Lévy Processes and Stochastic Partial Differential Equations

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Problem 1

The stochastic heat equation:

\[ \partial_t u(t, x) = (\Delta_x u)(t, x) + \dot{W}(t, x) \quad \forall t \in 0, x \in \mathbb{R}^d. \]
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Question: Why $\exists$ function solutions only when $d = 1$? (Walsh, Dalang–Frangos, Dalang, Pesat–Zabczyk)
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**Answer:** BM has local times only in \( d = 1 \).
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  (Walsh, Dalang–Frangos, Dalang, Pesat–Zabczyk)

- **Answer:** BM has local times only in \( d = 1 \).
  In fact, \( \exists \) function solutions in dimension \( 2 - \alpha \) for all \( \alpha \in (0, 2] \).
Problem 2

- Weakly interacting system of stochastic wave equations:

\[
\begin{align*}
\partial_{tt}u_i(t, x) &= (\partial_{xx} u_i)(t, x) + \sum_{j=1}^{d} Q_{ij} \dot{W}_j(t, x) \quad \forall x \in \mathbb{R}, \; t \geq 0, \\
u_i(0, x) &= \partial_t u_i(0, x) = 0,
\end{align*}
\]

\[\dot{W}_1, \ldots, \dot{W}_d := \text{i.i.d. white noises; } Q = (Q_{ij})_{i,j=1}^d \text{ invert.}\]
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Answer: Iff \(d < 4\).

(Orey–Pruitt, K, Dalang–Nualart; closely-related: LeGall)
The standard equation

\[
\dot{W} := \{\dot{W}(t, x)\}_{t \geq 0, x \in \mathbb{R}^d} \text{ space-time white noise:}
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The standard equation

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- \( \{ \dot{W}(A) \}_{A \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^d)} := \) a centered gaussian process.
- \( \text{Cov}(\dot{W}(A), \dot{W}(B)) = |A \cap B| \quad \forall A, B \subset \mathbb{R}_+ \times \mathbb{R}^d. \)
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- \text{Identification via Wiener integrals:} \\
\[ \int \phi d\dot{W} \simeq \int_0^\infty \int_{\mathbb{R}^d} \phi(t, x) \dot{W}(t, x) \, dx \, dt. \]
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- The stochastic heat equation: \( \exists (? )u := u(t, x) \)
  \([t \geq 0, x \in \mathbb{R}^d]:\)
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  \partial_t u(t, x) = (\Delta_x u)(t, x) + \dot{W}(t, x).
  \]

- **Fact 1:** Function-valued solution \( \exists \) iff \( d = 1 \).
- **Rough explanation:** \( \Delta_x \) smooths; \( \dot{W} \) makes rough.
The heat equation

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Consider \[ \partial_t u(t, x) = (\Delta_x u)(t, x) + f(t, x) \]
[f nice]
The heat equation

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Consider \( \partial_t u(t, x) = (\Delta_x u)(t, x) + f(t, x) \) [\( f \) nice]

Let

\[ Q_t(y) := \frac{1}{(4\pi t)^{d/2}} \exp \left( -\frac{\|y\|^2}{4t} \right) . \]
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Solution:
\[ u(t, x) = \int_0^t \int_{\mathbb{R}^d} Q_{t-s}(x-y)f(s, y) \, dy \, ds. \]
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Apply to “\( f := \dot{W} \)” [Mild solution; Walsh, 1986]
The stochastic heat equation

\[ \partial_t u(t,x) = (\Delta_x u)(t,x) + \dot{W}(t,x). \]
The stochastic heat equation

- \( \partial_t u(t, x) = (\Delta_x u)(t, x) + \dot{W}(t, x) \).
- **Mild solution:** If \( \exists \), then

\[
  u(t, x) = \int_0^t \int_{\mathbb{R}^d} Q_{t-s}(x - y) \dot{W}(dy \, ds).
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- **Weak solution**: A family \( u(t, \varphi) \), for nicely tempered \( \varphi \):
  \[
  u(t, \varphi) := \int_0^t \int_{\mathbb{R}^d} (Q_{t-s} \ast \varphi)(y) \dot{W}(dy \, ds). 
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  “\( u(t, \varphi) = \int_{\mathbb{R}^d} u(t, x) \varphi(x) \, dx. \)”
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  "\( u(t,\varphi) = \int_{\mathbb{R}^d} u(t,x)\varphi(x)\,dx \)."
- \( \varphi \mapsto u(t,\varphi) \) is a linear gaussian distribution [Itô; Menshov].
The stochastic heat equation

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Need:

\[ E \left( \left| u(t, \varphi) \right|^2 \right) = \int_0^t \int_{\mathbb{R}^d} \left| (Q_{t-s} \ast \varphi)(y) \right|^2 dy \, ds < \infty. \]
The stochastic heat equation

- Plancherel's theorem:

\[
\int_0^t \int_{\mathbb{R}^d} \left| (Q_{t-s} \ast \varphi)(y) \right|^2 dy ds
\]

\[
\leq \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \left| \hat{\varphi}(\xi) \right|^2 \left( 1 - e^{-\frac{2}{t} \|\xi\|^2} \right) d\xi
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- Weak solution exists if and only if \( \varphi \in H^{-2}(\mathbb{R}^d) \).

- Mild solution exists if and only if \( \delta_x \in H^{-2}(\mathbb{R}^d) \) for all \( x \in \mathbb{R}^d \) and \( d = 1 \).
The stochastic heat equation

- Plancherel’s theorem:

\[ \int_0^t \int_{\mathbb{R}^d} \left| (Q_{t-s} \ast \varphi)(y) \right|^2 \, dy \, ds \]

\[ = \left( \frac{1}{2\pi} \right)^{d/2} \int_{\mathbb{R}^d} \left| \hat{\varphi}(\xi) \right|^2 \, d\xi \]

\[ \leq 1 - \frac{2}{\left\| \xi \right\|^2} \int_0^t \int_{\mathbb{R}^d} \left| (Q_{t-s} \ast \varphi)(y) \right|^2 \, dy \, ds \]

\[ \leq 1 - e^{-\frac{2}{\left\| \xi \right\|^2} (t - s)} \]

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Plancherel’s theorem:

\[ \int_0^t \int_{\mathbb{R}^d} |(Q_{t-s} \ast \varphi)(y)|^2 \, dy \, ds = \frac{1}{(2\pi)^d} \int_0^t \int_{\mathbb{R}^d} e^{-2 \|\xi\|^2(t-s)} |\hat{\varphi}(\xi)|^2 \, d\xi \, ds \]
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\]

\[\therefore\text{ weak solution } \exists \text{ iff } \varphi \in H^{-2}(\mathbb{R}^d). \text{ In fact, }\]

\[
E \left( \left[ u(t, \varphi) \right]^2 \right) \preceq t \int_{\mathbb{R}^d} \frac{\left| \hat{\varphi}(\xi) \right|^2}{1 + t^2\|\xi\|^2} d\xi.
\]
The stochastic heat equation

▶ Plancherel’s theorem:

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▶ \therefore weak solution \exists iff \varphi \in H_{-2}(\mathbb{R}^d). In fact,

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\mathbb{E} \left( \left| u(t, \varphi) \right|^2 \right) \leq t \int_{\mathbb{R}^d} \frac{|\hat{\varphi}(\xi)|^2}{1 + t^2\|\xi\|^2} \, d\xi.
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- \( \therefore \) mild solution \( \exists \) iff \( \delta_x \in H_{-2}(\mathbb{R}^d) \) \( \forall x \in \mathbb{R}^d \iff d = 1. \)
(Dalang–Frangos) Replace $\dot{W}(t, x)$ by $\dot{F}(t, x)$, where $\dot{F}$ is a centered gaussian noise with $\text{Cov}(\int \phi d\dot{F}, \int \psi d\dot{F}) = \iint \phi(t, x)(s \wedge t)\kappa(\|x - y\|)\psi(s, y) dt \, dx \, ds \, dy$.

\[ \sum(s, t, \|x - y\|) \]
An explanation

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- \( \Sigma(s, t, \|x - y\|) \)

- (Dalang–Frangos, Pesat–Zabczyk, Dalang) There can exist solutions for \( d > 1 \), depending on \( \kappa \) [NASC].
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- Explains the roughening effect of white noise.

[analytic]
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- $\Sigma(s, t, \|x - y\|)$

- (Dalang–Frangos, Pesat–Zabczyk, Dalang) There can $\exists$ solutions for $d > 1$, depending on $\kappa$ [NASC].

- Explains the roughening effect of white noise.
  [analytic]

- We propose to explain the smoothing effect of $\Delta_x$.
  [probabilistic]
Lévy processes

- \( L := L^2\)-generator of a Lévy process \( X \) in \( \mathbb{R}^d \).
Lévy processes

- \( L := \text{generator of a Lévy process } X \text{ in } \mathbb{R}^d. \)
- **Normalization:** \( \mathbb{E} \exp(i\xi \cdot X(t)) = \exp(-t\Psi(\xi)), \hat{L}(\xi) = -\Psi(\xi). \)
  
  That is,
  \[
  \int_{\mathbb{R}^d} f(x)(Lg)(x) \, dx = -\int_{\mathbb{R}^d} \hat{f}(\xi) \hat{g}(\xi) \Psi(\xi) \, d\xi.
  \]
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  That is,
  $$\int_{\mathbb{R}^d} f(x)(Lg)(x) \, dx = -\int_{\mathbb{R}^d} \hat{f}(\xi) \hat{g}(\xi) \psi(\xi) \, d\xi.$$

- $\text{Dom}(L) := \{ f \in L^2(\mathbb{R}^d) : \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 \text{Re}\psi(\xi) \, d\xi < \infty \}$. 

- $\bar{X}$ is a Lévy process with char. exponent $2\text{Re}\Psi$. 

- $X'$ is an independent copy of $X$; $\bar{X}(t) := X(t) - X'(t)$.

Davar Khoshnevisan (Salt Lake City, Utah)
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- **Normalization:** \( \mathbb{E} \exp(i \xi \cdot X(t)) = \exp(-t \psi(\xi)), \, \hat{L}(\xi) = -\psi(\xi) \). That is,

  \[
  \int_{\mathbb{R}^d} f(x)(Lg)(x) \, dx = -\int_{\mathbb{R}^d} \hat{f}(\xi) \hat{g}(\xi) \psi(\xi) \, d\xi. 
  \]

- \( \text{Dom}(L) := \{ f \in L^2(\mathbb{R}^d) : \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 \Re \psi(\xi) \, d\xi < \infty \} \).

- \( X' := \) an independent copy of \( X \);
Lévy processes

- $L := L^2$-generator of a Lévy process $X$ in $\mathbb{R}^d$.
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  [Lévy]
- \( \bar{X} \) is a Lévy process with char. exponent \( 2\text{Re}\psi \).
Local times

\( \lambda^x_t := \text{local time of } \bar{X}, \text{ at place } x \text{ and time } t, \text{ when it exists.} \)
Local times

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- $\lambda^x_t = O_t(dx)/dx$, where $O_t(E) := \int_0^t 1_E(\bar{X}(s)) \, ds$. 

Theorem (Hawkes)

$\{\lambda^x_t\}_{t \geq 0}$, $x \in \mathbb{R}^d$ exists iff 

$$\int_{\mathbb{R}^d} \int_0^1 \xi d\xi + \Re \Psi(\xi) < \infty.$$
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**Theorem (Hawkes)**

$\{\lambda^x_t\}_{t \geq 0, x \in \mathbb{R}^d}$ exists iff
\[
\int_{\mathbb{R}^d} \frac{d\xi}{1 + \text{Re}\, \Psi(\xi)} < \infty.
\]
Examples and remarks

- Hawkes’ condition: \( \int_{\mathbb{R}^d} (1 + \text{Re}\psi(\xi))^{-1} \, d\xi < \infty \).
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- Suppose $d = 1$ and $\psi(\xi) = |\xi|(1 + i c \text{sgn}|\xi| \log |\xi|)$ for $0 \leq |c| \leq 2/\pi$. Then local times exist iff $c \neq 0$. 

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Consider the heat equation

\[ \partial_t u(t, x) = (L_x u)(t, x) + \dot{W}(t, x), \]

where \( L_x \) is the generator of a Lévy process on \( \mathbb{R}^d \), acting on the variable \( x \).
A heat equation

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Fundamental questions:

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Fundamental questions:

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- Etc.
A heat equation

Theorem (K–Foondun–Nualart)
Let $u$ denote the weak solution to the heat equation for $L$. Then, for all tempered functions $\varphi$ and all $t, \lambda > 0$,

$$
\frac{1 - e^{-2\lambda t}}{2} \mathcal{E}_\lambda(\varphi, \varphi) \leq \mathbb{E}\left(|u(t, \varphi)|^2\right) \leq \frac{e^{2\lambda t}}{2} \mathcal{E}_\lambda(\varphi, \varphi),
$$

where

$$
\mathcal{E}_\lambda(\varphi, \psi) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{\hat{\varphi}(\xi)\hat{\psi}(\xi)}{\lambda + \text{Re}\psi(\xi)} \, d\xi.
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Corollary

\( \exists \) function-valued solutions iff \( \bar{X} \) has local times. The solution is continuous iff \( x \mapsto \lambda_t^x \) is.
In fact, many [virtually all] of the properties of $x \mapsto u(t, x)$ are inherited from $x \mapsto \lambda_t^x$ and vice versa:
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A heat equation

- In fact, many \([\text{virtually all}]\) of the properties of \(x \mapsto u(t, x)\) are inherited from \(x \mapsto \lambda_t^x\) and vice versa:
  - Existence and continuity.
  - Hölder continuity.
  - \(p\)-variation of the paths . . . .
- \(\exists\) an embedding of the isomorphism theorem? [Dynkin; Brydges–Fröhlich–Spencer]
- A final **Theorem** (K–Foondun–Nualart): \(t \mapsto u(t, \varphi)\) has a continuous version iff

\[
\int_1^\infty \frac{\mathcal{E}_\lambda(\varphi, \varphi)}{\lambda \sqrt{|\log \lambda|}} \, d\lambda < \infty.
\]

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Recall:
\[ \partial_t u(t, x) = (L_x u)(t, x) + \dot{W}(t, x). \] (HE)

We chose \( L \) to be the generator of a Lévy process only because there we have NASC.

Could have \( L := \) generator of a nice Markov process.

Theorem (K–Foondun–Nualart)

Let \( L := \) Laplacian on a "nice" fractal of \( \text{dim} = 2 - \alpha \) for \( \alpha \in (0, 2] \).

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**Theorem (K–Foondun–Nualart)**

Let \( L := \) Laplacian on a “nice” fractal of \( \dim_H = 2 - \alpha \) for \( \alpha \in (0, 2] \). Then (HE) has function solutions that are in fact Hölder continuous.
Let $\dot{L}$ be space–time Lévy noise with values in $\mathbb{R}^d$. 

$E \exp (i \xi \cdot \dot{L}(A)) = \exp (-|A| \Psi(\xi))$ for $\xi \in \mathbb{R}^d$ and $A \in B(\mathbb{R}_+ \times \mathbb{R})$.

Assume:

$\Psi$ is nonnegative real.

$\forall a > 0 \exists A_a > 0$ such that $\Psi(a \xi) \geq A_a \Psi(\xi)$.

Gauge function $\exists$ and is finite, where $\Phi(\lambda) := \int_{\mathbb{R}^d} e^{-\lambda \Psi(\xi)} d\xi \forall \lambda > 0$. 

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Let $\dot{L} := -$ be space–time Lévy noise with values in $\mathbb{R}^d$.

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System of wave equations

- Let $\dot{L}$ be space–time Lévy noise with values in $\mathbb{R}^d$.
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[stable like]
Let $\dot{L} := L$ be space–time Lévy noise with values in $\mathbb{R}^d$.

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Zeros of the solution

\[
\begin{align*}
\partial_{tt} u_i(t, x) &= \partial_{xx} u_i(x, t) + \dot{L}_i(t, x), \\
u_i(0, x) &= \partial_t u_i(0, x) = 0.
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Theorem (K–Nualart)

TFAE:

1. With positive probab. $u_i(t, x) = 0$ for some $t > 0$ and $x \in \mathbb{R}$.
2. Almost surely, $u_i(t, x) = 0$ for some $t > 0$ and $x \in \mathbb{R}$.
3. $\int_0^1 \lambda \Phi(\lambda) d\lambda < \infty$.
4. If $\int_0^1 \lambda \Phi(\lambda) d\lambda < \infty$, then almost surely, $\dim H u - 1 \{0\} = 2 - \limsup_{\lambda \downarrow 0} \frac{\log \Phi(\lambda)}{\log (1/\lambda)}$. 

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Example

Suppose $\dot{L}_1, \ldots, \dot{L}_d$ are independent, $\dot{L}_j = \text{stable}(\alpha_j)$. Then:
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1. \( u \) has zeros iff \( \sum_{j=1}^{d} (1/\alpha_j) < 2 \).
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Example

Suppose $\dot{L}_1, \ldots, \dot{L}_d$ are independent, $\dot{L}_j = \text{stable}(\alpha_j)$. Then:

1. $u$ has zeros iff $\sum_{j=1}^{d} (1/\alpha_j) < 2$.
2. If $\sum_{j=1}^{d} (1/\alpha_j) < 2$, then $\dim_H u^{-1}\{0\} = 2 - \sum_{j=1}^{d} (1/\alpha_j)$. 
Idea of proof [existence part]

- WLOG consider $u(t, x)$ for $t, x \geq 0$. 

Appeal to K–Shieh–Xiao.

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WLOG consider $u(t, x)$ for $t, x \geq 0$.

$P\{0 \in u(G)\} \asymp P\{0 \in X(G)\}$, where

$$X(t, x) := X_1(t) + X_2(x),$$

where $X_1, X_2$ are i.i.d. Lévy processes, exponent $\Psi$.

[additive Lévy process]
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[additive Lévy process]

- Appeal to K–Shieh–Xiao.
Idea of proof [second part]

$$u(t, x) = \frac{1}{2} \dot{L}(C(t, x)),$$
where $C(t, x)$ is the “light cone” emanating from $(t, x)$.
Idea of proof [zero-one law part]

The zero set in the black triangle depends on the noise through its “backward light cone,” shaded black/pink.

Therefore, $P\{u^{-1}(\{0\}) \neq \emptyset\}$ is zero or one.