Lévy Processes

Recall that a Lévy process \( \{X_t\}_{t \geq 0} \) on \( \mathbb{R}^d \) is a cadlag stochastic process on \( \mathbb{R}^d \) such that \( X_0 = 0 \) and \( X \) has i.i.d. increments. We say that \( X \) is continuous if \( t \mapsto X_t \) is continuous. On the other hand, \( X \) is pure jump if \( t \mapsto X_t \) can move only when it jumps [this is not a fully rigorous definition, but will be made rigorous en route the Itô–Lévy construction of Lévy processes].

**Definition 1.** If \( X \) is a Lévy process, then its tail sigma-algebra is \( \mathcal{T} := \cap_{t \geq 0} \sigma(\{X_{r+t} - X_t\}_{r \geq 0}) \).

The following is a continuous-time analogue of the Kolmogorov zero-one law for sequences of i.i.d. random variables.

**Proposition 2** (Kolmogorov zero-one law). The tail sigma algebra of a Lévy process is trivial; i.e., \( \mathbb{P}(\Lambda) \in \{0, 1\} \) for all \( \Lambda \in \mathcal{T} \).

The Lévy–Itô construction

The following is the starting point of the classification of Lévy processes, and is also known as the Lévy–Khintchine formula; compare with the other Lévy–Khintchine formula (Theorem 6).

**Theorem 3** (The Lévy–Khintchine formula; Itô, 1942; Lévy, 1934). For every Lévy exponent \( \Psi \) on \( \mathbb{R}^d \) there exists a Lévy process \( X \) such that for all \( t \geq 0 \) and \( \xi \in \mathbb{R}^d \),

\[
\mathbb{E} e^{i\xi X_t} = e^{-\Psi(\xi)}.
\]  

(1)

Conversely, if \( X \) is a Lévy process on \( \mathbb{R}^d \) then (1) is valid for a Lévy exponent \( \Psi \).
In words, the collection of all Lévy processes on $\mathbb{R}^d$ is in one-to-one correspondence with the family of all infinitely-divisible laws on $\mathbb{R}^d$.

We saw already that if $X$ is a Lévy process, then $X_t$ [in fact, $X_t$ for every $t \geq 0$] is infinitely divisible. Therefore, it remains to prove that if $\Psi$ is a Lévy exponent, then there is a Lévy process $X$ whose exponent is $\Psi$. The proof follows the treatment of Itô (1942), and is divided into two parts.

**Isolating the pure-jump part.** Let $B := \{B_t\}_{t \geq 0}$ be a $d$-dimensional Brownian motion, and consider the Gaussian process defined by

$$W_t := \sigma B_t - a t. \quad (t \geq 0).$$

A direct computation shows that $W := \{W_t\}_{t \geq 0}$ is a continuous Lévy process with Lévy exponent

$$\Psi^{(c)}(\xi) = ia'\xi + \frac{1}{2} \|\sigma \xi\|^2 \quad \text{for all } \xi \in \mathbb{R}^d.$$

[$W$ is a Brownian motion with drift $-a$, where the coordinates of $W$ are possibly correlated, unless $\sigma$ is diagonal.] Therefore, it suffices to prove the following:

**Proposition 4.** There exists a pure-jump Lévy process $Z$ with characteristic exponent

$$\Psi^{(d)}(\xi) := \int_{\mathbb{R}^d} \left(1 - e^{i\xi \cdot z} + i(\xi : z) \mathbf{1}_{[0,1]}(\|z\|)\right) m(dz),$$

for all $\xi \in \mathbb{R}^d$.

Indeed, if this were so, then we could construct $W$ and $Z$ independently from one another, and set

$$X_t = W_t + Z_t \quad \text{for all } t \geq 0.$$

This proves Theorem 3, since $\Psi = \Psi^{(c)} + \Psi^{(d)}$. In fact, together with Theorem 6, this implies the following:

**Theorem 5.** (1) The only continuous Lévy processes are Brownian motions with drift, and; (2) The continuous [i.e., Gaussian] and pure-jump parts of an arbitrary Lévy process are independent from one another.

Therefore, it suffices to prove Proposition 4.

**Proof of Proposition 4.** Consider the measurable sets

$$A_{-1} := \left\{ z \in \mathbb{R}^d : \|z\| \geq 1 \right\}, \quad \text{and} \quad A_n := \left\{ z \in \mathbb{R}^d : 2^{-n+1} \leq \|z\| < 2^{-n} \right\}.$$
as \( n \) varies over all nonnegative integers. Now we can define stochastic processes \( \{X^{(n)}_{\cdot}\}_{n=-1}^{\infty} \) as follows: For all \( t \geq 0 \),

\[
X^{(n)}_t := \int_{A_{-1}} x \, \Pi_t(dx), \quad X^{(n)}_t := \int_{A_n} x \, \Pi_t(dx) - tm(A_n) \quad (n \geq 0).
\]

Thanks to the construction of Lecture 5 (pp. 26 and on), \( \{X^{(n)}_{\cdot}\}_{n=-1}^{\infty} \) are independent Lévy processes, and for all \( n \geq 0, t \geq 0, \) and \( \xi \in \mathbb{R}^d \),

\[
E e^{ix \cdot X^{(n)}_t} = e^{-t} \int_{A_n} \left( 1 - e^{i\xi \cdot z} + i(\xi \cdot z)1_{[0,1]}(\|z\|) \right) m(dz).
\]

Moreover, \( X^{(-1)} \) is a compound Poisson process with parameters \( m(\bullet \cap A_{-1})/m(A_{-1}) \) and \( \lambda = m(A_{-1}) \), for all \( n \geq 0 \), \( X^{(n)} \) is a compensated compound Poisson process with parameters \( m(\bullet \cap A_n)/m(A_n) \) and \( \lambda = m(A_n) \).

Now \( Y^{(n)}_t := \sum_{k=0}^{n} X^{[k]}_t \) defines a Lévy process with exponent

\[
\psi_n(\xi) := \int_{1>|z|>2^{-n+1}} \left( 1 - e^{i\xi \cdot z} + i(\xi \cdot z)1_{[0,1]}(\|z\|) \right) m(dz),
\]

valid for all \( \xi \in \mathbb{R}^d \) and \( n \geq 1 \). Our goal is to prove that there exists a process \( Y \) such that for all nonrandom \( T > 0 \),

\[
\sup_{t \in [0,T]} \left\| Y^{(n)}_t - Y_t \right\| \to 0 \quad \text{in } L^2(\mathcal{F}). \tag{2}
\]

Because \( Y^{(n)} \) is caglad for all \( n \), uniform convergence shows that \( Y \) is cadlag for all \( n \). In fact, the jumps of \( Y^{(n+1)} \) contain those of \( Y^{(n)} \), and this proves that \( Y \) is pure jump. And because the finite-dimensional distributions of \( Y^{(n)} \) converge to those of \( Y \), it follows then that \( Y \) is a Lévy process, independent of \( X^{(-1)} \), and with characteristic exponent

\[
\psi_{\infty}(\xi) = \lim_{n \to \infty} \psi_n(\xi) = \int_{1>|z|} \left( 1 - e^{i\xi \cdot z} + i(\xi \cdot z)1_{[0,1]}(\|z\|) \right) m(dz).
\]

[The formula for the limit holds by the dominated convergence theorem.] Sums of independent Lévy processes are themselves Lévy. And their exponents add. Therefore, \( X^{(-1)} + Y_t \) is Lévy with exponent \( \Psi^{(d)} \).

It remains to prove the existence of \( Y \). Let us choose and fix some \( T > 0 \), and note that for all \( j, k \geq 1 \) and \( t \geq 0 \),

\[
Y^{(n+k)}_t - Y^{(n)}_t = \sum_{j=k+1}^{n+k} \left( \int_{A_j} x \, \Pi_t(dx) - tm(A_j) \right),
\]
and the summands are independent because the $A_j$’s are disjoint. Since the left-hand side has mean zero, it follows that

$$E \left( \left\| Y_t^{(n+k)} - Y_t^{(n)} \right\|^2 \right) = \sum_{j=k+1}^{n+k} E \left( \left\| \int_{A_j} x \Pi_t(dx) - tm(A_j) \right\|^2 \right)$$

$$\leq 2^{d-1} t \sum_{j=k+1}^{n+k} \int_{A_j} \|x\|^2 m(dx) = 2^{d-1} t \int_{\cup_{j=k+1}^{n+k} A_j} \|x\|^2 m(dx);$$

see Theorem 3. Every one-dimensional mean-zero Lévy process is a mean-zero martingale [in the case of Brownian motion we have seen this in Math. 6040; the reasoning in the general case is exactly the same]. Therefore, $Y_t^{(n+k)} - Y_t^{(n)}$ is a mean-zero cadlag martingale (coordinatewise). Doob’s maximal inequality tells us that

$$E \left( \sup_{t \in [0,T]} \left\| Y_t^{(n+k)} - Y_t^{(n)} \right\|^2 \right) \leq 2^{d+1} T \int_{2^{-k} \leq \|x\| < 2^{k+1}} \|x\|^2 m(dx).$$

This and the definition of a Lévy measure (p. 3) together imply (2), whence the result.

### Problems for Lecture 6

1. Prove the Kolmogorov 0-1 law (page 29).

2. Prove that every Lévy process $X$ on $\mathbb{R}^d$ is a strong Markov process. That is, for all finite stopping times $T$ [in the natural filtration of $X$], $t_1, \ldots, t_k \geq 0$, and $A_1, \ldots, A_k \in \mathcal{B}(\mathbb{R}^d),$

$$P \left( \bigcap_{i=1}^k \left\{ X_{t_i+t} - X_t \in A_j \right\} \mid \mathcal{F}_T \right) = P \left( \bigcap_{i=1}^k \left\{ X_{t_i} \in A_j \right\} \right) \text{ a.s.}$$

(Hint: Follow the Math. 6040 proof of the strong Markov property of Brownian motion.)