Energy and Capacity

Polar and essentially-polar sets

Choose and fix a Borel set $G \subset \mathbb{R}^d$, and define the stopping time $T_G$ to be the entrance time of $G$:

$$T_G := \inf \{ s > 0 : X_s \in G \text{ or } X_{s^-} \in G \} \quad (\inf \emptyset := \infty). \quad (1)$$

In other words, $T_G$ is the first time, if ever, that the closure of the range of the process $X$ enters the set $G$.

**Definition 1.** A Borel set $G \subseteq \mathbb{R}^d$ is called polar if $P\{T_G < \infty\} = 0$; otherwise $G$ is said to be nonpolar. Similarly, $G$ is called essentially polar if $P\{T_{G-x} < \infty\} = 0$ for almost all $x \in \mathbb{R}^d$; otherwise $G$ is deemed essentially nonpolar.

We are abusing notation slightly; “essentially nonpolar” is being treated as an equivalent to “not essentially polar.”

We can note that

$$\int_{\mathbb{R}^d} P\{T_{G-x} < \infty\} \, dx = \int_{\mathbb{R}^d} P\left\{ X(\mathbb{R}_+) \cap (G-x) \neq \emptyset \right\} \, dx.$$ 

But $X(\mathbb{R}_+) \cap (G-x)$ is nonempty if and only if $x$ is an element of $G \ominus X(\mathbb{R}_+)$. Therefore, Fubini’s theorem tells us that

$G$ is essentially polar iff $E\left| G \ominus X(\mathbb{R}_+) \right| = 0.$

Or equivalently,

$G$ is essentially polar iff $E\left| X(\mathbb{R}_+) \ominus G \right| = 0.$
In particular, set $G := \{x\}$ to see that a singleton is essentially polar if and only if the range of $X(R_+)$ has positive Lebesgue measure with positive probability. [This ought to seem familiar!]

Our goal is to determine all essentially-polar sets, and relate them to polar sets in most interesting cases. To this end define for all $\lambda > 0$ and Borel probability measures $\nu$ and $\mu$ on $\mathbb{R}^d$ the following:

$$\mathcal{E}_\lambda(\mu, \nu) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{\mu}(\xi) \overline{\hat{\nu}(\xi)} \text{Re} \left( \frac{1}{\lambda + \Psi(\xi)} \right) d\xi,$$

And if $\mu(dx) = f(x) \, dx$ and $\nu(dx) = g(x) \, dx$, then we may write $\mathcal{E}_\lambda(f, g)$ in place of $\mathcal{E}_\lambda(\mu, \nu)$ as well. Also define

$$\text{Cap}_\lambda(G) := \left[ \inf_{\mu \in M_1(G)} \mathcal{E}_\lambda(\mu, \mu) \right]^{-1},$$

where $M_1(G)$ denotes the collection of all probability measures $\mu$ such that $\mu(G^c) = 0$, $\inf \sigma := \infty$, and $\infty^{-1} := 0$.

**Definition 2.** $\mathcal{E}_\lambda(\mu, \nu)$ is called the mutual $\lambda$-energy between $\mu$ and $\nu$, and $\text{Cap}_\lambda(G)$ the $\lambda$-capacity of $G$.

Our goal is to prove the following:

**Theorem 3.** If $\text{Cap}_\lambda(G) > 0$ then $G$ is essentially nonpolar. And if $\text{Cap}_\lambda(G) = 0$, then $G$ is polar.

Because of the preceding, we care mostly whether or not a given set $G$ has positive $\lambda$-capacity. Therefore, let me remind you that $\text{Cap}_\lambda(G) > 0$ if and only if there exists a probability measure $\mu$, supported in $G$, such that $\int_{\mathbb{R}^d} |\hat{\mu}(\xi)|^2 \text{Re}(1 + \Psi(\xi))^{-1} d\xi < \infty$.

Note that $\text{Cap}_\lambda(G) = \text{Cap}_\lambda(G + x)$ for all $x \in \mathbb{R}^d$. As a consequence of Theorem 3 we find then that $G$ is polar if and only if $\mathbb{P}\{T_{G-x} < \infty\} = 0$ for all $x \in \mathbb{R}^d$. That is: (a) All polar sets are essentially polar; and (b) The difference between polarity and essential polarity is about at most a Lebesgue-null set of shifts of $G$. As the following shows, there is in fact no difference in almost all cases of interest.

**Proposition 4.** Suppose $U_\lambda$ is absolutely continuous for some $\lambda > 0$. Then, a Borel set $G$ is essentially polar if and only if it is polar.

**An energy identity**

**Theorem 5** (Foondun and Khoshnevisan, 2010, Corollary 3.7). If $f$ is a probability density on $\mathbb{R}^d$, then

$$\int_{\mathbb{R}^d} (R_\lambda f)(x)f(x) \, dx = \mathcal{E}_\lambda(f, f) \quad \text{for all } \lambda > 0.$$
Proof. If \( f \in C_0(\mathbb{R}^d) \) with \( \hat{f} \in L^1(\mathbb{R}^d) \), then (4) follows from direct computations. Indeed, we can use the fact that \( \hat{u}_{\lambda}(\xi) = \text{Re}(\lambda + \Psi(\xi))^{-1} \geq 0 \) (see [2, p. 63]) together with Fubini’s theorem and find that

\[
\int_{\mathbb{R}^d} (R_{\lambda}f)(x)f(x) \, dx = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{f}(\xi)^2 \text{Re} \left( \frac{1}{\lambda + \Psi(\xi)} \right) \, d\xi. \tag{5}
\]

But in the present case, Fubini’s theorem is not applicable. Instead, we proceed in two steps: First we prove that

\[
\int_{\mathbb{R}^d} (R_{\lambda}f)(x)f(x) \, dx \geq \varepsilon \delta_{\lambda}(f,f).
\]

This holds trivially unless the left-hand side is finite, which we now assume is the case. Because \( f \) is a density function, Lusin’s theorem tells us that for all \( \delta > 0 \) there exists a compact set \( K_\delta \subset \mathbb{R}^d \) such that

\[
\int_{K_\delta^c} f(x) \, dx \leq \delta, \quad \text{and} \quad R_{\lambda}f \text{ is continuous on } K_\delta.
\]

In particular,

\[
\int_{\mathbb{R}^d} (R_{\lambda}f)(x)f(x) \, dx \geq \int_{K_\delta} (R_{\lambda}f)(x)f(x) \, dx = \lim_{\varepsilon \downarrow 0} \int_{K_\delta} [(R_{\lambda}f) * \varphi_\varepsilon](x)f(x) \, dx,
\]

where \( \varphi_\varepsilon \) denotes the density of \( B_\varepsilon \) for a \( d \)-dimensional Brownian motion \( B \). Let \( f_\delta := f \mathbb{1}_{K_\delta} \) and note that \( \hat{f}_\delta \rightarrow \hat{f} \), pointwise, as \( \delta \downarrow 0 \).

Since \( (R_{\lambda}f) * \varphi_\varepsilon = R_{\lambda}(f * \varphi_\varepsilon) \geq R_{\lambda}(f_\delta * \varphi_\varepsilon) \) and \( \varphi_\varepsilon = \varphi_{\varepsilon/2} * \varphi_{\varepsilon/2} \), we can apply Tonelli’s theorem to find that

\[
\int_{\mathbb{R}^d} (R_{\lambda}f)(x)f(x) \, dx \geq \liminf_{\varepsilon \downarrow 0} \int_{\mathbb{R}^d} (R_{\lambda}(f_\delta * \varphi_\varepsilon))(x)f_\delta(x) \, dx
\]

\[
= \liminf_{\varepsilon \downarrow 0} \int_{\mathbb{R}^d} (R_{\lambda}(f_\delta * \varphi_{\varepsilon/2}))(x)(f_\delta * \varphi_{\varepsilon/2})(x) \, dx
\]

\[
= \frac{1}{(2\pi)^d} \liminf_{\varepsilon \downarrow 0} \int_{\mathbb{R}^d} |\hat{f}_\delta(\xi)|^2 e^{-\varepsilon \|\xi\|^2/2} \text{Re} \left( \frac{1}{\lambda + \Psi(\xi)} \right) \, d\xi,
\]

thanks to (5). This proves that

\[
\int_{\mathbb{R}^d} (R_{\lambda}f)(x)f(x) \, dx \geq \frac{1}{(2\pi)^d} \liminf_{\delta \downarrow 0} \int_{\mathbb{R}^d} |\hat{f}_\delta(\xi)|^2 \text{Re} \left( \frac{1}{1 + \Psi(\xi)} \right) \, d\xi,
\]

and Fatou’s lemma proves (6). The converse bound is much easier: We merely note that, as above,

\[
\int_{\mathbb{R}^d} (R_{\lambda}(f * \varphi_\varepsilon))(x)f(x) \, dx = \int_{\mathbb{R}^d} (R_{\lambda}(f * \varphi_{\varepsilon/2}))(x)(f * \varphi_{\varepsilon/2})(x) \, dx
\]

\[
= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 e^{-\varepsilon \|\xi\|^2/2} \text{Re} \left( \frac{1}{\lambda + \Psi(\xi)} \right) \, d\xi.
\]
Then we let $\epsilon \downarrow 0$; the right-most term converges to $\delta_{\lambda}(f,f)$ by the dominated convergence theorem, and the lim inf of the left-most term is at most $\int_{\mathbb{R}^d} (R_{\lambda}f)(x)f(x)\,dx$ by Fatou’s lemma.

\section*{Proof of Theorem 3}

Theorem \ref{thm:energy_capacity} will follow immediately from Lemmas \ref{lem:energy_capacity} and \ref{lem:capacity} below.

Define
\[ (J_{\lambda}f)(x) := \int_0^{\infty} e^{-\lambda t}f(x + X_t)\,ds. \]

\begin{lemma}
For all $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ and $\lambda > 0$,
\[ \int_{\mathbb{R}^d} \mathbb{E} \left( (J_{\lambda}f)(x) \right) \,dx = \frac{1}{\lambda}, \quad \int_{\mathbb{R}^d} \mathbb{E} \left( (|J_{\lambda}f(x)|)^2 \right) \,dx = \frac{1}{\lambda} \delta_{\lambda}(f,f). \]
\end{lemma}

\begin{proof}
The first computation follows because $f$ is a probability density and hence $\int_{\mathbb{R}^d} (J_{\lambda}f)(x)\,dx = \lambda^{-1}$. Now we begin with the second computation of the lemma:
\begin{align*}
\mathbb{E} \left( (|J_{\lambda}f(x)|)^2 \right) & = 2 \int_0^{\infty} e^{-\lambda s} \,ds \int_s^{\infty} e^{-\lambda t} \,dt \mathbb{E} \left[ f(x + X_s) \cdot f(x + X_t) \right] \\
& = 2 \int_0^{\infty} e^{-\lambda s} \,ds \int_s^{\infty} e^{-\lambda t} \,dt \mathbb{E} \left[ f(x + X_s) \cdot (P_{1-s}f)(x + X_s) \right] \\
& = 2 \int_0^{\infty} e^{-\lambda s} \mathbb{E} \left[ f(x + X_s) \cdot (R_{\lambda}f)(x + X_s) \right] \,ds,
\end{align*}

thanks to the Markov property. Therefore,
\[ \int_{\mathbb{R}^d} \mathbb{E} \left( (|J_{\lambda}f(x)|)^2 \right) \,dx = \frac{1}{\lambda} \int_{\mathbb{R}^d} f(y) \cdot (R_{\lambda}f)(y) \,dy. \]

And the lemma follows from Theorem \ref{thm:markov_property}.
\end{proof}

\begin{lemma}
Regardless of the value of $\lambda > 0$,
\[ \mathbb{E}(|G \ominus X(\mathbb{R}^d)|) = \int_{\mathbb{R}^d} \mathbb{P} \left\{ T_{G-x} < \infty \right\} \,dx \geq \frac{1}{\lambda} \cdot \text{Cap}_{\lambda}(G). \]
\end{lemma}

\begin{remark}
It is important to note that $T_{G-x} < \infty$ if and only if the Lévy process $x + X_t$ [which starts at $x \in \mathbb{R}^d$ at time zero] ever hits $G$; more precisely, there exists $t > 0$ such that $x + X_t \in G$ or $x + X_{t-} \in G$. Therefore, the preceding states that if $G$ has positive $\lambda$-capacity, then $X$ hits $G$, starting from almost every starting point $x \in \mathbb{R}^d$. In fact, this property is one way of thinking about the essential nonpolarity of $G$.
\end{remark}
Proof. Let us begin with a simple fact from classical function theory.

The Paley–Zygmund inequality: \(^1\) Suppose \(Y : \mathbb{R}^d \times \Omega \to \mathbb{R}_+\) is nonnegative and measurable, \(Y \not\equiv 0\), \(Y \in L^2(\mathbb{R}^d \times \Omega)\), and \(\int_{\mathbb{R}^d} EY(x) \, dx = c > 0\). Then,
\[
\int_{\mathbb{R}^d} P \{ Y(x) > 0 \} \, dx \geq \frac{c^2}{\int_{\mathbb{R}^d} E (|Y(x)|^2) \, dx},
\]
where \(1/\infty := 0\).

Let \(f\) be a probability density that is supported on the closed [say] \(\epsilon\)-enlargement \(G^\epsilon\) of \(G\). We apply Lemma 6 together with the Paley-Zygmund inequality [with \(Y(x) := (f_x)(x)\)] and obtain
\[
\int_{\mathbb{R}^d} P \{ (f_x)(x) > 0 \} \, dx \geq \frac{1}{\lambda \cdot \mathcal{E}_\lambda (f, f)}.
\]
If \((f_x)(x) > 0\), then certainly \(x + X_s \in G^\epsilon\) for some \(s > 0\); i.e., \(T_{G^\epsilon-x} < \infty\). Therefore,
\[
\int_{\mathbb{R}^d} P \{ T_{G^\epsilon-x} < \infty \} \, dx \geq \frac{1}{\lambda} \cdot \sup_g \frac{1}{\mathcal{E}_\lambda (g, g)},
\]
where the supremum is taken over all probability densities \(g\) that are supported on \(G^\epsilon\). Let \(h_\epsilon\) be a probability density, supported on \(B(0, \epsilon)\), and observe that \(\rho * h_\epsilon\) is a probability density supported on \(G^\epsilon\) whenever \(\rho \in M_1(G)\). Because of \((2), \mathcal{E}_\lambda (\rho * h_\epsilon, \rho * h_\epsilon) \leq \mathcal{E}_\lambda (\rho, \rho)\), and hence
\[
\int_{\mathbb{R}^d} P \{ T_{G^\epsilon-x} < \infty \} \, dx \geq \frac{1}{\lambda} \cdot \text{Cap}_\lambda (G).
\]

Note that
\[
\bigcap_{\epsilon > 0} \{ T_{G^\epsilon-x} < \infty \} = \bigcap_{\epsilon > 0} \{ x + X(\mathbb{R}_+) \cap G^\epsilon \neq \emptyset \} = \left\{ x + \overline{X(\mathbb{R}_+)} \cap G \neq \emptyset \right\}.
\]
Therefore,
\[
\int_{\mathbb{R}^d} P \left\{ x + \overline{X(\mathbb{R}_+)} \cap G \neq \emptyset \right\} \, dx \geq \frac{1}{\lambda} \cdot \text{Cap}_\lambda (G).
\]
Now the left-hand side is the expectation of the Lebesgue measure of the random set \(G \ominus \overline{X(\mathbb{R}_+)}\) [check!]. Because \(X\) is cadlag, the set difference between \(X(\mathbb{R}_+)\) and its closure has zero measure (in fact, is countable).

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\(^1\)Here is the proof: By the Cauchy–Schwarz inequality,
\[
c = \int_{\mathbb{R}^d} EY(x) \, dx = \int_{\mathbb{R}^d, \omega} 1_{\{ f > 0 \}}(x, \omega) \cdot f(x, \omega) \, dx \, P(d\omega) \\
\leq \left( \int_{\mathbb{R}^d, \omega} 1_{\{ f > 0 \}}(x, \omega) \, dx \, P(d\omega) \right) \cdot \left( \int_{\mathbb{R}^d, \omega} |f(x, \omega)|^2 \, dx \, P(d\omega) \right)^{1/2} \\
= \left( \int_{\mathbb{R}^d} P\{ f(x) > 0 \} \, dx \right)^{1/2} \cdot \left( \int_{\mathbb{R}^d} E \{ |f(x)|^2 \} \, dx \right)^{1/2}.
\]

\(\square\)
Therefore, the Lebesgue measure of $G \cap \overline{X(R_+)}$ is the same as the Lebesgue measure of $G \cap X(R_+)$. This proves the result. 

**Lemma 9.** $P\{T_G \leq n\} \leq e^{\lambda n} \cdot \text{Cap}_\lambda(G)$ for all $n, \lambda > 0$.

**Proof.** This is trivial unless

$$P\{T_G \leq n\} > 0,$$

which we assume is the case.

For all measurable $f : R^d \to R_+$,

$$E \left( (J_\lambda f)(0) \mid \mathcal{F}_{T_G \wedge n} \right) \geq \int_0^{\infty} e^{-\lambda s} E \left[ f(X_s) \mid \mathcal{F}_{T_G \wedge n} \right] \, ds \cdot 1_{\{T_G \leq n\}}$$

$$= e^{-\lambda n} \int_0^{\infty} e^{-\lambda s} E \left[ f(X_s + T_G \wedge n) \mid \mathcal{F}_{T_G} \right] \, ds \cdot 1_{\{T_G \leq n\}}$$

$$= e^{-\lambda n} \int_0^{\infty} e^{-\lambda s} (P_s f)(X_{T_G \wedge n}) \, ds \cdot 1_{\{T_G \leq n\}},$$

thanks to the strong Markov property. Therefore,

$$E \left( (J_\lambda f)(0) \mid \mathcal{F}_{T_G \wedge n} \right) \geq e^{-\lambda n} (R_\lambda f)(X_{T_G}) \cdot 1_{\{T_G \leq n\}}.$$

The expectation of the term on the left-hand side is 1, thanks to Lemma 6 and the optional stopping theorem. Therefore,

$$1 \geq e^{-\lambda n} E \left[ (R_\lambda f)(X_{T_G}) \mid T_G \leq n \right] \cdot P\{T_G \leq n\}$$

$$= e^{-\lambda n} \int_{R^d} (R_\lambda f) \, d\rho \cdot P\{T_G \leq n\},$$

where $\rho(A) := P(X_{T_G} \in A \mid T_G \leq n)$. In accord with (7), $\rho \in M_1(G)$.

We apply the preceding with $f := \rho \ast \varphi_\epsilon$, where $\varphi_\epsilon$ denotes the density of $B_\epsilon$ for a $d$-dimensional Brownian motion. Because

$$\int_{R^d} (R_\lambda f) \, d\rho = \int_{R^d} (R_\lambda (\rho \ast \varphi_\epsilon)) (x) (\rho \ast \varphi_\epsilon) (x) \, dx,$$

it follows from Theorem 5 that

$$e^{\lambda n} \geq \frac{1}{(2\pi)^d} \int_{R^d} e^{-\epsilon \|\xi\|^2/2} |\hat{\rho}(\xi)|^2 \text{Re} \left( \frac{1}{\lambda + \Psi(\xi)} \right) \, d\xi \cdot P\{T_G \leq n\}.$$

Let $\epsilon \downarrow 0$ and appeal to the monotone convergence theorem to finish. 

**Problems for Lecture 13**

1. Prove that $\text{Cap}_\lambda(G) > 0$ for some $\lambda > 0$ iff $\text{Cap}_\lambda(G) > 0$ for all $\lambda > 0$.

2. Prove Proposition 4. (Hint: Inspect the proof of Theorem 10 on page 82.)