Energy and Capacity

Polar and essentially-polar sets

Choose and fix a Borel set $G \subset \mathbf{R}^d$, and define the stopping time T_G to be the *entrance time* of G:

$$T_G := \inf\{s > 0 : X_s \in G \text{ or } X_{s-} \in G\} \qquad (\inf\varnothing := \infty). \tag{1}$$

In other words, T_G is the first time, if ever, that the closure of the range of the process X enters the set G.

Definition 1. A Borel set $G \subseteq \mathbf{R}^d$ is called *polar* if $\mathrm{P}\{T_G < \infty\} = 0$; otherwise G is said to be *nonpolar*. Similarly, G is called *essentially polar* if $\mathrm{P}\{T_{G-x} < \infty\} = 0$ for almost all $x \in \mathbf{R}^d$; otherwise G is deemed *essentially nonpolar*. □

We are abusing notation slightly; "essentially nonpolar" is being treated as an equivalent to "not essentially polar."

We can note that

$$\int_{\mathbf{R}^d} \mathrm{P}\{T_{G-x} < \infty\}\,\mathrm{d}x = \int_{\mathbf{R}^d} \mathrm{P}\left\{\overline{X(\mathbf{R}_+)} \cap (G - x) \neq \varnothing\right\}\mathrm{d}x.$$

But $\overline{X(\mathbf{R}_+)} \cap (G - x)$ is nonempty if and only if x is an element of $G \ominus \overline{X(\mathbf{R}_+)}$. Therefore, Fubini's theorem tells us that

$$G \text{ is essentially polar iff } \mathrm{E}\left|G \ominus \overline{X(\mathbf{R}_+)}\right| = 0.$$

Or equivalently,

$$G \text{ is essentially polar iff } \mathrm{E}\left|\overline{X(\mathbf{R}_+)} \ominus G\right| = 0.$$

(Why?) In particular, set $G := \{x\}$ to see that a singleton is essentially polar if and only if the range of $X(\mathbf{R}_+)$ has positive Lebesgue measure with positive probability. [This ought to seem familiar!]

Our goal is to determine all essentially-polar sets, and relate them to polar sets in most interesting cases. To this end define for all $\lambda > 0$ and Borel probability measures ν and μ on $\mathbf{R}^d$ the following:

$$\mathcal{E}_\lambda(\mu\,,\nu) := \frac{1}{(2\pi)^d}\int_{\mathbf{R}^d} \hat\mu(\xi)\,\overline{\hat\nu(\xi)}\,\mathrm{Re}\left(\frac{1}{\lambda+\Psi(\xi)}\right)\mathrm{d}\xi. \tag{2}$$

And if $\mu(\mathrm{d}x) = f(x)\,\mathrm{d}x$ and $\nu(\mathrm{d}x) = g(x)\,\mathrm{d}x$, then we may write $\mathcal{E}_\lambda(f\,,g)$ in place of $\mathcal{E}_\lambda(\mu\,,\nu)$ as well. Also define

$$\mathrm{Cap}_\lambda(G) := \left[\inf_{\mu\in M_1(G)} \mathcal{E}_\lambda(\mu\,,\mu)\right]^{-1}, \tag{3}$$

where $M_1(G)$ denotes the collection of all probability measures μ such that $\mu(G^c) = 0$, $\inf\varnothing := \infty$, and $\infty^{-1} := 0$.

Definition 2. $\mathcal{E}_\lambda(\mu\,,\nu)$ is called the *mutual λ-energy between μ and ν*, and $\mathrm{Cap}_\lambda(G)$ the *λ-capacity* of G. □

Our goal is to prove the following:

Theorem 3. *If $\mathrm{Cap}_\lambda(G) > 0$ then G is essentially nonpolar. And if $\mathrm{Cap}_\lambda(G) = 0$, then G is polar.*

Because of the preceding, we care mostly whether or not a given set G has positive λ-capacity. Therefore, let me remind you that $\mathrm{Cap}_\lambda(G) > 0$ if and only if there exists a probability measure μ, supported in G, such that $\int_{\mathbf{R}^d} |\hat\mu(\xi)|^2 \mathrm{Re}(1+\Psi(\xi))^{-1}\,\mathrm{d}\xi < \infty$.

Note that $\mathrm{Cap}_\lambda(G) = \mathrm{Cap}_\lambda(G+x)$ for all $x \in \mathbf{R}^d$. As a consequence of Theorem 3 we find then that G is polar if and only if $\mathrm{P}\{T_{G-x} < \infty\} = 0$ for all $x \in \mathbf{R}^d$. That is: (a) All polar sets are essentially polar; and (b) The difference between polarity and essential polarity is about at most a Lebesgue-null set of shifts of G. As the following shows, there is in fact no difference in almost all cases of interest.

Proposition 4. *Suppose U_λ is absolutely continuous for some $\lambda > 0$. Then, a Borel set G is essentially polar if and only if it is polar.*

An energy identity

Theorem 5 (Foondun and Khoshnevisan, 2010, Corollary 3.7)**.** *If f is a probability density on $\mathbf{R}^d$, then*

$$\int_{\mathbf{R}^d} (R_\lambda f)(x)f(x)\,\mathrm{d}x = \mathcal{E}_\lambda(f\,,f) \qquad \textit{for all } \lambda > 0. \tag{4}$$

Proof. If $f \in C_0(\mathbf{R}^d)$ with $\hat{f} \in L^1(\mathbf{R}^d)$, then (4) follows from direct computations. Indeed, we can use the fact that $\hat{u}_\lambda(\xi) = \mathrm{Re}(\lambda + \Psi(\xi))^{-1} \geq 0$ [see (2, p. 63)] together with Fubini's theorem and find that

$$\int_{\mathbf{R}^d} (R_\lambda f)(x) f(x)\,\mathrm{d}x = \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} |\hat{f}(\xi)|^2 \mathrm{Re}\left(\frac{1}{\lambda + \Psi(\xi)}\right) \mathrm{d}\xi. \tag{5}$$

But in the present case, Fubini's theorem is not applicable. Instead, we proceed in two steps: First we prove that

$$\int_{\mathbf{R}^d} (R_\lambda f)(x) f(x)\,\mathrm{d}x \geq \mathcal{E}_\lambda(f\,, f). \tag{6}$$

This holds trivially unless the left-hand side is finite, which we now assume is the case. Because f is a density function, Lusin's theorem tells us that for all $\delta > 0$ there exists a compact set $K_\delta \subset \mathbf{R}^d$ such that

$$\int_{K_\delta^c} f(x)\,\mathrm{d}x \leq \delta, \quad \text{and} \quad R_\lambda f \text{ is continuous on } K_\delta.$$

In particular,

$$\int_{\mathbf{R}^d} (R_\lambda f)(x) f(x)\,\mathrm{d}x \geq \int_{K_\delta} (R_\lambda f)(x) f(x)\,\mathrm{d}x = \lim_{\epsilon \downarrow 0} \int_{K_\delta} \left((R_\lambda f) * \varphi_\epsilon\right)(x) f(x)\,\mathrm{d}x,$$

where φ_ϵ denotes the density of B_ϵ for a d-dimensional Brownian motion B. Let $f_\delta := f\mathbb{1}_{K_\delta}$ and note that $\hat{f}_\delta \to \hat{f}$, pointwise, as $\delta \downarrow 0$.

Since $(R_\lambda f) * \varphi_\epsilon = R_\lambda(f * \varphi_\epsilon) \geq R_\lambda(f_\delta * \varphi_\epsilon)$ and $\varphi_\epsilon = \varphi_{\epsilon/2} * \varphi_{\epsilon/2}$, we can apply Tonelli's theorem to find that

$$\begin{aligned}
\int_{\mathbf{R}^d} (R_\lambda f)(x) f(x)\,\mathrm{d}x &\geq \liminf_{\epsilon \downarrow 0} \int_{\mathbf{R}^d} \left(R_\lambda(f_\delta * \varphi_\epsilon)\right)(x) f_\delta(x)\,\mathrm{d}x \\
&= \liminf_{\epsilon \downarrow 0} \int_{\mathbf{R}^d} \left(R_\lambda(f_\delta * \varphi_{\epsilon/2})\right)(x) \left(f_\delta * \varphi_{\epsilon/2}\right)(x)\,\mathrm{d}x \\
&= \frac{1}{(2\pi)^d} \liminf_{\epsilon \downarrow 0} \int_{\mathbf{R}^d} |\hat{f}_\delta(\xi)|^2 \mathrm{e}^{-\epsilon\|\xi\|^2/2} \mathrm{Re}\left(\frac{1}{\lambda + \Psi(-\xi)}\right) \mathrm{d}\xi,
\end{aligned}$$

thanks to (5). This proves that

$$\int_{\mathbf{R}^d} (R_\lambda f)(x) f(x)\,\mathrm{d}x \geq \frac{1}{(2\pi)^d} \liminf_{\delta \downarrow 0} \int_{\mathbf{R}^d} |\hat{f}_\delta(\xi)|^2 \mathrm{Re}\left(\frac{1}{1 + \Psi(\xi)}\right) \mathrm{d}\xi,$$

and Fatou's lemma proves (6). The converse bound is much easier: We merely note that, as above,

$$\begin{aligned}
\int_{\mathbf{R}^d} \left(R_\lambda(f * \varphi_\epsilon)\right)(x) f(x)\,\mathrm{d}x &= \int_{\mathbf{R}^d} \left(R_\lambda(f * \varphi_{\epsilon/2})\right)(x) \left(f * \varphi_{\epsilon/2}\right)(x)\,\mathrm{d}x \\
&= \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} |\hat{f}(\xi)|^2 \mathrm{e}^{-\epsilon\|\xi\|^2/2} \mathrm{Re}\left(\frac{1}{\lambda + \Psi(\xi)}\right) \mathrm{d}\xi.
\end{aligned}$$

Then we let $\epsilon \downarrow 0$; the right-most term converges to $\mathscr{E}_\lambda(f\,,f)$ by the dominated convergence theorem, and the $\liminf$ of the left-most term is at most $\int_{\mathbf{R}^d}(R_\lambda f)(x)f(x)\,\mathrm{d}x$ by Fatou's lemma. □

Proof of Theorem 3

Theorem 3 will follow immediately from Lemmas 7 and 9 below.

Define

$$(J_\lambda f)(x) := \int_0^\infty \mathrm{e}^{-\lambda s} f(x + X_s)\,\mathrm{d}s.$$

Lemma 6. *For all $f \in L^1(\mathbf{R}^d) \cap L^2(\mathbf{R}^d)$ and $\lambda > 0$,*

$$\int_{\mathbf{R}^d} \mathrm{E}\left[(J_\lambda f)(x)\right]\,\mathrm{d}x = \frac{1}{\lambda}, \qquad \int_{\mathbf{R}^d} \mathrm{E}\left(|(J_\lambda f)(x)|^2\right)\,\mathrm{d}x = \frac{1}{\lambda}\mathscr{E}_\lambda(f\,,f).$$

Proof. The first computation follows because f is a probability density and hence $\int_{\mathbf{R}^d}(J_\lambda f)(x)\,\mathrm{d}x = \lambda^{-1}$. Now we begin with the second computation of the lemma:

$$\begin{aligned}
\mathrm{E}\left(|(J_\lambda f)(x)|^2\right) &= 2\int_0^\infty \mathrm{e}^{-\lambda s}\,\mathrm{d}s \int_s^\infty \mathrm{e}^{-\lambda t}\,\mathrm{d}t\ \mathrm{E}\left[f(x + X_s)\cdot f(x + X_t)\right]\\
&= 2\int_0^\infty \mathrm{e}^{-\lambda s}\,\mathrm{d}s \int_s^\infty \mathrm{e}^{-\lambda t}\,\mathrm{d}t\ \mathrm{E}\left[f(x + X_s)\cdot (P_{t-s}f)(x + X_s)\right]\\
&= 2\int_0^\infty \mathrm{e}^{-2\lambda s}\mathrm{E}\left[f(x + X_s)\cdot (R_\lambda f)(x + X_s)\right]\,\mathrm{d}s,
\end{aligned}$$

thanks to the Markov property. Therefore,

$$\int_{\mathbf{R}^d} \mathrm{E}\left(|(J_\lambda f)(x)|^2\right)\,\mathrm{d}x = \frac{1}{\lambda}\int_{\mathbf{R}^d} f(y)\cdot(R_\lambda f)(y)\,\mathrm{d}y.$$

And the lemma follows from Theorem 5. □

Lemma 7. *Regardless of the value of $\lambda > 0$,*

$$\mathrm{E}\left(|G \ominus X(\mathbf{R}_+)|\right) = \int_{\mathbf{R}^d} \mathrm{P}\left\{T_{G-x} < \infty\right\}\,\mathrm{d}x \geq \frac{1}{\lambda}\cdot \mathrm{Cap}_\lambda(G).$$

Remark 8. It is important to note that $T_{G-x} < \infty$ if and only if the Lévy process$x + X_t$ [which starts at $x \in \mathbf{R}^d$ at time zero] everh hits G; more precisely, there exists $t > 0$ such that $x + X_t \in G$ or $x + X_{t-} \in G$. Therefore, the preceding states that if G has positive λ-capacity, then X hits G, starting from almost every starting point $x \in \mathbf{R}^d$. In fact, this property is one way of thinking about the essential nonpolarity of G. □

Proof. Let us begin with a simple fact from classical function theory.

The Paley–Zygmund inequality.[1] *Suppose* $Y : \mathbf{R}^d \times \Omega \to \mathbf{R}_+$ *is nonnegative and measurable,* $Y \not\equiv 0$, $Y \in L^2(\mathbf{R}^d \times \Omega)$, *and* $\int_{\mathbf{R}^d} \mathrm{E}Y(x)\,\mathrm{d}x = c > 0$. *Then,*

$$\int_{\mathbf{R}^d} \mathrm{P}\left\{Y(x) > 0\right\} \mathrm{d}x \geq \frac{c^2}{\int_{\mathbf{R}^d} \mathrm{E}\left(|Y(x)|^2\right) \mathrm{d}x},$$

where $1/\infty := 0$.

Let f be a probability density that is supported on the closed [say] ϵ-enlargment G^ϵ of G. We apply Lemma 6 together with the Paley-Zygmund inequality [with $Y(x) := (J_\lambda f)(x)$] and obtain

$$\int_{\mathbf{R}^d} \mathrm{P}\left\{(J_\lambda f)(x) > 0\right\} \mathrm{d}x \geq \frac{1}{\lambda \cdot \mathcal{E}_\lambda(f\,, f)}.$$

If $(J_\lambda f)(x) > 0$, then certainly $x + X_s \in G^\epsilon$ for some $s > 0$; i.e., $T_{G^\epsilon - x} < \infty$. Therefore,

$$\int_{\mathbf{R}^d} \mathrm{P}\left\{T_{G^\epsilon - x} < \infty\right\} \mathrm{d}x \geq \frac{1}{\lambda} \cdot \sup_g \frac{1}{\mathcal{E}_\lambda(g\,, g)},$$

where the supremum is taken over all probability densities g that are supported on G^ϵ. Let h_ϵ be a probability density, supported on $B(0\,, \epsilon)$, and observe that $\rho * h_\epsilon$ is a probability density supported on G^ϵ whenever $\rho \in M_1(G)$. Because of (2), $\mathcal{E}_\lambda(\rho * h_\epsilon\,, \rho * h_\epsilon) \leq \mathcal{E}_\lambda(\rho\,, \rho)$, and hence

$$\int_{\mathbf{R}^d} \mathrm{P}\left\{T_{G^\epsilon - x} < \infty\right\} \mathrm{d}x \geq \frac{1}{\lambda} \cdot \mathrm{Cap}_\lambda(G).$$

Note that

$$\bigcap_{\epsilon > 0} \left\{T_{G^\epsilon - x} < \infty\right\} = \bigcap_{\epsilon > 0} \left\{x + X(\mathbf{R}_+) \cap G^\epsilon \neq \varnothing\right\} = \left\{x + \overline{X(\mathbf{R}_+)} \cap G \neq \varnothing\right\}.$$

Therefore,

$$\int_{\mathbf{R}^d} \mathrm{P}\left\{x + \overline{X(\mathbf{R}_+)} \cap G \neq \varnothing\right\} \mathrm{d}x \geq \frac{1}{\lambda} \cdot \mathrm{Cap}_\lambda(G).$$

Now the left-hand side is the expectation of the Lebesgue measure of the random set $G \ominus \overline{X(\mathbf{R}_+)}$ [check!]. Because X is cadlag, the set difference between $X(\mathbf{R}_+)$ and its closure has zero measure (in fact, is countable).

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[1] Here is the proof: By the Cauchy–Schwarz inequality,

$$\begin{aligned} c = \int_{\mathbf{R}^d} \mathrm{E}f(x)\,\mathrm{d}x &= \int_{\mathbf{R}^d \times \Omega} \mathbb{1}_{\{f>0\}}(x\,, \omega) \cdot f(x\,, \omega)\,\mathrm{d}x\,\mathrm{P}(\mathrm{d}\omega) \\ &\leq \left(\int_{\mathbf{R}^d \times \Omega} \mathbb{1}_{\{f>0\}}(x\,, \omega)\,\mathrm{d}x\,\mathrm{P}(\mathrm{d}\omega) \cdot \int_{\mathbf{R}^d \times \Omega} |f(x\,, \omega)|^2\,\mathrm{d}x\,\mathrm{P}(\mathrm{d}\omega)\right)^{1/2} \\ &= \left(\int_{\mathbf{R}^d} \mathrm{P}\{f(x) > 0\}\,\mathrm{d}x\right)^{1/2} \cdot \left(\int_{\mathbf{R}^d} \mathrm{E}\left(|f(x)|^2\right) \mathrm{d}x\right)^{1/2}. \end{aligned}$$

□

Therefore, the Lebesgue measure of $G \ominus \overline{X(\mathbf{R}_+)}$ is the same as the Lebesgue measure of $G \ominus X(\mathbf{R}_+)$. This proves the result. □

Lemma 9. $\mathrm{P}\{T_G \le n\} \le \mathrm{e}^{\lambda n} \cdot \mathrm{Cap}_\lambda(G)$ *for all* $n, \lambda > 0$.

Proof. This is trivial unless

$$\mathrm{P}\left\{T_G \le n\right\} > 0, \tag{7}$$

which we assume is the case.

For all measurable $f : \mathbf{R}^d \to \mathbf{R}_+$,

$$\begin{aligned}\mathrm{E}\left((J_\lambda f)(0) \mid \mathcal{F}_{T_G \wedge n}\right) &\ge \int_{T_G}^\infty \mathrm{e}^{-\lambda s} \mathrm{E}\left[f(X_s) \,\middle|\, \mathcal{F}_{T_G \wedge n}\right] \mathrm{d}s \cdot \mathbb{1}_{\{T_G \le n\}} \\ &= \mathrm{e}^{-\lambda n} \int_0^\infty \mathrm{e}^{-\lambda s} \mathrm{E}\left[f(X_{s+T_G\wedge n}) \,\middle|\, \mathcal{F}_{T_G}\right] \mathrm{d}s \cdot \mathbb{1}_{\{T_G \le n\}} \\ &= \mathrm{e}^{-\lambda n} \int_0^\infty \mathrm{e}^{-\lambda s} (P_s f)(X_{T_G \wedge n})\, \mathrm{d}s \cdot \mathbb{1}_{\{T_G \le n\}},\end{aligned}$$

thanks to the strong Markov property. Therefore,

$$\mathrm{E}\left((J_\lambda f)(0) \mid \mathcal{F}_{T_G \wedge n}\right) \ge \mathrm{e}^{-\lambda n} (R_\lambda f)(X_{T_G}) \cdot \mathbb{1}_{\{T_G \le n\}}.$$

The expectation of the term on the left-hand side is 1, thanks to Lemma 6 and the optional stopping theorem. Therefore,

$$\begin{aligned}1 &\ge \mathrm{e}^{-\lambda n} \mathrm{E}\left[(R_\lambda f)(X_{T_G}) \,\middle|\, T_G \le n\right] \cdot \mathrm{P}\{T_G \le n\} \\ &= \mathrm{e}^{-\lambda n} \int_{\mathbf{R}^d} (R_\lambda f)\, \mathrm{d}\rho \cdot \mathrm{P}\{T_G \le n\},\end{aligned}$$

where $\rho(A) := \mathrm{P}(X_{T_G} \in A \mid T_G \le n)$. In accord with (7), $\rho \in M_1(G)$.

We apply the preceding with $f := \rho * \varphi_\epsilon$, where φ_ϵ denotes the density of B_ϵ for a d-dimensional Brownian motion. Because

$$\int_{\mathbf{R}^d} (R_\lambda f)\, \mathrm{d}\rho = \int_{\mathbf{R}^d} \left(R_\lambda(\rho * \varphi_{\epsilon/2})\right)(x) \left(\rho * \varphi_{\epsilon/2}\right)(x)\, \mathrm{d}x,$$

it follows from Theorem 5 that

$$\mathrm{e}^{\lambda n} \ge \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} \mathrm{e}^{-\epsilon\|\xi\|^2/2} |\hat\rho(\xi)|^2 \mathrm{Re}\left(\frac{1}{\lambda + \Psi(\xi)}\right) \mathrm{d}\xi \cdot \mathrm{P}\left\{T_G \le n\right\}.$$

Let $\epsilon \downarrow 0$ and appeal to the monotone convergence theorem to finish. □

Problems for Lecture 13

1. Prove that $\mathrm{Cap}_\lambda(G) > 0$ for some $\lambda > 0$ iff $\mathrm{Cap}_\lambda(G) > 0$ for all $\lambda > 0$.

2. Prove Proposition 4. (Hint: Inspect the proof of Theorem 10 on page 82.)