^2\right) &= \int_0^t \mathrm{d}s\int_0^1 \mathrm{d}y\ |p_{t-s}(x\,,y)|^2\mathrm{E}\left(|\sigma(u(s\,,y)) - \sigma(u_0(y))|^2\right)\\ &\leqslant L\int_0^t \mathrm{d}s\int_0^1 \mathrm{d}y\ |p_{t-s}(x\,,y)|^2\mathrm{E}\left(|u(s\,,y)-u(0\,,y)|^2\right),\end{aligned}$$

where L denotes the Lipschitz constant of σ. According to (4.11), $\mathrm{E}(|u(s\,,y) - u(0\,,y)|^2) \leqslant Cs^\alpha$ where C is independent of s. We bound $p_{t-s}(x\,,y)$ from above by the free–space heat kernel $\Gamma(t-s\,,y-x)$ – see (4.6) – in order to see that

$$\begin{aligned}\mathrm{E}\left(|I(t\,,x) - (P_tu_0)(x) - I_2(t\,,x)|^2\right) &\leqslant CL\int_0^t s^\alpha\,\mathrm{d}s\int_{-\infty}^{\infty}\mathrm{d}y\ |\Gamma(t-s\,,x-y)|^2\\ &= C'\int_0^t \frac{s^\alpha}{\sqrt{t-s}}\,\mathrm{d}s = C''t^{\alpha+(1/2)}.\end{aligned}$$

Next we may observe that

$$\begin{aligned}\mathrm{E}\left(|I_1(t\,,x) - (P_tu_0)(x) - I_2(t\,,x)|^2\right) &= \int_0^t \mathrm{d}s\int_0^1 \mathrm{d}y\ |p_{t-s}(x\,,y)|^2\,|\sigma(u_0(x)) - \sigma(u_0(y))|^2\\ &\leqslant L\int_0^t \mathrm{d}s\int_0^1 \mathrm{d}y\ |p_{t-s}(x\,,y)|^2\,|u_0(x)-u_0(y)|^2\\ &\leqslant L'\int_0^t \mathrm{d}s\int_0^1 \mathrm{d}y\ |p_{t-s}(x\,,y)|^2|x-y|^{2\alpha},\end{aligned}$$

where L'/L denotes the Hölder exponent of u_0. Once again we estimate the heat kernel p by the free–space heat kernel Γ in order to obtain

$$\mathrm{E}\left(|I_1(t\,,x)-(P_tu_0)(x)-I_2(t\,,x)|^2\right)\leqslant L'\int_0^t \mathrm{d}s\int_{-\infty}^{\infty}\mathrm{d}y\,|\Gamma(t-s\,,y-x)|^2|x-y|^{2\alpha}=L''t^{\alpha+(1/2)},$$

uniformly for all $x\in[0\,,1]$ and $t>0$. The preceding two bounds together imply (5.9). □

5.4. **Comments on large-time asymptotics in the nonlinear setting.** One might wonder if the non-dissipation theorem, Proposition 5.1, has a non-linear generalization. This turns out to be a quite delicate matter, and will not be treated here. For answers in a physically-relevant special case keep an eye out for a preprint by Kim, Khoshnevisan, Mueller, and Shiu some time in the near future.

6. **Random initial data**

Let us return to the general SPDE (4.1) with general Lipschitz-continuous, non-random coefficient functions b and σ. Among other things, we have seen that if u is non random, $u(0)=u(1)=0$, and u is Hölder continuous, then (4.1) has a unique Hölder continuous solution $u=u(t\,,x)$. A quick inspection of our proof of Theorem 4.1 shows that in fact we could have $u(0)$ to be random, as long as it is independent of the white noise $\dot W$. In that case, essentially the same arguments as those used to prove Theorems 4.1, 4.2, and 4.3 can be used to establish the following generalization of those stated theorems.

Theorem 6.1. *Suppose the initial profile $u(0)=\{u(0\,,x)\}_{x\in[0,1]}$ is a random process and is independent of $\dot W$. Then:*

(1) *If $\sup_{x\in[0,1]}\mathrm{E}\left(|u(0\,,x)|^2\right)<\infty$, then (4.1) has a solution that is unique subject to $\sup_{x\in[0,1]}\sup_{t\in[0,T]}\mathrm{E}(|u(t\,,x)|^2)<\infty$ for all $T>0$.*
(2) *If, in addition, $\sup_{x\in[0,1]}\mathrm{E}(|u(0\,,x)|^k)<\infty$ for all $k\geqslant 2$, then*
$$\sup_{t\in[0,T]}\sup_{x\in[0,1]}\mathrm{E}\left(|u(t\,,x)|^k\right)<\infty\qquad\forall T>0,\ k\geqslant 2.$$
(3) *Suppose, yet in addition, that $u(0\,,0)=u(0\,,1)=0$ a.s. and that there exists $\alpha\in(0\,,1]$ such that for all $k\geqslant 2$ there exists $C_k>0$ such that*
$$\mathrm{E}\left(|u(0\,,x)-u(0\,,y)|^k\right)\leqslant C_k|x-y|^{k\alpha}\qquad\forall x,y\in[0\,,1].$$
Then, u has a Hölder-continuous modification.

Let us conclude this subsection with a neat application of Theorem 6.1.

Let B_0 denote a Brownian bridge, independent of white noise $\dot W$, and define

$$u_0(x):=\frac{\sigma}{\sqrt2}\,B_0(x)\qquad\forall x\in[0\,,1].\tag{6.1}$$

Thanks to the mild formulation of the solution – see (4.2) – the random process

$$u(t\,,x):=(P_tu_0)(x)+Z(t\,,x)\qquad[t>0,\ 0\leqslant x\leqslant 1]$$

is the unique solution to the SPDE,

$$\left[\begin{array}{ll}\dot u=u''+\sigma\dot W & \text{on }(0\,,\infty)\times[0\,,1],\text{ subject to}\\ u(0)=u_0 & \text{on }[0\,,1].\end{array}\right.\tag{6.2}$$

In other words, u solves (5.2), but starting from its "invariant measure" u_0. The following shows that $\{u(t)\}_{t\geqslant 0}$ is a stationary process. This should remind you of facts that you might know about the ergodic theory of [typically finite-dimensional] Markov processes.

Proposition 6.2. *Let u denote the solution to* (6.2)*, started from an independent Brownian bridge initial profile* (6.1)*. Then, the law of $u(t)$ is the same as the law of $u(0)$ for all $t \geqslant 0$.*

Proof. The process u is a centered, continuous Gaussian process. Moreover,

$$\mathrm{Cov}[u(t\,,x)\,,u(t\,,y)] = \mathrm{Cov}\left[(P_tu_0)(x)\,,(P_tu_0)(y)\right] + \mathrm{Cov}[Z(t\,,x)\,,Z(t\,,y)].$$

According to (5.5),

$$\mathrm{Cov}[Z(t\,,x)\,,Z(t\,,y)] = \frac{\sigma^2}{2\pi^2}\sum_{n=1}^\infty \psi_n(x)\psi_n(y)\cdot\frac{1-\mathrm{e}^{-2tn^2\pi^2}}{n^2}. \tag{6.3}$$

We compute the former quantity next:

$$\begin{aligned}\mathrm{Cov}\left[(P_tu_0)(x)\,,(P_tu_0)(y)\right] &= \frac{\sigma^2}{2}\int_0^1 p_t(x\,,w)\,\mathrm{d}w\int_0^1 p_t(y\,,z)\,\mathrm{d}z\ \mathrm{Cov}[B_0(w)\,,B_0(z)]\\ &= \frac{\sigma^2}{2}\int_0^1 p_t(x\,,w)\,\mathrm{d}w\int_0^1 p_t(y\,,z)\,\mathrm{d}z\ \sum_{n=1}^\infty\frac{\psi_n(w)\psi_n(z)}{n^2\pi^2};\end{aligned}$$

see (5.7). Plug in the eigenfunction expansion (1.3) of the heat kernel to see that the preceding is equal to

$$\begin{aligned}&\frac{\sigma^2}{2}\int_0^1\sum_{j=1}^\infty\psi_j(x)\psi_j(w)\mathrm{e}^{-j^2\pi^2t}\,\mathrm{d}w\int_0^1\sum_{k=1}^\infty\psi_k(y)\psi_k(z)\mathrm{e}^{-k^2\pi^2t}\,\mathrm{d}z\ \sum_{n=1}^\infty\frac{\psi_n(w)\psi_n(z)}{n^2\pi^2}\\ &=\frac{\sigma^2}{2\pi^2}\sum_{n=1}^\infty\frac{\psi_n(x)\psi_n(y)}{n^2}\mathrm{e}^{-2n^2\pi^2t}\qquad\text{(Fubini's theorem \& the orthonormality of the ψ_i's).}\end{aligned}$$

This and (6.3), in turn, together imply that

$$\mathrm{Cov}[u(t\,,x)\,,u(t\,,y)] = \frac{\sigma^2}{2\pi^2}\sum_{n=1}^\infty\frac{\psi_n(x)\psi_n(y)}{n^2} = \mathrm{Cov}[u_0(x)\,,u_0(y)]\qquad\forall t\geqslant 0;$$

see (5.7). □

As in the preceding let $u_0 := (\sigma/\sqrt{2})B_0$ where B_0 is a Brownian bridge that is independent of the white noise $\dot{W}$. We saw earlier that the process $u(t) := P_tu_0 + Z(t)$ has the same law as the process $x \mapsto (\sigma/\sqrt{2})B_0(x)$. Now,

$$(P_tu_0)(x) = \sum_{n=1}^\infty \psi_n(x)(u_0\,,\psi_n)\mathrm{e}^{-n^2\pi^2t}$$

is clearly a C^∞ function of x for every $t > 0$. Since $Z(t) = u(t) - (P_tu_0)$, it follows that at every fixed time $t > 0$, $Z(t)$ is a smooth perturbation of a scaled Brownian bridge $u(t) \overset{(d)}{=} (\sigma/\sqrt{2})B_0$.

7. The strong Markov property

For every $t \geqslant 0$, let $\mathcal{F}_t^0$ denote the sigma-algebra generated by all Wiener integrals of the form $\int_{[0,t]\times[0,1]} \varphi(x)\dot{W}(s\,,x)\,\mathrm{d}s\,\mathrm{d}x$, as φ ranges over all non-random functions in $L^2[0\,,1]$. Let $\mathcal{F}_t^1$ denote the P-completion of $\mathcal{F}_t^0$, and finally define $\mathcal{F}_t$ to be the right-continuous version of $\mathcal{F}_t^1$; that is,

$$\mathcal{F}_t := \bigcap_{s>t} \mathcal{F}_s^1 \qquad \forall t \geqslant 0.$$

Evidently, $\mathcal{F} := \{\mathcal{F}_t\}_{t\geqslant 0}$ is a filtration of sigma-algebras.

Let

$$X_t(\varphi) := \int_{[0,t]\times[0,1]} \varphi(x)\dot{W}(s\,,x)\,\mathrm{d}s\,\mathrm{d}x \qquad \forall t \geqslant 0,\ \varphi \in L^2[0\,,1].$$

Then, $X(\varphi)$ is a mean-zero Gaussian process that is indexed by $\mathbb{R}_+$ and whose covariance function is described as follows:

$$\mathrm{Cov}[X_s(\varphi)\,, X_t(\varphi)] = (s \wedge t)\|\varphi\|^2_{L[0,1]}.$$

In other words, $X(\varphi)$ is a Brownian motion, normalized so that its variance at time one is $\|\varphi\|^2_{L^2[0,1]}$. Thus, we can view $X := \{X_t\}_{t\geqslant 0}$ as an infinite-dimensional Brownian motion, taking values in the space of all linear functionals on $L^2[0\,,1]$.[2]

The filtration $\mathcal{F}$ is simply the right-continuous, augmented filtration that is associated to the infinite-dimensional Brownian motion X. This observation motivates the following.

Definition 7.1. We refer to $\mathcal{F}$ as the "Brownian filtration."

Let u continue to denote the solution to our SPDE (4.1). The construction of the Walsh stochastic integral ensures that the infinite-dimensional stochastic process $\{u(t)\}_{t\geqslant 0}$ is adapted with respect to the filtration $\mathcal{F}$.

Theorem 7.2 (The strong Markov property)**.** *Suppose $u(0)$ is independent of $\dot{W}$ and satisfies $\sup_{x\in[0,1]} \mathrm{E}(|u(0\,,x)|^k) < \infty$ for all $k \geqslant 2$. Suppose, in addition, that $u(0\,,0) = u(0\,,1) = 0$, and that there exists $\alpha \in (0\,,1]$ such that for every $k \geqslant 2$ there exists $C_k > 0$ such that*

$$\sup_{x\in[0,1]} \mathrm{E}\left(|u(0\,,x) - u(0\,,y)|^k\right) \leqslant C_k|x-y|^{k\alpha} \qquad \forall x, y \in [0\,,1].$$

Then, $\{u(t)\}_{t\geqslant 0}$ is a strong Markov process with respect to the filtration $\mathcal{F}$.

The reader should be warned that while the infinite-dimensional process $t \mapsto u(t)$ is Markovian, the finite-dimensional process $t \mapsto u(t\,,x)$ – where $x \in (0\,,1)$ is fixed – is generally non Markovian.

Sketch of the proof of Theorem 7.2. Choose and fix a time variable $\rho \geqslant 0$, and define

$$v(t\,,x) := u(t+\rho\,,x) \qquad \forall t \geqslant 0,\ x \in [0\,,1].$$

It is possible to adapt a fixed-point argument – see the proof of Theorem 4.1 – in order to prove that v solves (4.1) starting from random initial data $v(0) = u(\rho)$, and with the noise $\dot{W}$ replaced by a white noise ξ that is independent of $u(\rho)$. Formally speaking, the white noise ξ is obtained by setting

$$\xi(t\,,x) := \dot{W}(t+\rho\,,x) \qquad \forall t \geqslant 0, x \in [0\,,1].$$

[2]The infinite-dimensional process X is sometimes called *cylindrical Brownian motion.*

More precisely, we define ξ by computing every Wiener integral of the form $\int_{\mathbb{R}_+\times[0,1]}\varphi\,\mathrm{d}\xi$ as

$$\int_{[\rho,\infty)\times[0,1]}\varphi(t-\rho\,,x)\dot{W}(t\,,x)\,\mathrm{d}t\,\mathrm{d}x \qquad \forall\varphi\in L^2(\mathbb{R}_+\times[0\,,1]).$$

Elementary properties of Wiener integrals show that ξ is independent of $\mathcal{F}_\rho$.

Let P_w denote the law of the random field v starting from any random initial profile w that is independent of ξ and satisfies $\sup_{x\in[0,1]}\mathrm{E}(|w(x)|^2)<\infty$, and let E_w denote the associated expectation operator. Since $\{u(t)\}_{t\geqslant 0}$ is adapted to $\mathcal{F}$, ξ is independent of $\mathcal{F}_\rho$ and hence $u(\rho)$ is independent of ξ. Therefore, the uniqueness of the solution to (4.1) – see Theorem 4.1 – implies that

$$\mathrm{E}\left(F(u)\mid\mathcal{F}_\rho\right)=\mathrm{E}_{u(\rho)}[F(v)] \qquad \text{a.s. for all } \rho\geqslant 0,$$

for every measurable function $F: C((0\,,\infty)\times[0\,,1])\to\mathbb{R}$. This proves that the left-hand side is measurable with respect to the sigma-algebra generated by $u(\rho)$, which is another way to say that $\{u(t)\}_{t\geqslant 0}$ is a [weak] Markov process. The hypotheses of the theorem on $u(0)$ ensure that the solution u is continuous; see Theorem 6.1. Therefore, more-or-less standard arguments from Markov process theory show that the process $\{u(t)\}_{t\geqslant 0}$ is strong Markov. □

8. **A comparison principle**

Let D be a compact set in $\mathbb{R}^d$ that has a smooth boundary, and consider any function $h:\mathbb{R}_+\times D\to\mathbb{R}$ that is sufficiently smooth and satisfies the heat equation,

$$\dot{h}=\Delta h \qquad \text{on } D_T:=(0\,,T]\times D^\circ. \tag{8.1}$$

Then, according to the "strong maximum principle," h achieves its maximum on the "parabolic boundary" $\overline{D_T}\setminus D_T$ of D_T. More precisely put,

$$\sup_{(t,x)\in\overline{D_T}}h(t\,,x)=\sup_{\overline{D_T}\setminus D_T}h(t\,,x);$$

consult Evans [18] for instance.

We can consider $-h$ instead of h to see that the minimum of h is also achieved on $\overline{D_T}\setminus D_T$.

Now consider the problem (8.1) subject to the initial value, $h(0)=h_0$ on D°, where h_0 is a smooth and non-negative function on D°. Since h_0 has a minimum value $\geqslant 0$, and because $h-h_0$ also solves (8.1), we may apply the strong maximum principle to $h-h_0$ in order to see that $h(t\,,x)\geqslant\inf h_0\geqslant 0$ for all $(t\,,x)\in D_T$. This latter property is the "positivity principle" of the heat equation.

Because (8.1) is a linear PDE, we immediately obtain the following.

Proposition 8.1 (Comparison principle for PDEs)**.** *Suppose h and H respectively solve* (8.1) *on D_T with respective continuous initial profiles h_0 and H_0, and suppose $h_0(x)\leqslant H_0(x)$ for all $x\in D^\circ$. Then, $h(t\,,x)\leqslant H(t\,,x)$ for all $(t\,,x)\in D_T$.*

Proof. Because $H-h$ solves (8.1) and $H_0-h_0\geqslant 0$, the positivity principle of the heat equation completes the proof. □

Unfortunately, the noisy version (4.1) of (8.1) fails to satisfy the strong maximum principle. Nevertheless, it does satisfy a comparison principle.

Theorem 8.2 (Comparison principle for SPDEs)**.** *Suppose $u_0, U_0 : [0\,,1] \to \mathbb{R}$ are non random and bounded functions, and let u and U denote the respective solutions to the stochastic heat equation* (4.1) *with respective initial values u_0 and U_0. If, in addition, $u_0 \leqslant U_0$ pointwise on* $[0\,,1]$*, then* $\mathrm{P}\{u \leqslant U \text{ a.e.}\} = 1$.

Remark 8.3. (1) If u_0 and U_0 are Hölder continuous in $[0\,,1]$, and vanishing on the boundary, then u and U are a.s. continuous; see Theorem 4.3. Therefore, under those conditions, we have

$$u_0(x) \leqslant U_0(x) \quad \forall x \in [0\,,1] \qquad \Rightarrow \qquad \mathrm{P}\left\{u(t\,,x) \leqslant U(t\,,x) \quad \forall t > 0,\, x \in [0\,,1]\right\} = 1.$$

(2) Theorem 8.2 continues to hold if u_0 and U_0 are random, independent of $\dot{W}$, and satisfy $\sup_{x\in[0,1]} \mathrm{E}(|u_0(x)|^2) < \infty$ and $\sup_{x\in[0,1]} \mathrm{E}(|U_0(x)|^2) < \infty$; see Theorem 6.1.

Whereas Proposition 8.1 is an elementary fact, Theorem 8.2 is not. In order to avoid the several measure-theoretic details of the proof we merely sketch the argument. For the details of this argument (in different forms), see Donati–Martin and Pardoux [17], Geiß and Manthey [22], Kotelenez [26], Mueller [28], Shiga [29], ...and most particularly the recent article of Chen and Kim [9], where the details of the argument can be found.

8.1. **An approximate SPDE.** Define

$$W_\varepsilon(t\,,x) := \int_{[0,t]\times[0,1]} p_\varepsilon(x\,,z)\dot{W}(r\,,z)\,\mathrm{d}r\,\mathrm{d}z.$$

Then, W_ε is a mean-zero Gaussian process with covariance function

$$\begin{aligned}\mathrm{Cov}[W_\varepsilon(t\,,x)\,,W_\varepsilon(s\,,y)] &= (s\wedge t)\int_0^1 \mathrm{d}z\; p_\varepsilon(x\,,z)p_\varepsilon(y\,,z)\\ &= (s\wedge t)\int_0^1 \mathrm{d}z\; p_\varepsilon(x\,,z)p_\varepsilon(z\,,y)\\ &= (s\wedge t)p_{2\varepsilon}(x\,,y) \qquad \text{[the semigroup property]}.\end{aligned} \tag{8.2}$$

In other words, if we hold fixed $x \in [0\,,1]$ and $\varepsilon > 0$, then $t \mapsto W_\varepsilon(t\,,x)$ is a Brownian motion, normalized to have variance $p_{2\varepsilon}(x\,,x)$ at time one. Since $p_{2\varepsilon}(x\,,y)$ behaves as $\delta_0(x-y)$ for $\varepsilon \approx 0$, one expects the Itô differential $W_\varepsilon(\mathrm{d}t\,,x) := \dot{W}_\varepsilon(t\,,x)\,\mathrm{d}t$ to behave as $\dot{W}(t\,,x)\,\mathrm{d}t$, and hence $\dot{W}_\varepsilon(t\,,x) \approx \dot{W}(t\,,x)$ when $\varepsilon \approx 0$. And in any event, it is easy to see that for every predictable random field Φ, the stochastic integral process of Φ against $\dot{W}_\varepsilon$ is canonically defined as

$$\int_{[0,t]\times[0,1]} \Phi(s\,,y)\dot{W}_\varepsilon(s\,,y)\,\mathrm{d}s\,\mathrm{d}y := \int_0^1 \left(\int_0^t \Phi(s\,,y)\,\dot{W}_\varepsilon(s\,,y)\,\mathrm{d}s\right)\mathrm{d}y,$$

provided that the inside integral – which is an Itô integral is defined, and Lebesgue integrable as a function of y.

Now let us consider the SPDE (4.1) in the special case that $b \equiv 0$ in order to simplify the exposition, as before. That is,

$$\left[\begin{aligned}&\dot{u} = u'' + \sigma(u)\dot{W} \qquad \text{on } (0\,,\infty)\times[0\,,1], \text{ subject to}\\ &u(0) = u_0 \qquad \text{on } [0\,,1], \text{ and}\\ &u(t\,,0) = u(t\,,1) = 0 \qquad \forall t > 0.\end{aligned}\right. \tag{8.3}$$

We have seen that if u_0 is bounded and (say) non random, then u exists and is unique (subject to a natural moment condition). And that if u_0 vanishes on the boundary of $[0\,,1]$ and is Hölder continuous, then u is Hölder continuous as well.

Next let us consider the SPDE,

$$\begin{bmatrix} \dot u_\varepsilon = L_\varepsilon u_\varepsilon + \sigma(u_\varepsilon)\dot W_\varepsilon \qquad \text{on } (0\,,\infty)\times[0\,,1]\text{, subject to} \\ u_\varepsilon(0)=u_0 \qquad \text{on } [0\,,1]\text{, and} \\ u_\varepsilon(t\,,0)=u_\varepsilon(t\,,1)=0 \qquad \forall t>0, \end{bmatrix} \tag{8.4}$$

where L_ε is the following linear operator, and can be viewed as a "regularized second derivative operator":

$$(L_\varepsilon f)(x) := \frac{(P_\varepsilon f)(x)-f(x)}{\varepsilon} \qquad \forall f\in L^\infty[0\,,1],\ \varepsilon>0.$$

The following justifies the claim that L_ε is a regularized second derivative.

Proposition 8.4. *If $f\in C^4[0\,,1]$, then*

$$\int_0^1 \left|(L_\varepsilon f)(x)-f''(x)\right|^2\mathrm{d}x \leqslant 9\varepsilon^2\int_0^1 |f^{(iv)}(x)|^2\,\mathrm{d}x \qquad \forall\varepsilon>0.$$

Proof. Recall that $(f''\,,\psi_n)=(f\,,\psi_n'')=-n^2\pi^2(f\,,\psi_n)$. Also, $p_t(x\,,y)=p_t(y\,,x)$, and hence P_ε – hence also L_ε – is self-adjoint on $L^2[0\,,1]$. Because of (1.3), $(P_\varepsilon\psi_n)(x)=\exp\{-n^2\pi^2\varepsilon\}\psi_n(x)$. We combine these observations to find that

$$(L_\varepsilon f-f''\,,\psi_n)=(f\,,L_\varepsilon\psi_n)+n^2\pi^2(f\,,\psi_n)=\left(\frac{\mathrm{e}^{-n^2\pi^2\varepsilon}-1+n^2\pi^2\varepsilon}{\varepsilon}\right)(f\,,\psi_n).$$

Apply the Parseval identity to find that

$$\|L_\varepsilon f-f''\|^2_{L^2[0,1]}=\sum_{n=1}^\infty\left(\frac{\mathrm{e}^{-n^2\pi^2\varepsilon}-1+n^2\pi^2\varepsilon}{\varepsilon}\right)^2|(f\,,\psi_n)|^2\leqslant 9\pi^8\varepsilon^2\sum_{n=1}^\infty n^8|(f\,,\psi_n)|^2,$$

since $|\mathrm{e}^{-z}-1-z|\leqslant 3z^2$ for all $z\geqslant 0$. Because $(f^{(iv)}\,,\psi_n)=(f\,,\psi_n^{(iv)})=n^4\pi^4(f\,,\psi_n)$ for every positive integer n, it follows from the Parseval identity that $\pi^8\sum_{n=1}^\infty n^8|(f\,,\psi_n)|^2=\|f^{(iv)}\|^2_{L^2[0,1]}$. Together, these facts imply the proposition. ☐

8.2. **An approximate comparison principle.** Now let U_ε be the solution to (8.4) but with initial profile U_0 instead of u_0. We now prove that

$$U_\varepsilon(t\,,x)\geqslant u_\varepsilon(t\,,x) \qquad \text{a.s. for all } t>0 \text{ and } x\in[0\,,1]. \tag{8.5}$$

In order to prove (8.5) let us integrate (8.4) with respect to time in order to see that

$$u_\varepsilon(t\,,x)=u_0(x)+\int_0^t\left(L_\varepsilon[u_\varepsilon(s)]\right)(x)\,\mathrm{d}s+\int_0^t\sigma(u_\varepsilon(s\,,x))\,W_\varepsilon(\mathrm{d}t\,,x),$$

and there is an analogous expression for U_ε. Therefore,

$$D_\varepsilon(t\,,x):=U_\varepsilon(t\,,x)-u_\varepsilon(t\,,x)$$

satisfies

$$D_\varepsilon(t\,,x)=U_0(x)-u_0(x)+\int_0^t\left(L_\varepsilon[D_\varepsilon(s)]\right)(x)\,\mathrm{d}s+\int_0^t\left\{\sigma(U_\varepsilon(s\,,x))-\sigma(u_\varepsilon(s\,,x))\right\}W_\varepsilon(\mathrm{d}s\,,x).$$

By Itô's formula, if $F:\mathbb{R}\to\mathbb{R}$ is smooth and bounded, and vanishes on $(0\,,\infty)$, then

$$\begin{aligned}
\mathrm{E}\left[F\left(D_\varepsilon(t\,,x)\right)\right] &= \mathrm{E}\int_0^t F'(D_\varepsilon(s\,,x))\left(L_\varepsilon[D_\varepsilon(s)]\right)(x)\,\mathrm{d}s\\
&\qquad\qquad + \tfrac12 p_{2\varepsilon}(x\,,x)\mathrm{E}\int_0^t F''(D_\varepsilon(s\,,x))\left\{\sigma(U_\varepsilon(s\,,x))-\sigma(u_\varepsilon(s\,,x))\right\}^2\,\mathrm{d}s\\
&\leqslant \mathrm{E}\int_0^t F'(D_\varepsilon(s\,,x))\left(L_\varepsilon[D_\varepsilon(s)]\right)(x)\,\mathrm{d}s + \tfrac12 Lp_{2\varepsilon}(x\,,x)\mathrm{E}\int_0^t F''(D_\varepsilon(s\,,x))\left\{D_\varepsilon(s\,,x)\right\}^2\,\mathrm{d}s,
\end{aligned}$$

provided additionally that F is convex. [The coefficient $p_{2\varepsilon}(x\,,x)$ appears due to the quadratic variation of $W_\varepsilon(\bullet\,,x)$; see (8.2).] By the definition of L_ε,

$$\begin{aligned}
\mathrm{E}\left[F\left(D_\varepsilon(t\,,x)\right)\right] \leqslant \frac{1}{\varepsilon}\mathrm{E}\left[\int_0^t F'(D_\varepsilon(s\,,x))\left(P_\varepsilon[D_\varepsilon(s)]\right)(x)\,\mathrm{d}s\right] - \frac{1}{\varepsilon}\mathrm{E}\left[\int_0^t F'(D_\varepsilon(s\,,x))D_\varepsilon(s\,,x)\,\mathrm{d}s\right]\\
+ \frac{Lp_{2\varepsilon}(x\,,x)}{2\varepsilon}\mathrm{E}\left[\int_0^t F''(D_\varepsilon(s\,,x))\left\{D_\varepsilon(s\,,x)\right\}^2\,\mathrm{d}s\right].
\end{aligned}$$

By a standard approximation procedure, we may apply the preceding to the function $F(a):=a^-$; in that case, $F'=-\mathbf{1}_{(-\infty,0)}$ and $F''=\delta_0$ as a distribution. The second expectation on the right-hand side of the preceding display is non positive, and the third expectation vanishes. Thus,

$$\begin{aligned}
\mathrm{E}\left[(D_\varepsilon(t\,,x))^-\right] &\leqslant -\frac{1}{\varepsilon}\mathrm{E}\left[\int_0^t \mathbf{1}_{\{D_\varepsilon(s\,,x)<0\}}\left(P_\varepsilon[D_\varepsilon(s)]\right)(x)\,\mathrm{d}s\right]\\
&= -\frac{1}{\varepsilon}\mathrm{E}\left[\int_0^t \mathbf{1}_{\{D_\varepsilon(s\,,x)<0\}}\left(\int_0^1 p_\varepsilon(x\,,y)D_\varepsilon(s\,,y)\,\mathrm{d}y\right)\mathrm{d}s\right]\\
&\leqslant \frac{1}{\varepsilon}\int_0^t \mathrm{d}s\int_0^1 \mathrm{d}y\; p_\varepsilon(x\,,y)\mathrm{E}\left[(D_\varepsilon(s\,,y))^-\right].
\end{aligned}$$

Because $\int_0^1 p_\varepsilon(x\,,y)\,\mathrm{d}y\leqslant 1$, it follows that the function

$$Q(t):=\sup_{x\in[0,1]}\mathrm{E}\left[(D_\varepsilon(t\,,x))^-\right]\qquad [t>0]$$

satisfies $Q(t)\leqslant \varepsilon^{-1}\int_0^t Q(s)\,\mathrm{d}s$ for all $t\geqslant 0$, which implies that $Q\equiv 0$, owing to Gronwall's inequality. It follows immediately from this that $(D_\varepsilon(t\,,x))^-=0$ a.s. for all $t\geqslant 0$ and $x\in[0\,,1]$. This fact immediately implies (8.5).

8.3. **Putting it all together.** We are in position to briefly sketch the proof of the comparison theorem, Theorem 8.2.

First, recall that the mild solution u – see (4.2) – is also a weak solution; see (4.3). In a similar manner, using a stochastic Fubini argument, one can show that whenever $\phi:\mathbb{R}_+\times[0\,,1]\to\mathbb{R}$ is smooth and $\phi(t\,,0)=\phi(t\,,1)=0$ for all $T>t>0$,

$$\begin{aligned}
&\int_0^1 \phi(0\,,x)u_0(x)\,\mathrm{d}x - \int_{[0,T]\times[0,1]}\dot\phi(t\,,x)u_\varepsilon(t\,,x)\,\mathrm{d}t\,\mathrm{d}x\\
&= \int_{[0,T]\times[0,1]}(L_\varepsilon[\phi(t)])(x)u_\varepsilon(t\,,x)\,\mathrm{d}t\,\mathrm{d}x + \int_{[0,T]\times[0,1]}\phi(t\,,x)\sigma(u(t\,,x))\,\dot W_\varepsilon(t\,,x)\,\mathrm{d}t\,\mathrm{d}x.
\end{aligned}\tag{8.6}$$

It follows easily from Proposition 8.4 that $L_\varepsilon[\phi(t)] \to \phi''(t)$ as $\varepsilon \downarrow 0$, uniformly for all $t \in [0\,,T]$. We have argued, intuitively, that $\dot{W}_\varepsilon \approx \dot{W}$ when $\varepsilon \approx 0$. This can be proved to hold in the sense that $\int_{[0,T]\times[0,1]} \Phi(t\,,x)\dot{W}_\varepsilon(t\,,x)\,\mathrm{d}t\,\mathrm{d}x \to \int_{[0,T]\times[0,1]} \Phi(t\,,x)\dot{W}(t\,,x)\,\mathrm{d}t\,\mathrm{d}x$ in $L^2(\Omega)$ as $\varepsilon \downarrow 0$ for all predictable random fields Φ. Once this is accomplished, one can appeal to a fixed-point argument to show that $\int_{[0,T]\times[0,1]} \dot{\phi}(t\,,x)u_\varepsilon(t\,,x)\,\mathrm{d}t\,\mathrm{d}x \to \int_{[0,T]\times[0,1]} \dot{\phi}(t\,,x)u(t\,,x)\,\mathrm{d}t\,\mathrm{d}x$ in $L^2(\Omega)$ as $\varepsilon \downarrow 0$. The same fact holds if u_ε and u are respectively replaced by U_ε and U. I will skip the many technical details.

Since $\phi \in C^\infty(\mathbb{R}_+ \times [0\,,1])$ and $T > 0$ are arbitrary and $U_\varepsilon(t\,,x) \geqslant u_\varepsilon(t\,,x)$ the comparison result (Theorem 8.2) follows. □

It is possible to enhance Theorem 8.2 in different directions. Here is one such refinement. Its proof requires making small modifications to the derivation of Theorem 8.2, and is omitted. See Donati–Martin and Pardoux [17] and Geiß and Manthey [22] for more details.

Theorem 8.5. *Let u be the solution to* (4.1)*, and let U solve* (4.1) *in the case that* $(b\,,u_0)$ *are replaced with* $(B\,,U_0)$ *where $B : \mathbb{R} \to \mathbb{R}$ is non-random and Lipschitz continuous, and $U_0 : [0\,,1] \to \mathbb{R}$ is non-random and continuous. If, in addition, $b \leqslant B$ pointwise, and $u_0 \leqslant U_0$ pointwise, then* $\mathrm{P}\{u \leqslant U \text{ a.e.}\} = 1$.

9. Mueller's positivity principle

9.1. **Nonnegative solutions.** As was implied earlier, parabolic SPDEs possess positivity principles, though they do not satisfy a maximum principle. The following ready corollary of Theorem 8.5 is one such possibility for a positivity principle.

Corollary 9.1. *Let u be the solution to* (4.1)*, and suppose in addition that* $\sigma(0) = 0$ *and* $b \geqslant 0$ *pointwise. Then,* $\mathrm{P}\{u \geqslant 0 \text{ a.e.}\} = 1$.

Proof. First consider the case that $b = u_0 = 0$. Let $u_0(t\,,x) := 0$ and

$$u_{n+1}(t\,,x) = \int_{[0,t]\times[0,1]} p_{t-s}(x\,,y)\sigma(u_n(s\,,y))\dot{W}(s\,,y)\,\mathrm{d}s\,\mathrm{d}y.$$

Since $u_0 = 0$ and $\sigma(0) = 0$, it follows that $u_1 = 0$. Apply induction to see that $u_n \equiv 0$ for all $n \geqslant 0$. Since $u_n(t\,,x) \to u(t\,,x)$ in $L^2(\Omega)$ as $n \to \infty$ [see the proof of Theorem 4.1], it follows that $u(t\,,x) = 0$ when $u_0 = b = 0$.

In general, one can use Theorem 8.5 in order to compare the solution of (4.1) to the solution in the case that $b = u_0 = 0$. □

9.2. **Positive solutions.** As it turns out there is a much deeper version of Corollary 9.1 that we state and prove next. This result, and its many variants, form one of the cornerstones of the modern aspects of the theory of stochastic partial differential equations.

Theorem 9.2 (Mueller's positivity principle)**.** *Let u be the solution to* (4.1)*, and suppose in addition that u_0 is Hölder continuous and vanishes on* $\{0\,,1\}$*,* $u_0 > 0$ *on* $(0\,,1)$*, and* $\sigma(0) = 0$ *and* $b \geqslant 0$ *pointwise. Then,*

$$\mathrm{P}\left\{u(t\,,x) > 0 \quad \forall t > 0,\ x \in (0\,,1)\right\} = 1.$$

Proof. The proof generally follows ideas of Mueller [28]. The details of the method were worked out in Conus, Joseph, and Khoshnevisan [13], which is the reference that we follow (and adapt) here.

Choose and fix some $\eta \in (0\,,1/2)$. It suffices to prove that

$$\mathrm{P}\left\{u(t\,,x) > 0 \quad \forall t > 0,\, x \in (\eta\,,1-\eta)\right\} = 1. \tag{9.1}$$

We can apply the comparison theorem (Theorem 8.5), and compare u to the solution to (4.1) in the case that $b \equiv 0$. In this way, we may assume without loss in generality that $b \equiv 0$. Thus, u can be seen as the unique solution to the following:

$$u(t\,,x) = (P_t u_0)(x) + \int_{[0,t]\times[0,1]} p_{t-s}(x\,,y)\sigma(u(s\,,y))\dot{W}(s\,,y)\,\mathrm{d}s\,\mathrm{d}y.$$

There exists a $\delta > 0$ such that

$$u_0(x) \geqslant v_0(x) := \delta \mathbf{1}_{(\eta,1-\eta)}(x) \qquad \forall x \in [0\,,1].$$

Let $v(t\,,x)$ denote the solution to (4.1), but with initial profile v_0 instead of u_0. The comparison theorem (Theorem 8.2) ensures that $u(t\,,x) \geqslant v(t\,,x)$ for all $t > 0$ and $x \in [0\,,1]$. Thus, we may assume without loss in generality that $u_0(x) = \delta \mathbf{1}_{(\eta,1-\eta)}(x)$ for all $x \in [0\,,1]$.

Define $T_0 := 0$, and then for all $n \geqslant 0$ let

$$T_{n+1} := \inf\left\{t > T_n : \inf_{x\in(\eta,1-\eta)} u(t\,,x) \leqslant \mathrm{e}^{-(n+1)}\right\},$$

where $\inf \varnothing := \infty$. Then, $T_1 < T_2 < \ldots$ are stopping times with respect to the Brownian filtration $\mathcal{F}$. Since u is continuous (Theorem 4.3) and nonnegative (Corollary 9.1), the following holds for every $n \geqslant 1$:

$$u(T_n\,,x) \geqslant \mathrm{e}^{-n}\,\delta \mathbf{1}_{(\eta,1-\eta)}(x) \qquad \forall x \in [0\,,1], \quad \text{a.s. on } \{T_n < \infty\}.$$

We may apply the strong Markov property at time T_n (Theorem 7.2), together with the comparison principle, in order to see that $u(t + T_n\,,x) \geqslant V(t\,,x)$ for all $t > 0$ and $x \in [0\,,1]$, a.s. on $\{T_n < \infty\}$, where $V = V^{(n)}$ solves the SPDE (4.1) starting from $V(0\,,x) := \mathrm{e}^{-n}\,\delta \mathbf{1}_{(\eta,1-\eta)}(x)$ with a white noise that is independent of $\mathcal{F}_{T_n}$. In this connection, see also Theorem 6.1. Define

$$\tilde{V}(t\,,x) = \tilde{V}^{(n)}(t\,,x) := \mathrm{e}^{-n} V(t\,,x).$$

Then, $\tilde{V}$ solves (4.1) with initial data $\tilde{V}_0 := \delta \mathbf{1}_{(\eta,1-\eta)}$, where the function σ is now replaced by

$$\tilde{\sigma}(x) := \mathrm{e}^{n}\sigma(\mathrm{e}^{-n}x) \qquad \forall x \in \mathbb{R}.$$

The Lipschitz constant of $\tilde{\sigma}$ is at most that of σ; in particular, it does not depend on n. Thus, one can enhance Theorems 4.2 and 4.3 in order to see that for every $k \geqslant 2$, $\tau \in [0\,,1]$, and $\varrho \in (0\,,1/4)$, there exists real number $C = C(\varrho) > 0$ – independent of $(n\,,k)$ – such that

$$\mathrm{E}\left(\sup_{x\in[-1,1]}\sup_{s\in[0,\tau]}\left|\tilde{V}(s\,,x) - \tilde{V}(0\,,x)\right|^k\right) \leqslant C^k k^{k/2} \tau^{k\varrho} \exp\left\{Ck^3\tau\right\}.$$

Thus, Chebyshev's inequality ensures that for all integers $N \geqslant 1$, $k \geqslant 2$, and $n = 0, \ldots, N$, and every $t > 0$, the following is valid almost surely on $\{T_n \leqslant t\}$:

$$\begin{aligned}\mathrm{P}\left(T_{n+1} - T_n \leqslant 2t/N \mid \mathcal{F}_{T_n}\right) &\leqslant \mathrm{P}\left\{\sup_{x\in[0,1]}\sup_{s\in[0,2t/N]}\left|\tilde{V}(s\,,x) - \tilde{V}(0\,,x)\right| \geqslant 1 - \mathrm{e}^{-1}\right\}\\ &\leqslant \frac{C^k k^{k/2}\,(2t/N)^{k\varrho}\,\mathrm{e}^{2Ck^3t/N}}{(1-\mathrm{e}^{-1})^k}.\end{aligned}$$

Choose and fix an arbitrary $t > 0$. The preceding can be used with $k := A\sqrt{N \log N}$, for a large-enough choice of A – independent of $(n\,, N)$ – to see that there exists a real number $L = L(t\,,\eta) > 0$ – independent of $(n\,, N)$ – such that

$$\mathrm{P}\left(T_{n+1} - T_n \leqslant 2t/N \mid \mathcal{F}_{T_n}\right) \leqslant L\mathrm{e}^{-LN^{1/2}(\log N)^{3/2}} \qquad \forall N \geqslant 100,\ n = 0, \ldots, N.$$

If $T_N < t$, then by the triangle inequality, there exist at least $\lfloor N/2 \rfloor$ distinct values of $n \in \{0\,, \ldots, N-1\}$ such that $T_{n+1} - T_n \leqslant 2t/N$. [This argument is sometimes known as the "pigeonhole principle"; see Alon and Spencer [1].] Thus, we may apply repeatedly the preceding conditional probability estimate, together with a simple union bound, in order to see that for all $N \geqslant 100$,

$$\mathrm{P}\{T_N < t\} \leqslant \binom{N}{\lfloor N/2 \rfloor} \left(L\mathrm{e}^{-LN^{1/2}(\log N)^{3/2}}\right)^{\lfloor N/2 \rfloor}.$$

The above inequality and the Stirling formula – originally due to de Moivre – together imply the existence of a real number $K > 1$ such that

$$\mathrm{P}\left\{\inf_{x\in(\eta,1-\eta)} \inf_{s\in(0,t)} u(s\,,x) \leqslant \mathrm{e}^{-N}\right\} = \mathrm{P}\{T_N < t\} \leqslant K \exp\left(-\frac{(N\log N)^{3/2}}{K}\right),$$

for all $N \geqslant 100$. This verifies (9.1) and completes the proof. $\square$

Let us mention the following by-product of the proof.

Corollary 9.3. *Assume that the hypotheses of Theorem 9.2 are met. Then, for every $t > 0$ and $\eta \in (0\,,1)$ there exist real numbers $K = K(t\,,\eta) > 1$ and $\varepsilon_0 = \varepsilon_0(t\,,\eta)$ such that*

$$\mathrm{P}\left\{\inf_{x\in(\eta,1-\eta)} \inf_{s\in(0,t)} u(s\,,x) \leqslant \varepsilon\right\} \leqslant K \exp\left(-\frac{[\log(1/\varepsilon)\cdot\log\log(1/\varepsilon)]^{3/2}}{K}\right) \qquad \forall \varepsilon \in (0\,,\varepsilon_0).$$

Other, similar, strict-positivity results can be found in Chen and Huang [8], Chen and Kim [9], and Moreno-Flores [27].

10. **Deviation from linear growth**

Consider first the non-random reaction-diffusion equation,

$$\left[\begin{array}{ll} \dot{u} = u'' + b(u) & \text{on } (0\,,\infty)\times[0\,,1], \text{ subject to} \\ u(0) = u_0 & \text{on } [0\,,1], \end{array}\right. \tag{10.1}$$

where we assume, for the sake of simplicity, that u_0 is Hölder continuous and vanishes on $\{0\,,1\}$. There is a huge literature on this sort of equation. Perhaps the best-known result here is that if b is Lipschitz continuous, then (10.1) has a unique, Hölder-continuous solution. These results follow also from our stochastic PDE Theorems 4.1 and 4.3: Simply set $\sigma \equiv 0$.

Now let us suppose that b is merely locally Lipschitz continuous. Another well-known result is that, if in addition b has "linear growth," that is, if

$$\limsup_{|x|\to\infty} \frac{b(x)}{|x|} < \infty,$$

then (10.1) has a unique contiuous solution.

Suppose now that b is a locally Lipschitz, non-negative convex function that does not have linear growth. That is, $\limsup_{|x|\to\infty} |x|^{-1}|b(x)| = \infty$. It is well known that there are infinitely-many such reaction terms b for which the solution to (10.1) exists globally;

see for example the survey article by Bandle and Brunner [2] together with its voluminous bibliography. In fact, there are locally-Lipschitz functions, convex, non-negative functions b that satisfy the Osgood-type condition,

$$\int_1^\infty \frac{\mathrm{d}x}{b(x)} = \infty, \tag{10.2}$$

for which (10.1) is well posed.

By contrast, Bonder and Groisman [4] have proved the surprising fact that the stochastic version of (10.1) does not have this property.

Theorem 10.1 (Bonder and Groisman [4]). *Consider the stochastic PDE*

$$\left[\begin{array}{ll} \dot u = u'' + b(u) + \sigma \dot W & \textit{on } (0\,,\infty)\times[0\,,1], \ \textit{subject to} \\ u(0) = u_0 & \textit{on } [0\,,1], \end{array}\right. \tag{10.3}$$

where $\sigma > 0$, u_0 is Hölder continuous and vanishes on $\{0\,,1\}$, and $b : \mathbb{R} \to (0\,,\infty)$ is locally Lipschitz, convex, and satisfies (10.2)*. Then, the solution exists up to a finite stopping time T, is continuous, and blows up at time T in the sense that*

$$\lim_{t\uparrow T} \sup_{x\in[0,1]} |u(t\,,x)| = \infty \quad \textit{a.s.}$$

10.1. **Proof of existence and uniqueness.** For every integer $N \geqslant 1$ define

$$b_N(z) := \begin{cases} b(z) & \text{if } |z| \leqslant N, \\ b(N) & \text{if } z > N, \\ b(-N) & \text{if } z < N. \end{cases}$$

Every b_N is a bounded, globally Lipschitz-continuous function. Therefore, Theorems 4.1 and 4.3 together imply that the following SPDE has a unique mild solution $u_N = u_N(t\,,x)$ that is Hölder continuous:

$$\left[\begin{array}{ll} \dot u_N = u_N'' + b_N(u_N) + \sigma \dot W & \text{on } (0\,,\infty)\times[0\,,1], \ \text{subject to} \\ u_N(0) = u_0 & \text{on } [0\,,1]. \end{array}\right. \tag{10.4}$$

Define

$$T_N := \inf\left\{ t \geqslant 0 : \sup_{x\in[0,1]} |u_N(t\,,x)| \geqslant N \right\} \qquad [\inf\varnothing := \infty].$$

Then, every T_N is a stopping time with respect to the Brownian filtration $\mathcal{F}$. By the uniqueness portion of Theorem 4.1, $u_N(s\,,x) = u_{N+1}(s\,,x) = u_{N+2}(s\,,x) = \cdots$ for all $s \in [0\,,T_N)$ and $x \in [0\,,1]$ a.s. Let

$$u(s\,,x) := \liminf_{N\to\infty} u_N(s\,,x) \qquad \forall s \geqslant 0,\ x \in [0\,,1].$$

Choose and fix some $t > 0$ and note that $u(s\,,x) = u_N(s\,,x)$ for all $s \in [0\,,t]$ and $x \in [0\,,1]$, a.s. on the event $\{T_N > t\}$. It is not hard to check that u solves the SPDE (4.1) up to time t a.s. on the event $\{T_N > t\}$. Now consider the $\mathcal{F}$-stopping time

$$T_N' := \inf\left\{ t \geqslant 0 : \sup_{x\in[0,1]} |u(t\,,x)| \geqslant N \right\}.$$

Evidently, $T_N' \leqslant T_{N+1}'$, and hence $T := \lim_{N\to\infty} T_N'$ exists a.s. Of course, T might be finite with positive probability. But certainly $T > 0$ a.s.

The construction of the u_N's and u together ensure that $T_N' \leqslant T_N$. Therefore, $u(s\,,x) = u_N(s\,,x)$ for all $s \in [0\,,t]$ and $x \in [0\,,1]$ a.s. on $\{T_N' > t\}$. Let $N \to \infty$ to see that u is the unique continuous solution to (4.1) up to time t a.s. on $\{T > t\}$. Since $T > 0$ a.s., this shows that (4.1) has a unique short-time mild solution that is continuous [up to time to blow up, if there is such a time].

10.2. **Proof of Theorem 10.1.** Define

$$X_t := \int_0^1 u(t\,,x)\psi_1(x)\,\mathrm{d}x = \sqrt{2}\int_0^1 u(t\,,x)\sin(\pi x)\,\mathrm{d}x,$$

for all $t \geqslant 0$ such that $u(t)$ is defined. Since the mild solution is also a weak solution, and since $\psi_1 \geqslant 0$ on $[0\,,1]$ and vanishes on $\{0\,,1\}$, we may multiply (10.3) by ψ_1 and integrate $[\mathrm{d}t\,\mathrm{d}x]$ in order to see that

$$X_t = \int_0^t \mathrm{d}s \int_0^1 \mathrm{d}x\ \psi_1''(x)u(s\,,x) + \int_0^t \mathrm{d}s\int_0^1 \mathrm{d}x\ \psi_1(x)b(u(s\,,x)) \\ + \sigma\int_{[0,t]\times[0,1]} \psi_1(x)\,\dot{W}(s\,,x)\,\mathrm{d}s\,\mathrm{d}x.$$

The first term on the right-hand side is $-\pi^2\int_0^t X_s\,\mathrm{d}s$ since $\psi_1'' = -\pi^2\psi_1$. Because b is convex, Jensen's inequality then yields

$$X_t \geqslant -\pi^2\int_0^t X_s\,\mathrm{d}s + \int_0^t b(X_s)\,\mathrm{d}s + \sigma\beta_t,$$

where $\beta_t := \int_{[0,t]\times[0,1]}\psi_1(x)\,\dot{W}(s\,,x)\,\mathrm{d}s\,\mathrm{d}x$. The defining properties of Wiener integrals imply that β is a mean-zero Gaussian process with $\beta_0 = 0$ and

$$\mathrm{Cov}[\beta_t\,,\beta_T] = \int_0^{t\wedge T}\mathrm{d}s\int_0^2 \mathrm{d}x\ |\psi_1(x)|^2 = t\wedge T \qquad \forall t, T \geqslant 0.$$

Thus, β is a Brownian motion, and X solves the Itô-type stochastic differential inequality,

$$\mathrm{d}X_t \geqslant -\pi^2 X_t\,\mathrm{d}t + b(X_t)\,\mathrm{d}t + \sigma\mathrm{d}\beta_t,$$

subject to $X_0 = 0$. Now consider the Itô SDE, $\mathrm{d}Y_t = -\pi^2 Y_t\,\mathrm{d}t + b(Y_t)\,\mathrm{d}t + \sigma\mathrm{d}\beta_t$, subject to $Y_0 = 0$. That is,

$$Y_t = -\pi^2\int_0^t Y_s\,\mathrm{d}s + \int_0^t b(Y_s)\,\mathrm{d}s + \sigma\beta_t.$$

Let $F(t) := X_t - Y_t$ to see that

$$F(t) \geqslant -\pi^2\int_0^t F(s)\,\mathrm{d}s + \int_0^t \frac{b(X_s)-b(Y_s)}{X_s - Y_s}\,F(s)\,\mathrm{d}s,$$

for all $t \geqslant 0$ such that $u(t)$ is well defined.[3] In other words, F satisfies the differential inequality,

$$F'(t) \geqslant F(t)\left[-\pi^2 + \frac{b(X_t)-b(Y_t)}{X_t - Y_t}\right],$$

[3]We are defining $B(x\,,y) := (b(x) - b(y))/(x-y)$ to be zero when $x = y$. Since b is locally Lipschitz continuous, the two-variable function B is locally bounded.

for all $t \geqslant 0$ such that $u(t)$ is well defined. Since $F(0) = 0$, it follows from this differential inequality that

$$F(t) \geqslant F(0) \exp\left\{ -\pi^2 t + \int_0^t \frac{b(X_s) - b(Y_s)}{X_s - Y_s}\, \mathrm{d}s \right\} = 0,$$

for all $t \geqslant 0$ such that $u(t)$ is well defined. In particular, $X_t \geqslant Y_t$ for all such $t \geqslant 0$. Because

$$|X_t| \leqslant \sup_{x\in[0,1]} |u(t\,,x)| \cdot \int_0^1 \psi_1(x)\, \mathrm{d}x = \frac{2\sqrt{2}}{\pi} \sup_{x\in[0,1]} |u(t\,,x)|,$$

it remains to verify that $Y_t = +\infty$ for all sufficiently-large [random] $t > 0$. But Y is a nice diffusion, and since $\int_1^\infty \mathrm{d}x/b(x) = \infty$, one can appeal to Feller's test for explosions of diffusions (see Itô and McKean [24]) to verify that, almost surely, $Y_t = \infty$ for all $t > 0$ sufficiently large. We skip the remaining details. □

Thus, we learn from Theorem 10.1 that if there exists $\varepsilon > 0$ such that $b(z) \asymp |z|(\log |z|)^{1+\varepsilon}$ for all sufficiently-large z, then under the additional conditions of Theorem 10.1 we have finite-time blowup. This condition is in a sense sharp, as the following shows.

Theorem 10.2 (Dalang, Khoshnevisan, and Zhang, in process)**.** *Suppose $b \geqslant 0$ is locally Lipschitz and satisfies $b(z) = O(|z| \log |z|)$ as $|z| \to \infty$. Then, the stochastic PDE* (10.3) *has a unique mild solution for all time.*

We will not prove Theorem 10.2 here. It will become publicly available some time in the near future. Although the following sort of argument does not play a role in the actual proof of Theorem 10.2, we offer the following as anecdotal evidence for the truth of Theorem 10.2: The diffusion Y of the proof of Theorem 10.1 does not blow up in finite time if $b(z) = O(|z| \log |z|)$; see Fang and Zhang [19].

11. **Stability**

We conclude these notes with a basic theorem on stability of the solution to (4.1). I had originally intended to include this as a last example of a set of useful renewal-theoretic techniques in SPDEs, but was not able to because of time constraints.

The following essentially states that if u and v solve (4.1) with respective initial profiles u_0 and v_0, and if $u_0 \approx v_0$, then $u(t) \approx v(t)$ for all $t \geqslant 0$. The following makes these remarks more precise.

Theorem 11.1. *Let u solve* (4.1)*, and let v solve* (4.1) *with u_0 replaced by v_0, both with $b \equiv 0$ (to keep things simpler). Suppose $\sup_{x\in[0,1]} |u_0(x) - v_0(x)| \leqslant \delta$ for some $\delta > 0$. Then, for every $\varepsilon \in (0\,, 1)$,*

$$\sup_{x\in[0,1]} \mathrm{E}\left(|u(t\,,x) - v(t\,,x)|^2\right) \leqslant 4\mathrm{e}^{4Lt}\, \delta^2 \qquad \forall t \geqslant 0,$$

where $L :=$ the Lipschitz constant of σ.

Remark 11.2. One can write the conclusion of Theorem 11.1 in the following equivalent, perhaps less cryptic, form:

$$\sup_{x\in[0,1]} \|u(t\,,x) - v(t\,,x)\|_{L^2(\Omega)} \leqslant 2\mathrm{e}^{2Lt} \sup_{x\in[0,1]} |u_0(x) - v_0(x)|.$$

Remark 11.3. If $b \not\equiv 0$ then a similar statement continues to hold. We leave the details to the interested reader as exercise.

It is possible to formulate much stronger theorems that are likely valid under more stringent requirements on u_0 and v_0. For example, it should be possible to prove that if, in addition, u_0 and v_0 both vanish on $\{0\,,1\}$ and are both Hölder continuous, then for every $T > 0$ there exists a real number C_T – independently of $\sup_{x\in[0,1]} |u_0(x) - v_0(x)|$ – such that

$$\mathrm{E}\left(\sup_{t\in[0,T]}\sup_{x\in[0,1]} |u(t\,,x) - v(t\,,x)|^2\right) \leqslant C_T \sup_{x\in[0,1]} |u_0(x) - v_0(x)|^2.$$

I have not tried to verify the details of this assertion, and will leave it to you as [a potentially challenging] exercise.

First we prove the following technical result, which is a certain generalization of the Gronwall inequality.

Lemma 11.4 (Foondun and Khoshnevisan [20])**.** *Suppose $F, G : \mathbb{R}_+ \to \mathbb{R}_+$ are locally bounded and measurable functions that satisfy the following for some constant $a \geqslant 0$:*

$$F(t) \leqslant a + \int_0^t F(s)G(t-s)\,\mathrm{d}s \qquad \forall t \geqslant 0.$$

Then, $F(t) \leqslant (1-\varepsilon)^{-1} a \exp\{\beta(\varepsilon)t\}$ for all $t \geqslant 0$ and $\varepsilon \in (0\,,1)$, where

$$\beta(\varepsilon) := \inf\left\{\alpha > 0 :\ \int_0^\infty \mathrm{e}^{-\alpha s} G(s)\,\mathrm{d}s \leqslant \varepsilon\right\} \qquad [\inf\varnothing := \infty].$$

Remark 11.5. Before we prove this result, we digress to point out a non-trivial connection to classical renewal theory. Georgiou, Joseph, Khoshnevisan, and Shiu [23] have shown that $F(t) \leqslant R(t)$, where R solves the Choquet–Deny type convolution equation,

$$R(t) = a + \int_0^t R(s)G(t-s)\,\mathrm{d}s := a + (R * G)(t) \qquad \forall t > 0.$$

Note that $\exp\{-\beta(1)t\}G(t)$ is frequently a probability density function. Define two new functions, $R_0(t) := \exp\{-\beta(1)t\}R(t)$ and $G_0(t) := \exp\{-\beta(1)t\}G(t)$. Then, $R_0(t) = a\exp\{-\beta(1)t\} + (F_0 * G_0)(t)$ is a renewal equation of classical probability. Blackwell's renewal theorem easily shows that $\ell := \lim_{t\to\infty} R_0(t)$ exists and is finite. Therefore, $R(t) \sim \ell\exp\{-\beta(1)t\}$ as $t \to \infty$. In particular, $F(t) \leqslant (\ell + o(1))\exp\{-\beta(1)t\}$. Because $\beta(1) = \lim_{\varepsilon\uparrow 1}\beta(\varepsilon)$, this shows that the exponential part of the bound in Lemma 11.4 is essentially unimproveable. A similar observation was made earlier by Döring and Savov [16].

Proof of Lemma 11.4. Define, for all $T > 0$, $F_\beta(T) := \sup_{t\in[0,T]} \mathrm{e}^{-\beta t}F(t)$. For all $t \in [0\,,T]$,

$$F(t) \leqslant a + F_\beta(T)\cdot\int_0^t \mathrm{e}^{\beta s}G(t-s)\,\mathrm{d}s = a + F_\beta(T)\mathrm{e}^{\beta t}\cdot\int_0^t \mathrm{e}^{-\beta s}G(s)\,\mathrm{d}s.$$

Multiply both sides by $\exp\{-\beta t\}$ and optimize to see that

$$F_\beta(T) \leqslant a + F_\beta(T)\int_0^\infty \mathrm{e}^{-\beta s}G(s)\,\mathrm{d}s = a + \varepsilon F_\beta(T).$$

Thus, $F_\beta(T) \leqslant a/(1-\varepsilon)$. Let $T \uparrow \infty$ to deduce the result. $\square$

Now we prove Theorem 11.1.

Proof of Theorem 11.1. Define $D(t\,,x) := u(t\,,x) - v(t\,,x)$ for all $t \geqslant 0$ and $x \in [0\,,1]$. Then, we can write u and v in mild form (4.2) to see that

$$D(t\,,x) = (P_t D_0)(x) + \int_{[0,t]\times[0,1]} p_{t-s}(x\,,y)\,[\sigma(u(s\,,y)) - \sigma(v(s\,,y))]\,\dot{W}(s\,,y)\,\mathrm{d}s\,\mathrm{d}y.$$

Note that

$$|(P_t D_0)(x)| \leqslant \int_0^1 p_t(x\,,y)|D_0(y)|\,\mathrm{d}y \leqslant \sup_{y\in[0,1]} |D_0(y)| = \delta,$$

since $\int_0^1 p_t(x\,,y)\,\mathrm{d}y = \mathrm{P}_x\{\tau > t\} \leqslant 1$. Therefore, by the Minkowski inequality,

$$\begin{aligned}\|D(t\,,x)\|_{L^2(\Omega)} &\leqslant \delta + \sqrt{L\int_0^t \mathrm{d}s \int_0^1 \mathrm{d}y\ |p_{t-s}(x\,,y)|^2 \|D(s\,,y)\|^2_{L^2(\Omega)}}\\ &\leqslant \delta + \sqrt{L\int_0^t \mathrm{d}s \int_0^1 \mathrm{d}y\ |\Gamma(t-s\,,x-y)|^2 M(s)},\end{aligned}$$

where $M(t) := \sup_{x\in[0,1]} \mathrm{E}(|D(t\,,x)|^2)$ and Γ is the free-space heat kernel, defined earlier in (4.6). By the latter's semigroup properties,

$$\int_0^1 |\Gamma(t-s\,,x-y)|^2\,\mathrm{d}y \leqslant [\Gamma(t-s) * \Gamma(t-s)]\,(0) = \Gamma(2[t-s]\,,0) = \frac{1}{\sqrt{8\pi(t-s)}}.$$

Thus, we find that

$$\sqrt{M(t)} \leqslant \delta + \sqrt{\frac{L}{\sqrt{8\pi}}\int_0^t \frac{M(s)}{\sqrt{t-s}}\,\mathrm{d}s}.$$

Since $(p+q)^2 \leqslant 2p^2 + 2q^2$, this yields

$$M(t) \leqslant 2\delta^2 + \frac{L}{\sqrt{2\pi}}\int_0^t \frac{M(s)}{\sqrt{t-s}}\,\mathrm{d}s \leqslant 2\delta^2 + \frac{L}{\sqrt{\pi}}\int_0^t \frac{M(s)}{\sqrt{t-s}}\,\mathrm{d}s \qquad \forall t > 0.$$

Apply Lemma 11.4 to see that

$$M(t) \leqslant 4\delta^2 \mathrm{e}^{L\beta(1/2)t/\sqrt{\pi}} \qquad \forall t \geqslant 0,$$

where

$$\beta(1/2) := \inf\left\{\alpha > 0 : \int_0^\infty \frac{\mathrm{e}^{-\alpha s}}{\sqrt{s}}\,\mathrm{d}s \leqslant \tfrac{1}{2}\right\} = 4\sqrt{\pi}.$$

This completes the proof. □

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