An Introduction to Parabolic SPDEs

Lectures in Probability and Stochastic Processes XI
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The Heat Equation

\[ u = u(t, x) \quad t \geq 0 \quad 0 \leq x \leq 1 \]
The Heat Equation

- \( u = u(t, x) \) \( \quad t \geq 0 \quad 0 \leq x \leq 1 \)
- The heat equation on \([0, 1]\) with Dirichlet zero-boundary and initial value \(u_0 \in L^2[0, 1]\) is

\[
\begin{align*}
\dot{u} &= u'' \quad \text{on} \quad (0, \infty) \times [0, 1], \\
\quad u(0) &= u_0, \\
\quad u(t, 0) &= u(t, 1) = 0 \quad \forall t > 0.
\end{align*}
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The Heat Equation

1. \( u = u(t, x) \quad t \geq 0 \quad 0 \leq x \leq 1 \)
2. The heat equation on \([0, 1]\) with Dirichlet zero-boundary and initial value \( u_0 \in L^2[0, 1] \) is

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3. Define \( e_n(x) := \sqrt{2} \sin(n\pi x) \) for \( 0 \leq x \leq 1 \).
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- \(u_0 = \sum_{n=1}^{\infty} (u_0, e_n) e_n\)

\[\langle f, g \rangle := \int_0^1 f(x)g(x) \, dx\]
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  \( = -n^2\pi^2 u_n(t) \Rightarrow u_n(t) = u_n(0)e^{-n^2\pi^2 t} \)
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The Heat Equation

\[ e_n(x) = \sqrt{2} \sin(n\pi x) \]

**Theorem**

Consider the heat equation

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where \( u_0 \in L^2[0, 1] \). Then,

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u(t, x) = \sum_{n=1}^{\infty} \langle u_0, e_n \rangle e^{-n^2\pi^2 t} e_n(x).
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u(t, x) = \sum_{n=1}^{\infty} (u_0, e_n) e^{-n^2\pi^2t} e_n(x).
\]

- The series converges uniformly for \((t, x) \in [\varepsilon, \infty) \times [0, 1]\) for every \( \varepsilon > 0 \).
The Heat Equation

Graph showing the solution to the heat equation with time and temperature axes.

Legend:
- \(<\)
- \(<\)
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- \(\geq\)
- \(\leq\)
- \(-\)
- \(\rightarrow\)
- \(\leftarrow\)
- \(+\)
The Heat Equation

\[ e_n(x) = \sqrt{2} \sin(n\pi x) \]

- Define the heat kernel, \( p_t(x, y) := \sum_{n=1}^{\infty} e_n(x)e_n(y)e^{-n^2\pi^2t}. \)
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- Define the **heat kernel**, \( p_t(x, y) := \sum_{n=1}^{\infty} e_n(x)e_n(y)e^{-n^2\pi^2 t} \).
- \( (P_t\varphi)(x) := \int_0^1 p_t(x, y)\varphi(y) \, dy = \sum_{n=1}^{\infty} e_n(x)(\varphi, e_n) \exp(-n^2\pi^2 t) \) solves our heat equation.
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- \( \{P_t\}_{t \geq 0} \) is a semigroup: \( P_{t+s} = P_tP_s = P_sP_t \) \( \forall s, t \geq 0 \), provided that we define \( P_0\varphi := \varphi. \) 

  \[ \iff (P_{t+s}\varphi)(x) = [P_t(P_s\varphi)](x). \]
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- \( \{P_t\}_{t\geq 0} \) is known as the heat semigroup.
**The Heat Equation**

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- \( \{P_t\}_{t \geq 0} \) is known as the **heat semigroup**.
- \( p_t(x,y) := \) the transition probability that Brownian motion goes from \( x \) to \( y \) in \( t \) units of time before hitting \( \{0,1\} \).
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- Therefore, \( \forall x \in [0, 1] \) and \( t \geq 0 \),

\[ (P_t \varphi)(x) = E_x [\varphi(W(t)); T > t], \]

where

\[ T := \inf \{t > 0 : W(t) \notin [0, 1]\} \quad \text{[} \inf \emptyset := \infty \text{].} \]
**White Noise on \([0, 1]\)**

- Let \(X_1, X_2, \ldots\) be i.i.d. \(N(0, 1)\)s

---

**Lemma**

\[ W'_{N} (\phi) = \sum_{N}^{\infty} N \sum_{n=1}^{N} X_n (\phi e_n) \] is linear.

**Proof.**

Since \(\|\phi\|_{L^2(0,1)}^2 = \sum_{n=1}^{\infty} (\phi e_n)^2\),

\[
E (|W'_N (\phi) - W'_N (\phi + M\phi)|^2) = N + M \sum_{n=N+1}^{\infty} (\phi e_n)^2 \to 0 \quad \text{as} \quad N, M \to \infty.
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\[ W' (\phi) := \lim_{N \to \infty} W'_N (\phi) = \sum_{n=1}^{\infty} X_n (\phi e_n). \]
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$\{W'_N(\varphi)\}_{N=1}^{\infty}$ is a Cauchy sequence in $L^2(\Omega)$. 
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- $W'(\varphi) := \lim_{N \to \infty} W'_N(\varphi) = \sum_{n=1}^{\infty} X_n(\varphi, e_n)$. 

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$W'(\varphi) := \sum_{n=1}^{\infty} X_n(\varphi, e_n)$

**Definition**

$W' := \{W'(\varphi)\}_{\varphi \in L^2[0,1]}$ is called **white noise**.
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**Proposition**

$W'$ is a centered Gaussian random field with $E[W'(\varphi) W'(\psi)] = \langle \psi \rangle$ for all $\psi \in L^2[0,1]$.
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**Proof.**

$E[W'(\varphi) W'(\psi)] = \sum_{n=1}^{\infty} (\varphi, e_n) (\psi, e_n)$. 

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$W'(\varphi) := \sum_{n=1}^{\infty} X_n(\varphi, e_n)$

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  $= \sum_{n=1}^{\infty} X_n \int_{0}^{x} e_n(y) \, dy$
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- $\forall x \in [0, 1]: W(x) := W'(\mathbb{1}_{[0,x]}) = \sum_{n=1}^{\infty} X_n(\mathbb{1}_{[0,x]}, e_n) = \sum_{n=1}^{\infty} X_n \int_0^x e_n(y) \, dy$
- $E[W(x)W(y)] = (\mathbb{1}_{[0,x]}, \mathbb{1}_{[0,y]}) = \min(x, y)$; i.e., $W = \text{Brownian motion on } [0, 1]$. 

Informally, $W'(x) = \sum_{n=1}^{\infty} X_n e_n(x)$ and $E[W'(x)W'(y)] = \delta_0(x - y)$. 

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**White Noise on** $[0, 1]$

$W'(\varphi) := \sum_{n=1}^{\infty} X_n(\varphi, e_n)$

- $\forall x \in [0, 1]: W(x) := W'(\mathbb{1}_{[0,x]}) = \sum_{n=1}^{\infty} X_n(\mathbb{1}_{[0,x]}, e_n)$
  
  $= \sum_{n=1}^{\infty} X_n \int_0^x e_n(y) \, dy$

- $E[W(x)W(y)] = (\mathbb{1}_{[0,x]}, \mathbb{1}_{[0,y]}) = \min(x, y)$; i.e., $W =$ Brownian motion on $[0, 1]$.

- Also, $\forall \varphi : [0, 1] \rightarrow \mathbb{R}$ smooth,
White Noise on $[0, 1]$

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- $\forall x \in [0, 1] : W(x) := W' (1_{[0,x]}) = \sum_{n=1}^{\infty} X_n (1_{[0,x]}, e_n)$
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White Noise on \([0,1]\)

\[ W'(\varphi) := \sum_{n=1}^{\infty} X_n(\varphi, e_n) \]

- \(\forall x \in [0,1]: W(x) := W'(\mathbb{1}_{[0,x]}) = \sum_{n=1}^{\infty} X_n(\mathbb{1}_{[0,x]}, e_n)\)
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$W'$ is the distributional derivative of $W$.
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$\Rightarrow$ $W'$ is the distributional derivative of $W$. 

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**White Noise on** $[0, 1]$

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$\Rightarrow W'$ is the distributional derivative of $W$.

$W'(\varphi) := \int_{0}^{1} \varphi dW := \int_{0}^{1} \varphi(x)W'(x) dx$.  [Wiener integral]
White Noise on $[0, 1]$

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$\Rightarrow W'$ is the distributional derivative of $W$.

$W'(\varphi) := \int_0^1 \varphi \, dW := \int_0^1 \varphi(x)W'(x) \, dx$. [Wiener integral]

Informally, $W'(x) = \sum_{n=1}^{\infty} X_n e_n(x)$ & $E[W'(x)W'(y)] = \delta_0(x - y)$. 
White Noise on \([0, 1]\)

- Pointwise, we should scale \(W(x + \varepsilon) - W(x)\) by \(\sqrt{\varepsilon}\); that is,

\[
\frac{W(x + \varepsilon) - W(x)}{\sqrt{\varepsilon}} \overset{(d)}{=} W(1).
\]

Weakly, we should scale \(W(x + \varepsilon) - W(x)\) by \(\varepsilon\); that is, for all \(\phi\) smooth on \([0, 1]\) that vanish on the boundary, and all \(\varepsilon > 0\) small,

\[
\int_0^1 \phi(x) \cdot W(x + \varepsilon) - W(x) \varepsilon \, dx = \int_0^1 \phi(x) - \phi(x - \varepsilon) \varepsilon W(x) \, dx.
\]

\(a.s.\) \(\rightarrow jmn\) \(\phi'\) as \(\varepsilon \downarrow 0 = W' = \int_0^1 \phi \, dW\).
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**White Noise on** [0, 1]

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\overset{a.s.}{\longrightarrow} - (\phi', W) \quad \text{as } \varepsilon \downarrow 0
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  \]
  \[\xrightarrow{\text{a.s.}} - (\varphi', W) \quad \text{as } \varepsilon \downarrow 0\]
  \[= W'(\varphi) = \int_0^1 \varphi \, dW.\]
The Itô Integral

- $\dot{W}(\varphi) = \int_0^1 \varphi \, dW$ has been defined for all $\varphi \in L^2[0,1]$ — this is the Wiener integral.
The Itô Integral

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- Also \( \int_0^x \varphi \, dW := \int_0^1 \varphi \mathbb{1}_{[0,x]} \, dW \) for \( 0 \leq x \leq 1 \).
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- There exists a generalization that covers some random $\varphi$ — this is the Itô integral, which we next recall quickly.
The Itô Integral

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- Let \( \mathcal{F}_x \supseteq \sigma \{ W(y); \ y \leq x \} \) [completed and augmented].

**Simple functions**

\[ \Phi(x) := \sum_{n=1}^N \Phi_n(x) \] for \( \Phi_n \in L^2(\Omega/\mathcal{F}_a/P) \):

- \( \dot{W}(\Phi) := \sum_{n=1}^N \dot{W}(\Phi_n) \) (Elementary functions)

**Fact:** This definition is coherent.

\[ E[\dot{W}(\Phi) \dot{W}(\Psi)] = E[(\Phi \Psi)] \] (Itô isometry).
The Itô Integral

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- Let \( \mathcal{F}_x \supseteq \sigma\{W(y); \ y \leq x\} \) [completed and augmented].
- (Simple functions) \( \Phi(x) := X \mathbb{1}_{[a,b]}(x) \) for \( X \in L^2(\Omega, \mathcal{F}_a, \mathbb{P}) \):
  \[
  \dot{W}(\Phi) := \int \Phi \, dW := X \left[ W(b) - W(a) \right]
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- (Elementary functions) $\Phi = \sum_{n=1}^N \Phi_n$ where $\Phi_1, \ldots, \Phi_N$ are simple functions with disjoint support: $\dot{W}(\Phi) := \sum_{n=1}^N \dot{W}(\Phi_n)$. 

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$E[\dot{W}(\Phi) \dot{W}(\Psi)] = E[\Phi \Psi]$ [Itô isometry].
The Itô Integral

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- Fact: This definition is coherent.
- \( E \left[ \dot{W}(\Phi) \dot{W}(\Psi) \right] = E[(\Phi, \Psi)] \) [Itô isometry].
The Itô Integral

- Equivalently,

\[ E(\|\dot{W}(\Phi) - \dot{W}(\Psi)\|^2) = E(\|\Phi - \Psi\|_{L^2[0,1]}^2) = \|\Phi - \Psi\|_{L^2(\Omega \times [0,1])}^2. \]
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- Let \( \mathcal{E} \) denote the collection of all \textit{elementary functions} endowed with inner product \( \langle \Phi, \Psi \rangle := E[\langle \Phi, \Psi \rangle] \).
The Itô Integral

- Equivalently,
  \[ E(|\dot{W}(\Phi) - \dot{W}(\Psi)|^2) = E(\| \Phi - \Psi \|^2_{L^2[0,1]}) = \| \Phi - \Psi \|^2_{L^2(\Omega \times [0,1])}. \]

- Let \( \mathcal{E} \) denote the collection of all \textit{elementary functions} endowed with inner product \( \langle \Phi, \Psi \rangle := E[(\Phi, \Psi)] \).

- \( \dot{W} \) is a linear isometry from \( \mathcal{E} \) to \( L^2(\Omega \times [0,1]) \).
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- Let \( \mathcal{E} \) denote the collection of all *elementary functions* endowed with inner product \( \langle \Phi, \Psi \rangle := E[\langle \Phi, \Psi \rangle] \).
- \( \dot{W} \) is a linear isometry from \( \mathcal{E} \) to \( L^2(\Omega \times [0,1]) \).
- Let \( \mathcal{P} \) denote the completion of \( \mathcal{E} \) in the norm of \( L^2(\Omega \times [0,1]) \).
The Itô Integral

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- \( \dot{W} \) is a linear isometry from \( \mathcal{E} \) to \( L^2(\Omega \times [0,1]) \).
- Let \( \mathcal{P} \) denote the completion of \( \mathcal{E} \) in the norm of \( L^2(\Omega \times [0,1]) \).
- \( \dot{W} \) continuously extends to a linear isometry from the collection \( \mathcal{P} \) of all \emph{Predictable processes} to \( L^2(\Omega \times [0,1]) \).
The Itô Integral

- Equivalently,
  \[ E(|\dot{W}(\Phi) - \dot{W}(\Psi)|^2) = E(\|\Phi - \Psi\|_{L^2[0,1]}^2) = \|\Phi - \Psi\|_{L^2(\Omega \times [0,1])}^2. \]

- Let \( \mathcal{E} \) denote the collection of all \textit{elementary functions} endowed with inner product \( \langle \Phi, \Psi \rangle := E(\Phi, \Psi). \)

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\textbf{Theorem (Itô, Meyer, …)}

\textit{If} \( \Phi \) \textit{is} \( \mathcal{F} \)-\textit{adapted and càdlàg and} \( \|\Phi\|_{L^2(\Omega \times [0,1])}^2 = E(\|\Phi\|_{L^2[0,1]}^2) < \infty, \)
\textit{then} \( \Phi \in \mathcal{P}. \)
The Itô Integral

- Equivalently,
  \[ E(|\dot{W}(\Phi) - \dot{W}(\Psi)|^2) = E(\|\Phi - \Psi\|^2_{L^2[0,1]}) = \|\Phi - \Psi\|^2_{L^2(\Omega \times [0,1])}. \]

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Theorem (Itô, Meyer, ...) 

\( \text{If } \Phi \text{ is } \mathcal{F} \text{-adapted and càdlàg and } \|\Phi\|^2_{L^2(\Omega \times [0,1])} = E(\|\Phi\|^2_{L^2[0,1]}) < \infty, \text{ then } \Phi \in \mathcal{P}. \)

- \( \forall \Phi \in \mathcal{P} : \quad \dot{W}(\Phi) = \int_0^1 \Phi \, dW = \int_0^1 \Phi(x) W'(x) \, dx \) [Itô’s integral].
The Itô Integral

- Equivalently,

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\]

- Let \( E \) denote the collection of all \textit{elementary functions} endowed with inner product \( \langle \Phi, \Psi \rangle := E((\Phi, \Psi)). \)

- \( \dot{W} \) is a linear isometry from \( E \) to \( L^2(\Omega \times [0,1]) \).

- Let \( P \) denote the completion of \( E \) in the norm of \( L^2(\Omega \times [0,1]) \).

- \( \dot{W} \) continuously extends to a linear isometry from the collection \( P \) of all \textit{predictable processes} to \( L^2(\Omega \times [0,1]) \).

**Theorem (Itô, Meyer, ...)**

If \( \Phi \) is \( \mathcal{F} \)-adapted and càdlàg and \( \|\Phi\|_{L^2(\Omega \times [0,1])}^2 = E(\|\Phi\|_{L^2[0,1]}^2) < \infty \), then \( \Phi \in P \).

- \( \forall \Phi \in P : \quad \dot{W}(\Phi) = \int_0^1 \Phi \, dW = \int_0^1 \Phi(x) W'(x) \, dx \) [Itô’s integral].

- \( \int_0^x \Phi \, dW := \int_0^1 \Phi 1_{[0,x]} \, dW \) for all \( 0 \leq x \leq 1 \).
The Itô Integral

Theorem (Itô, Meyer, …)

If $\Phi \in \mathcal{P}$, then $M(x) := \int_0^x \Phi \, dW$ ($0 \leq x \leq 1$) defines a continuous $L^2$-martingale with quadratic variation,

$$\langle M \rangle(x) = \int_0^x |\Phi(y)|^2 \, dy \quad (0 \leq x \leq 1).$$
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**Theorem (Itô, Meyer, ...)**

If \( \Phi \in \mathcal{P} \), then \( M(x) := \int_0^x \Phi \, dW \) \((0 \leq x \leq 1)\) defines a continuous \( L^2 \)-martingale with quadratic variation,

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- We have ensured by default that if \( \Phi \in \mathcal{P} \), then \( \mathbb{E}(\|\Phi\|_{L^2[0,1]}^2) < \infty \).
The Itô Integral

**Theorem (Itô, Meyer, ...)**

If $\Phi \in \mathcal{P}$, then $M(x) := \int_0^x \Phi \, dW$ ($0 \leq x \leq 1$) defines a continuous $L^2$-martingale with quadratic variation,

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- We have ensured by default that if $\Phi \in \mathcal{P}$, then $E(\|\Phi\|^2_{L^2[0,1]}) < \infty$.
- Being an element of $\mathcal{P}$ requires more than the above integrability.
The Itô Integral

**Theorem (Itô, Meyer, ...)**

If $\Phi \in \mathcal{P}$, then $M(x) := \int_0^x \Phi \, dW$ ($0 \leq x \leq 1$) defines a continuous $L^2$-martingale with quadratic variation, 

$$\langle M \rangle(x) = \int_0^x |\Phi(y)|^2 \, dy \quad (0 \leq x \leq 1).$$

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- Being an element of $\mathcal{P}$ requires more than the above integrability.
- This notion of predictability is slightly non-standard.
**Space-Time White Noise**

- Let $X_1, X_2, \ldots$ be i.i.d. Brownian motions on $\mathbb{R}_+$
Space-Time White Noise

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- Define $\forall t > 0$ and $x \in [0, 1]$,

$$W(t, x) := \sum_{n=1}^{\infty} X_n(t) (1_{[0,x]} e_n).$$
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Wish to define space-time white noise $\dot{W}$ as the weak mixed derivative $\partial_t \partial_x W$ of $W$. 

Space-Time White Noise

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**Proposition**

- Informally, \( \dot{W}(t, x) = \sum_{n=1}^{\infty} \dot{X}_n(t)e_n(x) \) and \( \mathbb{E}[\dot{W}(t, x) \, \dot{W}(s, y)] = \delta_0(t-s)\delta_0(x-y). \)
Space-Time White Noise

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- \( \varphi \mapsto \dot{W}(\varphi) \) is linear.
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- \( \dot{W}(\Phi) = \int_{\mathbb{R}_+ \times [0, 1]} \Phi \, dW = \int_{\mathbb{R}_+ \times [0, 1]} \Phi(t, x) \dot{W}(t, x) \, dt \, dx \).
- \( \int_{F} \Phi \, dW := \int_{\mathbb{R}_+ \times [0, 1]} \Phi 1_F \, dW \) for all Borel sets \( F \subset \mathbb{R}_+ \times [0, 1] \).
\[ \dot{W}(\Phi) = \sum_{n=1}^{\infty} \int_{0}^{\infty} (\Phi(t), e_n) dX_n(t) \]

**Theorem (Pardoux, Krylov–Rozovskiǐ, Walsh, …)**

For every predictable random field \( \Phi = \{\Phi(t, x); t \geq 0, x \in [0, 1]\} \),

\[ M(t) := \int_{[0,t] \times [0,1]} \Phi \, dW \quad (t \geq 0) \]

is a centered continuous \( L^2 \)-martingale with quadratic variation,

\[ \langle M \rangle(t) = \int_{0}^{t} ds \int_{0}^{1} dy \ |\Phi(s, y)\|^2 = \int_{0}^{t} \|\Phi(s)\|_{L^2[0,1]}^2 \, ds. \]