1. The result

The goal of this note is to prove the following classical theorem. Throughout,
\[ \kappa_s(x) := \|x\|-s \text{ for all } x \in \mathbb{R}^d. \]

**Theorem 1.1.** If \( \alpha \in (0, d) \), then the \( L^2 \)-Fourier transform of \( \kappa_\alpha \) is \( c\kappa_{d-\alpha} \) for a constant \( c = c_{d,\alpha} \in (0, \infty) \).

**Remark 1.2.** \( \kappa_\alpha \) is not in \( L^1 \). Therefore, the Fourier transform is in the sense of \( L^2 \): For all rapidly-decreasing Schwartz functions \( \varphi : \mathbb{R}^d \to \mathbb{R} \),
\[ \int_{\mathbb{R}^d} \varphi(x) \kappa_\alpha(x) \, dx = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{\varphi}(\xi) \hat{\kappa}_{d-\alpha}(\xi) \, d\xi. \]

**Proof.** We begin with the following identity: If \( \beta > -1 \) and \( \theta > 0 \), then
\[ \int_0^\infty e^{-t\|\xi\|^2} t^\beta \, dt = \frac{\Gamma(1 + \beta)}{\|\xi\|^{2+2\beta}}. \]

Therefore, consider an arbitrary function \( \varphi : \mathbb{R}^d \to \mathbb{R} \) such that \( \varphi \) and its Fourier transform decay faster than any polynomial at infinity. [This defines the class \( \mathcal{S}(\mathbb{R}^d) \) of rapidly-decreasing functions of L. Schwartz, and is dense in \( C(\mathbb{R}^d) \).] Then, we apply the preceding with \( \beta := (\alpha - 2)/2 \) to find that
\[ \int_{\mathbb{R}^d} \varphi(\xi) \|\xi\|^\alpha \, d\xi = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{(\alpha-2)/2} \left( \int_{\mathbb{R}^d} \varphi(\xi) e^{-t\|\xi\|^2} \, d\xi \right) \, dt. \]

We apply the Parseval identity to the middle integral to obtain
\[ \left( \int_{\mathbb{R}^d} \varphi(\xi) e^{-t\|\xi\|^2} \, d\xi \right) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{\varphi}(\xi) \frac{e^{-\|\xi\|^2/(2t)}}{(2\pi t)^{d/2}} \, d\xi. \]

We plug this back into (4) to find that
\[ \int_{\mathbb{R}^d} \varphi(\xi) \|\xi\|^\alpha \, d\xi = c_1 \int_0^\infty t^{(\alpha-2)/2} \left( \int_{\mathbb{R}^d} \hat{\varphi}(\xi) \frac{e^{-\|\xi\|^2/(2t)}}{(2\pi t)^{d/2}} \, d\xi \right) \, dt. \]
The decay properties of \( \varphi \) and the gaussian allow us to interchange the integrals once again. Thus, we find that

\[
\int_{\mathbb{R}^d} \frac{\varphi(\xi)}{\|\xi\|^\alpha} d\xi = c_1 \int_{\mathbb{R}^d} \overline{\varphi}(\xi) \left( \int_0^\infty t^{(\alpha-2)/2} \frac{e^{-\|\xi\|^2/(2t)}}{(2\pi t)^{d/2}} dt \right) d\xi. 
\]

A change of variables shows that

\[
\int_0^\infty t^{(\alpha-2)/2} \frac{e^{-\|\xi\|^2/(2t)}}{(2\pi t)^{d/2}} dt = \|\xi\|^{\alpha-d} \int_0^\infty s^{(\alpha-2)/2} \frac{e^{-1/(2s)}}{(2\pi s)^{d/2}} ds,
\]

which has the form \( c_2 \kappa_{d-\alpha}(\xi) \). Since \( \alpha \in (0, d) \), the last integral is finite. Let \( c_{d,\alpha} := (2\pi)^d c_1 c_2 \) to deduce (2) and hence the theorem. \( \square \)