An Introduction to Parabolic SPDEs

Lectures in Probability and Stochastic Processes XI
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Consider the heat equation

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where $u_0 \in L^2[0,1]$. *Then,*

$$u(t,x) = \sum_{n=1}^{\infty} (u_0, \mathbf{e}_n) e^{-n^2\pi^2 t} \mathbf{e}_n(x).$$

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- The series converges uniformly for $(t,x) \in [\varepsilon,\infty)\times[0,1]$ for every $\varepsilon > 0$.

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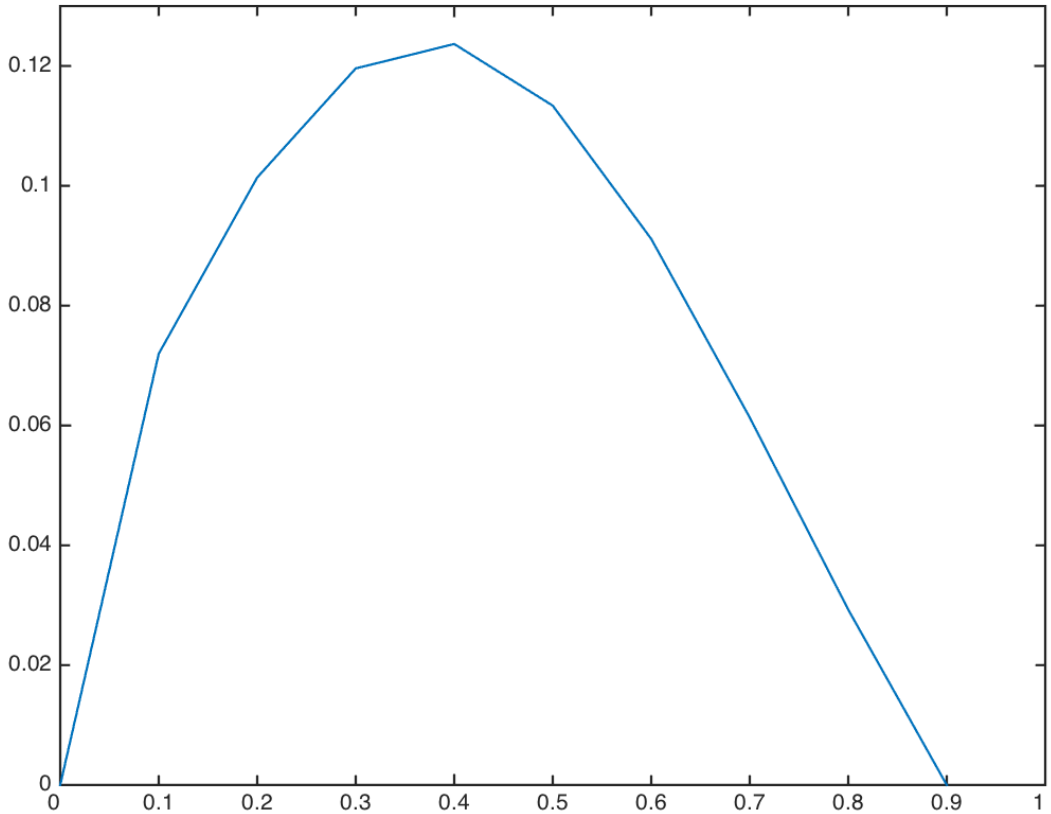

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- $\{P_t\}_{t\geqslant 0}$ is a semigroup: $P_{t+s} = P_tP_s = P_sP_t\ \forall s,t \geqslant 0$, provided that we define $P_0\varphi := \varphi$. $\quad [\Leftrightarrow\ (P_{t+s}\varphi)(x) = [P_t(P_s\varphi)](x).]$

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- $p_t(x\,,y) :=$ the transition probability that Brownian motion goes from x to y in t units of time before hitting $\{0\,,1\}$.
- Therefore, $\forall x \in [0\,,1]$ and $t \geqslant 0$,

$$(P_t\varphi)(x) = \mathrm{E}_x\left[\varphi(W(t));\ T > t\right],$$

where

$$T := \inf\{t > 0:\ W(t) \notin [0\,,1]\} \qquad [\inf\varnothing := \infty].$$

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Since $\|\varphi\|^2_{L^2[0,1]} = \sum_{n=1}^\infty (\varphi\,, \mathbf{e}_n)^2$,

$$\mathrm{E}\left(\left|W'_N(\varphi) - W'_{N+M}(\varphi)\right|^2\right) = \sum_{n=N+1}^{N+M} (\varphi\,, \mathbf{e}_n)^2 \to 0,$$

as $N, M \to \infty$. □

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- $W'(\varphi) := \lim_{N\to\infty} W'_N(\varphi) = \sum_{n=1}^\infty X_n(\varphi\,, \mathbf{e}_n)$.

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- Also, $\forall \varphi : [0\,,1] \to \mathbb{R}$ smooth,

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- This notion of predictability is slightly non-standard.

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- $W :=$ *space-time Brownian sheet.*

Space-Time White Noise

- Let $X_1, X_2, \ldots$ be i.i.d. Brownian motions on $\mathbb{R}_+$
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- Wish to define space-time white noise $\dot{W}$ as the weak mixed derivative $\partial_t \partial_x W$ of W.

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- Informally, $\dot{W}(t\,,x) = \sum_{n=1}^{\infty} \dot{X}_n(t)\mathbf{e}_n(x)$ and $\mathrm{E}[\dot{W}(t\,,x)\dot{W}(s\,,y)] = \delta_0(t-s)\delta_0(x-y)$.

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- $\int_F \Phi\,dW := \int_{\mathbb{R}_+ \times [0,1]} \Phi \mathbb{1}_F\,dW$ for all Borel sets $F \subset \mathbb{R}_+ \times [0\,,1]$.

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Theorem (Pardoux, Krylov–Rozovskiĭ, Walsh, ...)

For every predictable random field $\Phi = \{\Phi(t\,, x);\ t \geqslant 0, x \in [0\,,1]\}$,

$$M(t) := \int_{[0,t]\times[0,1]} \Phi\, dW \qquad (t \geqslant 0)$$

is a centered continuous L^2*-martingale with quadratic variation,*

$$\langle M \rangle(t) = \int_0^t ds \int_0^1 dy\ |\Phi(s\,, y)|^2 = \int_0^t \|\Phi(s)\|^2_{L^2[0,1]}\, ds.$$