Some Examples

Our immediate goal is to see some examples of Lévy processes, and/or infinitely-divisible laws on \(\mathbb{R}^d\).

Uniform motion

Choose and fix a nonrandom \(a \in \mathbb{R}^d\) and define

\[ X_t := at \quad \text{for all } t \geq 0. \quad (1) \]

Then, \(\{X_t\}_{t \geq 0}\) is a [nonrandom] Lévy process with Lévy triple \((\mu, 0, 0)\). The process \(\{X_t\}_{t \geq 0}\) denotes uniform motion in the direction of \(a\).

Poisson processes on the real line

If \(N = \text{Poiss}(\lambda)\) for some \(\lambda > 0\), then

\[ \mathbb{E} e^{i \xi N} = \sum_{n=0}^{\infty} e^{i \xi n} \cdot \frac{e^{-\lambda} \lambda^n}{n!} = \exp \left( -\lambda \left( 1 - e^{i \xi} \right) \right). \quad (2) \]

That is, \(\mathbb{E} \exp(i \xi N) = \exp(-\Psi(\xi))\), where

\[ \Psi(\xi) = \int_{\mathbb{R}} \left( 1 - e^{i \xi z} - i \xi z \mathbf{1}_{(0,1)}(|z|) \right) m(dz), \]

and \(m(dz) := \lambda \delta_{\{1\}}(dz)\). The corresponding Lévy process is called the Poisson process with intensity parameter \(\lambda\).
Nonstandard Brownian motion with drift

The Lévy triple \((0, I, 0)\), where "I" denotes the \((d \times d)\) identity matrix, belongs to a vector of \(d\) i.i.d. standard-normal random variables, and the corresponding Lévy process is \([\text{standard}] \ d\)-dimensional Brownian motion. We can generalize this example easily: Choose and fix a vector \(a \in \mathbb{R}^d\), and a \((d \times d)\) matrix \(\sigma\), and consider the Lévy triple \((a, \sigma, 0)\). The corresponding Lévy exponent is

\[
\Psi(\xi) = i(a \cdot \xi) + \frac{1}{2} \|\sigma \xi\|^2.
\]

Therefore, \(\Psi\) is the Lévy exponent of random vector \(X\) in \(\mathbb{R}^d\) if and only if \(X = -a + \sigma Z\) where \(Z\) is a vector of \(d\) i.i.d. standard-normal random variables. The corresponding Lévy process is described by \(W_t := -\alpha t + \sigma B_t\), where \(B\) is standard Brownian motion \([\text{check}]\). The \(j\)th coordinate of \(W\) is a Brownian motion with mean \(-\alpha_j\) and variance \(\nu_j^2 := (\sigma \sigma^\top)_{j,j}\), and the coordinates of \(W\) are not in general independent. Since \(\lim_{t \to \infty} (W_t/t) = -\alpha\) a.s. by the law of large numbers for Brownian motion, \(-\alpha\) is called the "drift" of the nonstandard Brownian motion \(W\).

Isotropic stable laws

Choose and fix a number \(\alpha\). An isotropic stable law of index \(\alpha\) is the infinitely divisible distribution with Lévy exponent \(\Psi(\xi) = c \|\xi\|^\alpha\), where \(c \in \mathbb{R}\) is a fixed constant. The corresponding Lévy process is called the isotropic stable process with index \(\alpha\). We consider only random vectors with Lévy exponent \(\exp(-c \|\xi\|^\alpha)\) in this discussion.

Of course, \(c = 0\) leads to \(\Psi = 0\), which is the exponent of the infinitely divisible, but trivial, random variable \(X = 0\). Therefore, we study only \(c \neq 0\). Also, we need \(|\exp(-\Psi(\xi))| = \exp(-c \|\xi\|^\alpha) \leq 1\), and this means that \(c\) cannot be negative. Finally, \(\Psi(0) = 0\), and hence we have \(\alpha > 0\).

Lemma 1 (Lévy). \(\Psi(\xi) = c \|\xi\|^\alpha\) is a Lévy exponent iff \(\alpha \in (0, 2]\).

The case \(\alpha = 2\) is the Gaussian case we just saw. And \(\alpha = 1\) is also noteworthy; the resulting distribution is the "isotropic [or symmetric, in dimension one] Cauchy distribution" whose density is

\[
f(x) = \frac{\Gamma((d+1)/2)\alpha^{(d+1)/2}}{\pi^{d/2} \Gamma(d/2) \sigma^{d/2}} \left(1 + \frac{\|x\|^2}{\alpha^2}\right)^{-(d+1)/2} \quad \text{for all } x \in \mathbb{R}^d.
\]

Proof of Lemma 1. Exercise 8 below shows that \(|\Psi(\xi)| = O(\|\xi\|^2)\) as \(\|\xi\| \to \infty\); therefore, \(\alpha \leq 2\). Since \(\alpha = 2\) is Gaussian, we limit ourselves to \(\alpha < 2\).
Next let us consider \( \alpha \in (0, 2) \) and of course \( \alpha > 0 \) in that case. A change of variables shows that for all \( \xi \in \mathbb{R}^d \),

\[
\int_{\mathbb{R}^d} \left( 1 - e^{i \xi \cdot z} - i \langle \xi, z \rangle \mathbb{I}_{(0, 1)}(\|z\|) \right) \frac{dz}{\|z\|^{d+\alpha}} = \int_{\mathbb{R}^d} \left( 1 - \cos(\xi \cdot z) \right) \frac{dz}{\|z\|^{d+\alpha}} \propto \|\xi\|^\alpha.
\]

(The first identity is justified because the left-most integral is a radial function of \( \xi \), and hence real.) Therefore, we can choose \( C \in (0, \infty) \) so that

\[
m(\text{d}x) = C \|x\|^{-(d+\alpha)} \text{d}x
\]

is the Lévy measure with exponent \( \Psi(\xi) = \exp(-c \|\xi\|^\alpha) \) if \( \beta < \alpha \).

Sometimes, a reasonable knowledge of the Lévy exponent of an infinitely-divisible law yields insight into its structure. Here is a first example; Skorohod (1961) contains much more precise [and very useful] estimates of the tails of stable laws.

**Proposition 2.** If \( X \) has an isotropic stable distribution with \( \alpha \in (0, 2) \), then for all \( \beta > 0 \), \( E(\|X\|^{\beta}) < \infty \) if \( \beta < \alpha \).

In other words, except in the case that \( \alpha = 2 \), the decay of the tail of an isotropic stable law is slow. It is also possible to say more about the tails of the distribution (Theorem 14, page 50). But that requires a more sophisticated analysis.

**Proof.** Because \( E \exp(i z \cdot X) = \exp(-c \|z\|^\alpha) = E \cos(z \cdot X) \) for some \( c > 0 \), we may apply \( (4) \) in the following form:

\[
\int_{\mathbb{R}^d} \left( 1 - e^{i \xi \cdot z} \right) \frac{dz}{\|z\|^{d+\alpha}} = \int_{\mathbb{R}^d} \left( 1 - \cos(i \xi \cdot z) \right) \frac{dz}{\|z\|^{d+\alpha}} \propto \|\xi\|^\alpha.
\]

Replace \( \xi \) by \( X \) and take expectations to obtain

\[
E \left( \|X\|^\beta \right) \propto E \int_{\mathbb{R}^d} \left( 1 - \cos(z \cdot X) \right) \frac{dz}{\|z\|^{d+\beta}} = \int_{\mathbb{R}^d} \left( 1 - e^{-c \|z\|^\alpha} \right) \frac{dz}{\|z\|^{d+\beta}} \propto \int_0^\infty \left( 1 - e^{-r^\alpha} \right) \frac{dr}{r^{\beta+1}}.
\]

Now

\[
\int_1^\infty \left( 1 - e^{-r^\alpha} \right) \frac{dr}{r^{\beta+1}} \leq \int_1^\infty \frac{dr}{r^{\beta+1}} < \infty \quad \text{for every } \beta > 0.
\]

Therefore, \( E(\|X\|^{\beta}) < \infty \) if \( \int_0^1 (1 - e^{-r^\alpha}) r^{\beta+1} < \infty \). The result follows from this and the fact that \( (\theta/2) \leq 1 - e^{-\theta} \leq \theta \) for all \( \theta \in (0, 1) \). \( \square \)
The asymmetric Cauchy distribution on the line

We can specialize the preceding example to \( d = \alpha = 1 \) and obtain the symmetric Cauchy law \( \mu \) on the line. Of course the density of \( \mu \) is known as well.\(^1\) But more significantly, we have learned that the Lévy triple of \( \mu \) is \((0, 0, m)\), where \( m(dz) \propto z^{-2} dz \). This suggests that perhaps we can create an asymmetric variation of the Cauchy law by considering a Lévy triple of the form \((0, 0, m)\) for a Lévy measure of the form

\[
\frac{m(dz)}{dz} = \frac{c_1}{z^2} \mathbf{1}_{[0,\infty)}(z) + \frac{c_2}{z^2} \mathbf{1}_{(-\infty,0)}(z),
\]

where \( c_1 \neq c_2 \) are both positive and \( c_0 \) is selected carefully. This can in fact be done, as we will see next.

**Theorem 3.** For every \( c, c_1, c_2 > 0 \) and \( \theta \in [-2/\pi, 2/\pi] \) there exists \( c_0 \in \mathbb{R} \) and an infinitely-divisible Borel probability measure \( \mu \) on \( \mathbb{R} \) such that: (i) The Lévy triple of \( \mu \) has the form \((c_0, 0, m)\) for \( m \) as above; and (ii) \( \hat{\mu}(\xi) = \exp\{-c|\xi| - ic\theta \xi \log |\xi|\} \) for \( \xi \in \mathbb{R} \setminus \{0\}.\(^2\)

The measure \( \mu \) given above is called the Cauchy distribution on \( \mathbb{R} \) with asymmetry parameter \( \theta \) [and scale parameter \( c \)]. When \( \theta = 0 \), \( \mu \) is the symmetric Cauchy law. When \( |\theta| = 2/\pi \), \( \mu \) is called the completely asymmetric Cauchy law on \( \mathbb{R} \).

The Gamma distribution on the half line

It is easy to see that the Gamma \((\alpha, \lambda)\) distribution on \((0, \infty)\) is infinitely divisible for every \( \alpha, \lambda > 0 \). Next we identify its Lévy triple.

**Proposition 4.** If \( \mu \) is a Gamma \((\alpha, \lambda)\) distribution on \( \mathbb{R}_+ \) for some \( \alpha, \lambda > 0 \), then \( \mu \) is infinitely divisible with \( \alpha := 0, \alpha := 0, \) and Lévy measure \( m \) and Lévy exponent \( \Psi \) respectively given by

\[
m(dx) = \frac{\alpha e^{-\lambda x}}{x} dx \cdot \mathbf{1}_{[0,\infty)}(x), \quad \Psi(\xi) = \alpha \log \left( 1 - \frac{i\xi}{\lambda} \right),
\]

where “log” denotes the principle branch of the logarithm.

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1. It is \( f(x) = 1/[\pi(1 + x^2)] \) for \(-\infty < x < \infty\).
2. I am making a little fuss about \( \xi \notin \mathbb{R} \setminus \{0\} \), since the function \( \xi \log |\xi| \) is defined only for \( \xi \in \mathbb{R} \setminus \{0\} \). But of course \( \hat{\mu}(0) = 1 \) because \( \mu \) is a probability measure. Alternatively, we can define the function \( \xi \log |\xi| \) continuously on all of \([0, \infty)\) by letting \( \lim_{\xi \to 0} \xi \log |\xi| = 0 \). If so, then the stated formula for \( \hat{\mu}(\xi) \) is valid for all \( \xi \in \mathbb{R} \).
Adding independent Lévy processes

Finally, let me mention the following device which can be used to generate new Lévy processes from old ones:

Lemma 5. If \( \{X_t\}_{t \geq 0} \) and \( \{Y_t\}_{t \geq 0} \) are independent Lévy processes on \( \mathbb{R}^d \) with respective triples \( (\sigma, \alpha, m) \) and \( (\bar{\sigma}, \bar{\alpha}, \bar{m}) \), then \( \{X_t + Y_t\}_{t \geq 0} \) is a Lévy process on \( \mathbb{R}^d \) with Lévy triple \( (A, \Sigma, M) \), where \( A := a + \bar{a} \) and \( M := m + \bar{m} \), and \( \Sigma \) can be chosen in any fashion as long as \( \Sigma' \Sigma = \sigma' \sigma + \bar{\sigma}' \bar{\sigma} \).

This is elementary and can be checked by directly verifying the defining properties of Lévy processes. However, I emphasize that (i) there is always such a \( \Sigma \) and (ii) even though \( \Sigma \) might not be defined uniquely, \( \Sigma' \Sigma \) is.

Problems for Lecture 2

1. Prove that if \( q := \int_0^1 z^{-2}(z - \sin z) \, dz - \int_1^\infty z^{-2} \sin z \, dz \), then for all \( \xi > 0 \),

\[
\int_0^\infty \left( 1 - e^{i \xi x} + i \xi z I_{[0,1]}(z) \right) \frac{dz}{x^2} = \frac{\pi \xi^3}{2} + i \xi \ln \xi + i q \xi.
\]

Deduce Theorem 3 from this identity.

2. This problem outlines the proof of Proposition 4.

   (1) Suppose \( f : (0, \infty) \to \mathbb{R} \) has a continuous derivative \( f' \in L^1(\mathbb{R}) \), and \( f(0+) := \lim_{x \to 0} f(x) \) and \( f(\infty-) := \lim_{x \to \infty} f(x) \) exist and are finite. Prove that for all \( \lambda, \rho > 0 \),

\[
\int_0^\infty \left[ f(\lambda x) - f((\lambda + \rho) x) \right] \frac{dx}{x} = \left[ f(0+) - f(\infty-) \right] \ln \left( 1 + \frac{\rho}{\lambda} \right).
\]

   This is the Frullani integral identity.

   (2) Prove that for all \( \lambda > 0 \) and \( \xi \in \mathbb{R} \),

\[
1 - \frac{i \xi}{\lambda} = \exp \left( \int_0^\infty e^{-\lambda x} \left( 1 - e^{i \xi x} \right) \frac{dx}{x} \right),
\]

and deduce Proposition 4 from this identity.

3 (Stable scaling). Prove that if \( X \) is an isotropic stable process in \( \mathbb{R}^d \) with index \( \alpha \in (0, 2) \), then \( Y_r := R^{-\alpha} X_r \) defines a Lévy process for every \( R > 0 \) fixed. Explicitly compute the Lévy exponent of \( Y \). Is the same result true when \( R \) depends on \( f \)? What happens if you consider instead an asymmetric Cauchy process on the line?

\[\text{(5)}\text{This follows readily from the fact that } \sigma' \sigma + \bar{\sigma}' \bar{\sigma} \text{ is a nonnegative-definite matrix.}\]