Gaussian Processes

1. Basic Notions

Let T be a set, and $X := \{X_t\}_{t\in T}$ a stochastic process, defined on a suitable probability space $(\Omega, \mathcal{F}, \mathbb{P})$, that is indexed by T.

Definition 1.1. We say that X is a *Gaussian process indexed by T* when $(X_{t_1}, \dots, X_{t_n})$ is a Gaussian random vector for every $t_1, \dots, t_n \in T$ and $n \geqslant 1$. The distribution of X—that is the Borel measure $\mathbb{R}^T \ni A \mapsto \mu(A) := \mathbb{P}\{X \in A\}$—is called a *Gaussian measure*.

Lemma 1.2. *Suppose $X := (X_1, \dots, X_n)$ is a Gaussian random vector. If we set $T := \{1, \dots, n\}$, then the stochastic process $\{X_t\}_{t\in T}$ is a Gaussian process. Conversely, if $\{X_t\}_{t\in T}$ is a Gaussian process, then $(X_1, \dots, X_n)$ is a Gaussian random vector.*

The proof is left as exercise.

Definition 1.3. If X is a Gaussian process indexed by T, then we define $\mu(t) := \mathbb{E}(X_t)$ $[t \in T]$ and $C(s, t) := \mathrm{Cov}(X_s, X_t)$ for all $s, t \in T$. The functions μ and C are called the *mean* and *covariance* functions of X respectively.

Lemma 1.4. *A symmetric $n \times n$ real matrix C is the covariance of some Gaussian random vector if and only if C is positive semidefinite. The latter property means that*

$$z'Cz = \sum_{i=1}^{n} \sum_{j=1}^{n} z_i z_j C_{i,j} \geqslant 0 \qquad \textit{for all } z_1, \dots, z_n \in \mathbb{R}.$$

Proof. Consult any textbook on multivariate normal distributions. $\square$

Corollary 1.5. *A function $C : T \times T \to \mathbb{R}$ is the covariance function of some T-indexed Gaussian process if and only if $(C(t_i\,, t_j))_{1\leqslant i,j\leqslant n}$ is a positive semidefinite matrix for all $t_1, \ldots, t_n \in T$.*

Definition 1.6. From now on we will say that a function $C : T \times T \to \mathbb{R}$ is *positive semidefinite* when $(C(t_i\,, t_j))_{1\leqslant i,j\leqslant n}$ is a positive semidefinite matrix for all $t_1, \ldots, t_n \in T$.

Note that we understand the structure of every Gaussian process by looking only at finitely-many Gaussian random variables at a time. As a result, the theory of Gaussian processes does not depend *a priori* on the topological structure of the indexing set T. In this sense, the theory of Gaussian processes is quite different from Markov processes, martingales, etc. In those theories, it is essential that T is a totally-ordered set [such as $\mathbb{R}$ or $\mathbb{R}_+$], for example. Here, T can in principle be any set. Still, it can happen that X has particularly-nice structure when T is Euclidean, or more generally, has some nice group structure. We anticipate this possibility and introduce the following.

Definition 1.7. Suppose T is an abelian group and $\{X_t\}_{t\in T}$ a Gaussian process indexed by T. Then we use the additive notation for T, and say that X is *stationary* when $(X_{t_1}, \ldots, X_{t_k})$ and $(X_{s+t_1}\,, \ldots, X_{s+t_k})$ have the same law for all $s, t_1, \ldots, t_k \in T$.

Lemma 1.8. *Let T be an abelian group and let $X := \{X_t\}_{t\in T}$ denote a T-indexed Gaussian process with mean function m and covariance function C. Then X is stationary if and only if m and C are "translation invariant." That means that*

$$m(s+t) = m(t) \quad \text{and } C(t_1\,, t_2) = C(s+t_1\,, s+t_2) \text{ for all } s, t_1, t_2 \in \mathbb{R}^M.$$

The proof is left as exercise.

2. Examples of Gaussian Processes

§1. Brownian Motion. By *Brownian motion* X, we mean a Gaussian process, indexed by $\mathbb{R}_+ := [0\,,\infty)$, with mean function 0 and covariance function

$$C(s\,,t) := \min(s\,,t) \qquad [s,t \geqslant 0].$$

In order to justify this definition, it suffices to prove that C is a positive semidefinite function on $T \times T = \mathbb{R}^2_+$. Suppose $z_1, \ldots, z_n \in \mathbb{R}$ and

$t_1, \ldots, t_n \geqslant 0$. Then,

$$\sum_{i=1}^n \sum_{j=1}^n z_i \overline{z_j} C(t_i\,, t_j) = \sum_{i=1}^n \sum_{j=1}^n z_i \overline{z_j} \int_0^\infty \mathbb{1}_{[0,t_i]}(s) \mathbb{1}_{[0,t_j]}(s)\,\mathrm{d}s$$
$$= \int_0^\infty \left(\sum_{i=1}^n z_i \mathbb{1}_{[0,t_i]}(s) \right) \overline{\left(\sum_{j=1}^n z_j \mathbb{1}_{[0,t_j]}(s) \right)} \mathrm{d}s$$
$$= \int_0^\infty \left| \sum_{i=1}^n z_i \mathbb{1}_{[0,t_i]}(s)\,\mathrm{d}s \right|^2 \geqslant 0.$$

Therefore, Brownian motion exists.

§2. The Brownian Bridge. A *Brownian bridge* is a mean-zero Gaussian process, indexed by $[0\,,1]$, and with covariance

$$C(s\,,t) = \min(s\,,t) - st \qquad [0 \leqslant s, t \leqslant 1]. \tag{6.1}$$ Cov:BB

The most elegant proof of existence, that I am aware of, is due to J. L. Doob: Let B be a Brownian motion, and define

$$X_t := B_t - tB_1 \quad [0 \leqslant t \leqslant 1].$$

Then, $X := \{X_t\}_{0 \leqslant t \leqslant 1}$ is a mean-zero Gaussian process that is indexed by $[0\,,1]$ and has the covariance function of (6.1).

subsec:OU

§3. The Ornstein–Uhlenbeck Process. An *Ornstein–Uhlenbeck process* is a stationary Gaussian process X indexed by $\mathbb{R}_+$ with mean function 0 and covariance

$$C(s\,,t) = \mathrm{e}^{-|t-s|} \qquad [s, t \geqslant 0].$$

It remains to prove that C is a positive semidefinite function. The proof rests on the following well-known formula:[1]

$$\mathrm{e}^{-|x|} = \frac{1}{\pi} \int_{-\infty}^\infty \frac{\mathrm{e}^{ix\alpha}}{1+\alpha^2}\,\mathrm{d}\alpha \qquad [x \in \mathbb{R}]. \tag{6.2}$$ FT:Cauchy

Thanks to (6.2),

$$\sum_{j=1}^n \sum_{k=1}^n z_j \overline{z_k} C(t_j\,, t_k) = \frac{1}{\pi} \int_{-\infty}^\infty \frac{\mathrm{d}\alpha}{1+\alpha^2} \sum_{j=1}^n \sum_{k=1}^n z_j \overline{z_k} \mathrm{e}^{i\alpha(t_j - t_k)}$$
$$= \frac{1}{\pi} \int_{-\infty}^\infty \frac{\mathrm{d}\alpha}{1+\alpha^2} \left| \sum_{j=1}^n z_j \mathrm{e}^{i\alpha t_j} \right|^2 \geqslant 0.$$

[1] In other words, if Y has a standard Cauchy distribution on the line, then its characteristic function is $\mathbb{E}\exp(ixY) = \exp(-|x|)$.

§4. Brownian Sheet. An N-parameter *Brownian sheet* X is a Gaussian process, indexed by $\mathbb{R}_+^N := [0\,,\infty)^N$, whose mean function is zero and covariance function is

$$C(\boldsymbol{s}\,,\boldsymbol{t}) = \prod_{j=1}^{n} \min(s^j\,,t^j) \qquad [\boldsymbol{s} := (s^1,\ldots,s^N), \boldsymbol{t} := (t^1,\ldots,t^N) \in \mathbb{R}_+^N].$$

Clearly, a 1-parameter Brownian sheet is Brownian motion; in that case, the existence problem has been addressed. In general, we may argue as follows: For all $z_1,\ldots,z_n \in \mathbb{R}$ and $\boldsymbol{t}_1,\ldots,\boldsymbol{t}_n \in \mathbb{R}_+^N$,

$$\begin{aligned}\sum_{j=1}^n\sum_{k=1}^n z_jz_k\prod_{\ell=1}^N \min(s_j^\ell\,,s_k^\ell) &= \sum_{j=1}^n\sum_{k=1}^n z_jz_k\prod_{\ell=1}^N\int_0^\infty \mathbb{1}_{[0,s_j^\ell]}(r)\mathbb{1}_{[0,s_k^\ell]}(r)\,\mathrm{d}r\\ &= \sum_{j=1}^n\sum_{k=1}^n z_j\overline{z_k}\int_{\mathbb{R}_+^N}\prod_{\ell=1}^N \mathbb{1}_{[0,s_j^\ell]}(r^\ell)\mathbb{1}_{[0,s_k^\ell]}(r^\ell)\,\mathrm{d}r.\end{aligned}$$

It is harmless to take the complex conjugate of z_k since z_k is real valued. But now $z_k^{1/N} = \overline{z_k}^{1/N}$ is in general complex-valued, and we may write

$$\begin{aligned}\sum_{j=1}^n\sum_{k=1}^n z_jz_k\prod_{\ell=1}^N \min(s_j^\ell\,,s_k^\ell) &= \int_{\mathbb{R}_+^N}\sum_{j=1}^n\sum_{k=1}^n\prod_{\ell=1}^N (z_j\overline{z_k})^{1/N}\mathbb{1}_{[0,s_j^\ell]}(r^\ell)\mathbb{1}_{[0,s_k^\ell]}(r^\ell)\,\mathrm{d}r\\ &= \int_{\mathbb{R}_+^N}\left|\sum_{j=1}^n\prod_{\ell=1}^N z_j^{1/N}\mathbb{1}_{[0,s_j^\ell]}(r^\ell)\right|^2\mathrm{d}r \geqslant 0.\end{aligned}$$

This proves that the Brownian sheet exists.

§5. Fractional Brownian Motion. A *fractional Brownian motion* [or fBm] is a Gaussian process indexed by $\mathbb{R}_+$ that has mean function 0, $X_0 := 0$, and covariance function given by

$$\mathbb{E}(|X_t - X_s|^2) = |t-s|^{2\alpha} \qquad [s,t \geqslant 0], \tag{6.3}$$

Var:fBm

for some constant $\alpha > 0$. The constant α is called the *Hurst parameter* of X.

Note that (6.3) indeed yields the covariance function of X: Since $\operatorname{Var}(X_t) = \mathbb{E}(|X_t - X_0|^2) = t^{2\alpha}$,

$$|t-s|^{2\alpha} = \mathbb{E}\left(X_t^2 + X_s^2 - 2X_sX_t\right) = t^{2\alpha} + s^{2\alpha} - 2\operatorname{Cov}(X_s\,,X_t).$$

Therefore,

$$\operatorname{Cov}(X_s\,,X_t) = \frac{t^{2\alpha} + s^{2\alpha} - |t-s|^{2\alpha}}{2} \qquad [s,t \geqslant 0]. \tag{6.4}$$

Cov:fBm

Direct inspection shows that (6.4) does not define a positive-definite function C when $\alpha \leqslant 0$. This is why we have limited ourselves to the case that $\alpha > 0$.

Note that an fBm with Hurst index $\alpha = 1/2$ is a Brownian motion. The reason is the following elementary identity:

$$\frac{t+s-|t-s|}{2} = \min(s\,,t) \qquad [s,t \geqslant 0],$$

which can be verified by considering the cases $s \geqslant t$ and $t \geqslant s$ separately.

The more interesting "if" portion of the following is due to Mandelbrot and Van Ness (1968).

th:fBm:exists **Theorem 2.1.** *An fBm with Hurst index α exists if and only if $\alpha \leqslant 1$.*

Fractional Brownian motion with Hurst index $\alpha = 1$ is a trivial process in the following sense: Let N be a standard normal random variable, and define $X_t := tN$. Then, $X := \{X_t\}_{t \geqslant 0}$ is fBm with index $\alpha = 1$. For this reason, many experts do not refer to the $\alpha = 1$ case as fractional Brownian motion, and reserve the teminology fBm for the case that $\alpha \in (0\,,1)$. Also, fractional Brownian motion with Hurst index $\alpha = 1/2$ is Brownian motion.

Proof. First we examine the case that $\alpha < 1$. Our goal is to prove that

$$C(s\,,t) := \frac{t^{2\alpha} + s^{2\alpha} - |t-s|^{2\alpha}}{2}$$

is a covariance function.

Consider the function

$$\Phi(t\,,r) := (t-r)_+^{\alpha-(1/2)} - (-r)_+^{\alpha-(1/2)}, \tag{6.5}$$ Phi1

defined for all $t \geqslant 0$ and $r \in \mathbb{R}$, where $a_+ := \max(a\,,0)$ for all $a \in \mathbb{R}$. Direct inspection yields that $\int_{-\infty}^{\infty} [\Phi(t\,,r)]^2\,\mathrm{d}r < \infty$, since $\alpha < 1$, and in fact a second computation on the side yields

$$\int_{-\infty}^{\infty} \Phi(t\,,r)\Phi(s\,,r)\,\mathrm{d}r = \kappa C(s\,,t) \qquad \text{for all } s,t \geqslant 0, \tag{6.6}$$ Phi2

where κ is a positive and finite constant that depends only on α. In particular,

$$\begin{aligned}\sum_{i=1}^n \sum_{j=2}^n z_i z_j C(t_i\,,t_j) &= \frac{1}{\kappa} \sum_{i=1}^n \sum_{j=2}^n z_i z_j \int_{-\infty}^{\infty} \Phi(t_i\,,r)\Phi(t_j\,,r)\,\mathrm{d}r \\ &= \frac{1}{\kappa} \int_{-\infty}^{\infty} \left[\sum_{i=1}^n z_i \Phi(t_i\,,r)\right]^2 \mathrm{d}r \geqslant 0.\end{aligned}$$

This proves the Theorem in the case that $\alpha < 1$. We have seen already that theorem holds [easily] when $\alpha = 1$. Therefore, we now consider $\alpha > 1$, and strive to prove that fBm does not exist in this case.

The proof hinges on a technical fact which we state without proof; this and much more will be proved later on in Theorem 2.3 on page 97. Recall that $\bar{Y}$ is a *modification* of Y when $\mathbb{P}\{Y_t = \bar{Y}_t\} = 1$ for all t.

pr:KCT:Gauss

Proposition 2.2. *Let $Y := \{Y_t\}_{t\in[0,\tau]}$ denote a Gaussian process indexed by $T := [0\,,\tau]$, where $\tau > 0$ is a fixed constant. Suppose there exists a finite constant C and a constant $\eta > 0$ such that*

$$\mathbb{E}\left(|Y_t - Y_s|^2\right) \leqslant C|t-s|^\eta \qquad \textit{for all } 0 \leqslant s,t \leqslant \tau.$$

Then Y has a Hölder-continuous modification $\bar{Y}$. Moreover, for every non-random constant $\rho \in (0\,,\eta/2)$,

$$\sup_{0\leqslant s\neq t\leqslant\tau} \frac{|\bar{Y}_t - \bar{Y}_s|}{|t-s|^\rho} < \infty \qquad \textit{almost surely.} \tag{6.7}$$

eq:KCT:Gauss

We use Proposition 2.2 in the following way: Suppose to the contrary that there existed an fBm X with Hurst parameter $\alpha > 1$. By Proposition 2.2, X would have a continuous modification $\bar{X}$ such that for all $\rho \in (0\,,\alpha)$ and $\tau > 0$,

$$V(\tau) := \sup_{0\leqslant s\neq t\leqslant\tau} \frac{|\bar{X}_t - \bar{X}_s|}{|t-s|^\rho} < \infty \qquad \text{almost surely.}$$

Choose $\rho \in (1\,,\alpha)$ and observe that

$$\left|\bar{X}_t - \bar{X}_s\right| \leqslant V(\tau)|t-s|^\rho \qquad \text{for all } s,t \in [0\,,\tau],$$

almost surely for all $\tau > 0$. Divide both side by $|t-s|$ and let $s \to t$ in order to see that $\bar{X}$ is differentiable and its derivative is zero everywhere, a.s. Since $\bar{X}_0 = X_0 = 0$ a.s., it then follows that $\bar{X}_t = 0$ a.s. for all $t \geqslant 0$. In particular, $\mathbb{P}\{X_t = 0\} = 1$ for all $t \geqslant 0$. Since the variance of X_t is supposed to be $t^{2\alpha}$, we are led to a contradiction. □

subsec:WN

§6. White Noise and Wiener Integrals. Let $\mathbb{H}$ be a complex Hilbert space with norm $\|\ldots\|_{\mathbb{H}}$ and corresponding inner product $\langle\cdot\,,\cdot\rangle_{\mathbb{H}}$.

Definition 2.3. A *white noise* indexed by $T = \mathbb{H}$ is a Gaussian process $\{\xi(h)\}_{h\in\mathbb{H}}$, indexed by $\mathbb{H}$, with mean function 0 and covariance function,

$$C(h_1\,,h_2) = \langle h_1\,,h_2\rangle_{\mathbb{H}} \qquad [h_1,h_2 \in \mathbb{H}].$$

The proof of existence is fairly elementary: For all $z_1, \ldots, z_n \in \mathbb{R}$ and $h_1, \ldots, h_n \in \mathbb{H}$,

$$\begin{aligned}\sum_{j=1}^n \sum_{k=1}^n z_j z_k C(h_j\,, h_k) &= \sum_{j=1}^n \sum_{k=1}^n z_j z_k \langle h_j\,, h_k\rangle_{\mathbb{H}} \\ &= \left\langle \sum_{j=1}^n z_j h_j\,, \sum_{k=1}^n z_k h_k \right\rangle_{\mathbb{H}} = \left\| \sum_{j=1}^n z_j h_j \right\|_{\mathbb{H}}^2,\end{aligned}$$

which is clearly $\geqslant 0$.

The following simple result is one of the centerpieces of this section, and plays an important role in the sequel.

lem:WN:Lin **Lemma 2.4.** *For every $a_1, \ldots, a_m \in \mathbb{R}$ and $h_1, \cdots, h_m \in \mathbb{H}$,*

$$\xi\left(\sum_{j=1}^m a_j h_j\right) = \sum_{j=1}^m a_j \xi(h_j) \qquad \text{a.s.}$$

Proof. We plan to prove that: (a) For all $a \in \mathbb{R}$ and $h \in \mathbb{H}$,

$$\xi(ah) = a\xi(h) \qquad \text{a.s.;} \tag{6.8}$$ WN:Lin1

and (b) For all $h_1, h_2 \in \mathbb{H}$,

$$\xi(h_1 + h_2) = \xi(h_1) + \xi(h_2) \qquad \text{a.s.} \tag{6.9}$$ WN:Lin2

Together, (6.8) and (6.9) imply the lemma with $m = 2$; the general case follows from this case, after we apply induction. Let us prove (6.8) then:

$$\begin{aligned}\mathbb{E}\left(|\xi(ah) - a\xi(h)|^2\right) &= \mathbb{E}\left(|\xi(ah)|^2\right) + a^2\,\mathbb{E}\left(|\xi(h)|^2\right) - 2a\mathrm{Cov}\left(\xi(ah)\,, \xi(h)\right) \\ &= \|ah\|_{\mathbb{H}}^2 + a^2\|h\|_{\mathbb{H}}^2 - 2a\langle ah\,, h\rangle_{\mathbb{H}} = 0.\end{aligned}$$

This proves (6.8). As regards (6.9), we note that

$$\begin{aligned}&\mathbb{E}\left(|\xi(h_1 + h_2) - \xi(h_1) - \xi(h_2)|^2\right) \\ &= \mathbb{E}\left(|\xi(h_1 + h_2)|^2\right) + \mathbb{E}\left(|\xi(h_1) + \xi(h_2)|^2\right) - 2\mathrm{Cov}\left(\xi(h_1 + h_2)\,, \xi(h_1) + \xi(h_2)\right) \\ &= \|h_1 + h_2\|_{\mathbb{H}}^2 + \|h_1\|_{\mathbb{H}}^2 + \|h_2\|_{\mathbb{H}}^2 + 2\langle h_1\,, h_2\rangle_{\mathbb{H}} \\ &\qquad\qquad - 2\left[\langle h_1 + h_2\,, h_1\rangle_{\mathbb{H}} + \langle h_1 + h_2\,, h_2\rangle_{\mathbb{H}}\right] \\ &= \|h_1 + h_2\|_{\mathbb{H}}^2 - 2\langle h_1\,, h_2\rangle_{\mathbb{H}} - \|h_1\|_{\mathbb{H}}^2 - \|h_2\|_{\mathbb{H}}^2,\end{aligned}$$

which is zero, thanks to the Pythagorean rule on $\mathbb{H}$. This proves (6.9) an hence the lemma. □

Lemma 2.4 can be rewritten in the following essentially-equivalent form.

th:Wiener

Theorem 2.5 (Wiener)**.** *The map $\xi : \mathbb{H} \to L^2(\Omega, \mathcal{F}, \mathbb{P}) := L^2(\mathbb{P})$ is a linear Hilbert-space isometry.*

Because of its isometry property, white noise is also referred to as the *iso-normal* or *iso-gaussian* process.

Very often, the Hilbert space $\mathbb{H}$ is an L^2-space itself; say, $\mathbb{H} = L^2(\mu) := L^2(A, \mathcal{A}, \mu)$. Then, we can think of $\xi(h)$ as an $L^2(\mathbb{P})$-valued integral of $h \in \mathbb{H}$. In such a case, we sometimes adopt an integral notation; namely,

$$\int h(x)\, \xi(\mathrm{d}x) := \int h\, \mathrm{d}\xi := \xi(h).$$

This operation has all but one of the properties of integrals: The triangle inequality does not hold.[2]

Definition 2.6. The random variable $\int h\, \mathrm{d}\xi$ is called the *Wiener integral* of $h \in \mathbb{H} = L^2(\mu)$. One also defines definite Wiener integrals as follows: For all $h \in L^2(\mu)$ and $E \in \mathcal{A}$,

$$\int_E h(x)\, \xi(\mathrm{d}x) := \int_E h\, \mathrm{d}\xi := \xi(h \mathbb{1}_E).$$

This is a rational definition since $\|h\mathbb{1}_E\|_{L^2(\mu)} \leqslant \|h\|_{L^2(\mu)} < \infty$.

An important property of white noise is that, since it is a Hilbert-space isometry, it maps orthogonal elements of $\mathbb{H}$ to orthogonal elements of $L^2(\mathbb{P})$. In other words:

$$\mathbb{E}[\xi(h_1)\xi(h_2)] = 0 \quad \text{if and only if} \quad (h_1, h_2)_{\mathbb{H}} = 0.$$

Because $(\xi(h_1), \xi(h_2))$ is a Gaussian random vector of uncorrelated coordindates, we find that

$$\xi(h_1) \text{ and } \xi(h_2) \text{ are independent if and only if } (h_1, h_2)_{\mathbb{H}} = 0.$$

The following is a ready consequence of this rationale.

pr:uncorr:indep

Proposition 2.7. *If $\mathbb{H}_1, \mathbb{H}_2, \ldots$ are orthogonal subspaces of $\mathbb{H}$, then*

$$\{\xi(h)\}_{h \in \mathbb{H}_i} \qquad i = 1, 2, \ldots$$

are independent Gaussian processes.

The following highlights the strength of the preceding result.

pr:KL

Proposition 2.8. *Let $\{\psi_i\}_{i=1}^\infty$ be a complete orthonormal basis for $\mathbb{H}$. Then, we can find a sequence of i.i.d. standard normal random variables $X_1, X_2, \ldots$ such that*

$$\xi(h) = \sum_{j=1}^\infty c_j X_j,$$

[2] In fact, $|\xi(h)| \geqslant 0$ a.s., whereas $\xi(|h|)$ is negative with probability 1/2.

where $c_j := \langle h\,, \psi_j\rangle_{\mathbb{H}}$ *and the sum converges in* $L^2(\mathbb{P})$.

Remark 2.9. Proposition 2.8 yields a 1-1 identification of the white noise ξ with the i.i.d. sequence $\{X_i\}_{i=1}^{\infty}$. Therefore, in the setting of Proposition 2.8, some people refer to a sequence of i.i.d. standard normal random variables as white noise.

Proof. Thanks to Proposition 2.7, $X_j := \xi(\psi_j)$ defines an i.i.d. sequence of standard normal random variables. According to the Riesz–Fischer theorem

$$h = \sum_{j=1}^{\infty} c_j \psi_j \qquad \text{for every } h \in \mathbb{H},$$

where the sum converges in $\mathbb{H}$. Therefore, Theorem 2.5 ensures that

$$\xi(h) = \sum_{j=1}^{\infty} c_j \xi(\psi_j) = \sum_{j=1}^{\infty} c_j X_j \qquad \text{for every } h \in \mathbb{H},$$

where the sum converges in $L^2(\mathbb{P})$. We have implicitly used the following ready consequence of Wiener's isometry [Theorem 2.5]: If $h_n \to h$ in $\mathbb{H}$ then $\xi(h_n) \to \xi(h)$ in $L^2(\mathbb{P})$. It might help to recall that the reason is simply that $\|\xi(h_n - h)\|_{L^2(\mathbb{P})} = \|h_n - h\|_{\mathbb{H}}$. □

Next we work out a few examples of Hilbert spaces that arise in the literature.

Example 2.10 (Zero-Dimensional Hilbert Spaces)**.** We can identify $\mathbb{H} = \{0\}$ with a Hilbert space in a canonical way. In this case, white noise indexed by $\mathbb{H}$ is just a normal random variable with mean zero and variance 0 [i.e., $\xi(0) := 0$].

Example 2.11 (Finite-Dimensional Hilbert Spaces)**.** Choose and fix an integer $n \geqslant 1$. The space $\mathbb{H} := \mathbb{R}^n$ is a real Hilbert space with inner product $\langle a\,, b\rangle_{\mathbb{H}} := \sum_{j=1}^n a_j b_j$ and norm $\|a\|_{\mathbb{H}}^2 := \sum_{j=1}^n a_j^2$. Let ξ denote white noise indexed by $\mathbb{H} = \mathbb{R}^n$ and define a random vector $X := (X_1\,, \ldots, X_n)$ via

$$X_j := \xi(\mathbf{e}_j) \qquad j = 1, 2, \ldots, n,$$

where $\mathbf{e}_1 := (1\,, 0\,, \ldots, 0)', \ldots, \mathbf{e}_n := (0\,, \ldots 0\,, 1)'$ denote the usual orthonormal basis elements of $\mathbb{R}^n$. According to Proposition 2.8 and its proof, $X_1, \ldots, X_n$ are i.i.d. standard normal random variables and for every n-vector $a := (a_1\,, \ldots, a_n)$,

$$\xi(a) = \sum_{j=1}^{n} a_j X_j = a'X. \tag{6.10}$$

MVN

Now consider m points $a_1, \ldots, a_m \in \mathbb{R}^n$ and write the jth coordinate of a_i and a_i^j. Define

$$Y := \begin{pmatrix} \xi(a_1) \\ \vdots \\ \xi(a_m) \end{pmatrix}.$$

Then Y is a mean-zero Gaussian random vector with covariance matrix $A'A$ where A is an $m \times m$ matrix whose jth column is a_j. Then we can apply (6.10) to see that $Y = A'X$. In other words, every multivariate normal random vector with mean vector $\mathbf{0}$ and covariance matrix $A'A$ can be written as a linear combination $A'X$ of i.i.d. standard normals.

Example 2.12 (Lebesgue Spaces)**.** Consider the usual Lebesgue space $\mathbb{H} := L^2(\mathbb{R}_+)$. Since $\mathbb{1}_{[0,t]} \in L^2(\mathbb{R}_+)$ for all $t \geqslant 0$, we can define a mean-zero Gaussian process $B := \{B_t\}_{t \geqslant 0}$ by setting

$$B_t := \xi(\mathbb{1}_{[0,t]}) = \int_0^t \mathrm{d}\xi. \tag{6.11}$$

B:xi

Then, B is a Brownian motion because

$$\mathbb{E}[B_s B_t] = \left\langle \mathbb{1}_{[0,t]} \,, \mathbb{1}_{[0,s]} \right\rangle_{L^2(\mathbb{R}_+)} = \min(s\,, t).$$

Since $\mathbb{E}(|B_t - B_s|^2) = |t - s|$, Kolmgorov's continuity theorem [Proposition 2.2] shows that B has a continuous modification $\bar{B}$. Of course, $\bar{B}$ is also a Brownian motion, but it has continuous trajectories [Wiener's Brownian motion]. Some authors intepret (6.11) somewhat loosely and present white noise as the derivative of Brownian motion. This viewpoint can be made rigorous in the following way: White noise is the weak derivative of Brownian motion, in the sense of distribution theory. We will not delve into this matter further though.

I will close this example by mentioning, to those that know Wiener and Itô's theories of stochastic integration against Brownian motion, that the Wiener integral $\int_0^\infty \varphi_s \, \mathrm{d}B_s$ of a non-random function $\varphi \in L^2(\mathbb{R}_+)$ is the same object as $\int_0^\infty \varphi \, \mathrm{d}\xi = \xi(\varphi)$ here. Indeed, it suffices to prove this assertion when $\varphi_s = \mathbb{1}_{[0,t]}(s)$ for some fixed number $t > 0$. But then the assertion is just our definition (6.11) of the Brownian motion B.

Example 2.13 (Lebesgue Spaces, Continued)**.** Here is a fairly general receipe for constructing mean-zero Gaussian processes from white noise: Suppose we could write

$$C(s\,, t) = \int K(s\,, r) K(t\,, r)\, \mu(\mathrm{d}r) \qquad [s, t \in T],$$

where μ is a locally-finite measure on some measure space $(A\,, \mathcal{A})$, and $K : A \times T \to \mathbb{R}$ is a function such that $K(t\,, \bullet) \in L^2(\mu)$ for all $t \in T$. Then, the receipe is this: Let ξ be white noise on $\mathbb{H} := L^2(\mu)$, and define

$$X_t := \int K(t\,, r)\,\xi(\mathrm{d}r) \qquad [t \in T].$$

Then, $X := \{X_t\}_{t\in T}$ defines a mean-zero T-indexed Gaussian process with covariance function C. Here are some examples of how we can use this idea to build mean-zero Gaussian processes from white noise.

(1) Let $A := \mathbb{R}_+$, $\mu :=$ Lebesgue measure, and $K(t\,, r) := \mathbb{1}_{[0,t]}(r)$. These choices lead us to the same white-noise construction of Brownian motion as the previous example.

(2) Given a number $\alpha \in (0\,, 1)$, let ξ be a white noise on $\mathbb{H} := L^2(\mathbb{R})$. Because of (6.6) and our general discussion, earlier in this example, we find that

$$X_t := \frac{1}{\kappa}\int_{\mathbb{R}} \left[(t-r)_+^{\alpha-(1/2)} - (-r)_+^{\alpha-(1/2)}\right] \xi(\mathrm{d}r) \qquad [t \geqslant 0]$$

defines an fBm with Hurst index α.

(3) For a more interesting example, consider the covariance function of the Ornstein–Uhlenbeck process whose covariance function is, we recall,

$$C(s\,, t) = \mathrm{e}^{-|t-s|} \qquad [s, t \geqslant 0].$$

Define

$$\mu(\mathrm{d}a) := \frac{1}{\pi}\frac{\mathrm{d}a}{1+a^2} \qquad [-\infty < a < \infty].$$

According to (6.2), and thanks to symmetry,

$$\begin{aligned} C(s\,, t) &= \int \mathrm{e}^{i(t-s)r}\,\mu(\mathrm{d}r) = \int \cos(tr - sr)\,\mu(\mathrm{d}r) \\ &= \int \cos(tr)\cos(sr)\,\mu(\mathrm{d}r) + \int \sin(tr)\sin(sr)\,\mu(\mathrm{d}r). \end{aligned}$$

Now we follow our general discussion, let ξ and ξ' are two independent white noises on $L^2(\mu)$, and then define

$$X_t := \int \cos(tr)\,\xi(\mathrm{d}r) - \int \sin(tr)\,\xi'(\mathrm{d}r) \qquad [t \geqslant 0].$$

Then, $X := \{X_t\}_{t\geqslant 0}$ is an Ornstein–Uhlenbeck process.[3]

[3] One could just as easily put a plus sign in place of the minus sign here. The rationale for this particular way of writing is that if we study the "complex-valued white noise" $\zeta := \xi + i\xi'$, where ξ' is an independent copy of ξ, then $X_t = \mathrm{Re}\int \exp(itr)\,\zeta(\mathrm{d}r)$. A fully-rigorous discussion requires facts about "complex-valued" Gaussian processes, which I will not develop here.