Recurrence and Transience

The recurrence/transience dichotomy

Definition 1. We say that \( X \) is recurrent if \( \lim \inf_{t \to \infty} \|X_t\| = 0 \) a.s. We say that \( X \) is transient if \( \lim \inf_{t \to \infty} \|X_t\| = \infty \) a.s.

We have the following dichotomy result.

Theorem 2. The following are equivalent:

1. \( X \) is transient;
2. \( X \) is not recurrent;
3. \( U_0(B(0, r)) < \infty \) for all \( r > 0 \).

Theorem 2 has the following important dichotomous implication: Either \( X \) is transient, or \( X \) is recurrent [and not both]. The proof relies on a convenient series of equivalences.

Proposition 3. The following are equivalent:

1. \( \sup_{x \in \mathbb{R}^d} U_0(B(x, r)) < \infty \) for all \( r > 0 \);
2. \( U_0(B(0, r)) < \infty \) for all \( r > 0 \);
3. \( U_0(B(0, r)) < \infty \) for some \( r > 0 \);
4. \( \int_0^\infty \mathbf{1}_{B(0, r)}(X_s) \, ds < \infty \) a.s. for all \( r > 0 \);
5. \( \lim_{z \uparrow \infty} \sup_{x \in B(0, 2r)} P\{\int_0^\infty \mathbf{1}_{B(x, r)}(X_s) \, ds > z\} < 1 \) for all \( r > 0 \).
Proof. Let

\[ J(x, r) := \int_0^\infty 1_{B(x, r)}(X_s) \, ds. \]

Since \( U_0(B(0, r)) = EJ(0, r) \), we readily obtain (1)\( \Rightarrow \) (2) and (2)\( \Rightarrow \) (3). Also (2)\( \Rightarrow \) (1), because \( U_\lambda(B(x, r)) \leq U_\lambda(B(0, 2r)) \) uniformly in \( x \), thanks to Proposition 3 (page 64); we can let \( \lambda \downarrow 0 \) to obtain (2) from (1).

Suppose (3) holds; i.e., \( U_0(B(0, r)) < \infty \) for some \( r > 0 \). It is not hard to see that there exists a number \( N \geq 1 \) such that \( B(0, 2r) \) is a union of \( N \) balls of radius \( r \). [The key observation is that \( N \) does not depend on \( r \) by scaling.] Consequently, \( U_0(B(0, 2r)) \leq N \sup_{x \in \mathbb{R}^d} U_0(B(x, r/2)) \leq NU_0(B(0, r)) \) thanks to Proposition 3. This shows that (3)\( \Rightarrow \) (2), and hence (1)–(3) are equivalent.

Next we prove that (1) and (5) are equivalent: Chebyshev's inequality tells us that (1)\( \Rightarrow \) (5). Therefore, we are concerned with the complementary implication.

Suppose (5) holds, and fix some \( r > 0 \). We can find \( \gamma > 0 \) and \( \delta \in (0, 1) \) such that

\[ \sup_{x \in B(0, 2r)} P\{ J(x, r) > \gamma \} \leq \delta. \]

Choose and fix \( a \in B(0, 2r) \), and define

\[ T := \inf \{ s > 0 : \int_0^s 1_{B(a, r)}(X) \, ds > \gamma \} = \inf \{ s = \infty \}. \]

For every integer \( n \geq 0 \), it is not hard to see that if \( \int_0^\infty 1_{B(a, r)}(X_t) \, dt > (n + 1)\gamma \), then certainly \( T < \infty \); this follows because the process \( X \) has cadlag paths and \( B(a, r) \) is open. Moreover, \( \int_0^T 1_{B(a, r)}(X_s) \, ds = \gamma \) a.s. on \( \{ T < \infty \} \). Therefore, we can write

\[ P \left( \int_0^\infty 1_{B(a, r)}(X_s) \, ds > (n + 1)\gamma \right) = P \left( T < \infty, \int_T^\infty 1_{B(a, r)}(X_s) \, ds > n\gamma \right). \]

Because \( X \) is cadlag and \( B(a, r) \) is open, it follows that \( X_T \in B(a, r) \) a.s. on \( \{ T < \infty \} \). Therefore, the strong Markov property implies that a.s.,

\[ P \left( \int_0^\infty 1_{B(a, r)}(X_s) \, ds > n\gamma \bigg\vert \mathcal{F}_T \right) \leq \sup_{x \in B(0, 2r)} P \{ J(x, r) > n\gamma \}. \]

We can iterate this to find that

\[ \sup_{a \in B(0, 2r)} P \{ J(a, r) > (n + 1)\gamma \} \leq \left( \sup_{x \in B(0, 2r)} P \{ J(x, r) > \gamma \} \right)^{n+1} \leq \delta^{n+1}. \]
The recurrence/transience dichotomy

This shows, in particular, that (5) ⇒ (1); in fact,

$$\sup_{a \in B(0, 2r)} E \left[ J(a, r) \right] \leq \sum_{n=0}^{\infty} \sup_{x \in B(0, 2r)} P \left\{ J(x, r) > n r \right\} < \infty.$$  

And (4) ⇒ (5) because $J(x, r) \leq J(0, 3r)$ uniformly for all $x \in B(0, 2r)$. Since (1) ⇒ (4), this proves the equivalence of (1), (2), (3), (4), and (5).  

We now derive the recurrence–transience dichotomy.

**Proof of Theorem 2.** Clearly, (1) ⇒ (2). And if $X$ is transient, then the last hitting time $L := \sup \{ t > 0 : X_t \in B(0, r) \}$ of the ball $B(0, r)$ is a.s. finite. Therefore, $J(0, r) \leq L < \infty$ a.s., and Proposition 3 implies that (1) ⇒ (3).

Next, we suppose that (3) holds, so that $J(0, r) < \infty$ a.s. for all $r > 0$. If (2) did not hold, that is if $X$ were recurrent, then

$$T_n := \inf \{ s > n : X_s \in B(0, r/2) \} \quad (\inf \varnothing := \infty)$$

would be finite a.s. for all $n \geq 1$. And by the Markov property,

$$P \left\{ \int_n^{\infty} 1_{B(0, r)}(X_t) \, dt \geq z \right\} \geq P \left\{ \int_{T_n}^{\infty} 1_{B(0, r)}(X_t) \, dt \geq z \right\} = P \left\{ \int_0^{\infty} 1_{B(0, r)}(X_{T_n+t}) \, dt \geq z \right\}.$$  

Because $X_{T_n} \in B(0, r/2)$ a.s., it would follow from the strong Markov property that

$$P \left\{ \int_n^{\infty} 1_{B(0, r)}(X_t) \, dt \geq z \right\} \geq P \left\{ \int_0^{\infty} 1_{B(0, r/2)}(X_t) \, dt \geq z \right\} = P \{ J(0, r/2) \geq z \}.$$  

The left-hand side tends to zero as $n$ goes to $\infty$, for all $z > 0$. Therefore, $J(0, r/2) = 0$ a.s. Because $B(0, r/2)$ is open, $X_0 = 0$, and $X$ has cadlag paths, this leads us to a contradiction; i.e., (3) ⇒ (2). It remains to prove that (3) ⇒ (1).

Let us assume that (2) holds; i.e., that $X$ is recurrent. Then the following are all a.s.-finite stopping times:

$$T_1 := \inf \{ s > 0 : \|X_s\| > r \},$$  

$$T_2 := \inf \{ s > T_1 : \|X_s\| < r/2 \},$$  

$$S_2 := \inf \{ s > 0 : \|X_{T_2+t} - X_{T_2}\| > r/2 \},$$  

$$T_3 := \inf \{ s > T_2 + S_2 : \|X_s\| < r/2 \},$$  

$$S_3 := \inf \{ s > 0 : \|X_{T_3+s} - X_{T_3}\| > r/2 \},$$  

$$S_4 := \inf \{ s > 0 : X_{T_4+s} - X_{T_4} \}$$
etc. Because \( X \) is assumed to be recurrent, these are all a.s.-finite stopping times. And it is easy to see that

\[
J(0,r) = \int_0^\infty 1_{B(0,r)}(X_s) \, ds \geq T_1 + \sum_{j=1}^\infty S_j \geq \sum_{j=1}^\infty S_j.
\]

By the strong Markov property, the \( S_j \)'s are i.i.d. And since \( X \) is c.d.l.g, the \( S_j \)'s are a.s. strictly positive. From this we can deduce that \( \sum_{j=1}^\infty S_j = \infty \) a.s., \(^1\) and hence (3) \( \Rightarrow \) (2). This completes the proof. \( \square \)

**The Port–Stone theorem**

The following well-known result of Port and Stone \( (1967, 1971) \) characterizes recurrence in terms of the Lévy exponent \( \Psi. \)

**Theorem 4** (Port and Stone, 1967, 1971). \( X \) is transient iff \( \Re(1/\Psi) \) is locally integrable near the origin.

**Example 5.** Consider standard Brownian motion in \( \mathbb{R}^d \), so that \( \Psi(\xi) = \frac{1}{2} \|\xi\|^2 \). Then, for every \( R > 0 \),

\[
\int_{B(0,R)} \Re \left( \frac{1}{\Psi(\xi)} \right) \, d\xi = \int_{B(0,R)} \frac{2}{\|\xi\|^2} \, d\xi \propto \int_0^R r^{d-3} \, dr
\]

is infinite if and only if \( d \leq 2 \). Thus, standard Brownian motion is recurrent iff \( d \leq 2 \). On the other hand, Brownian motion with nonzero drift is transient in all dimensions \( d \geq 1 \) \( [\text{why?}] \).

Somewhat more generally, if \( X \) denotes an isotropic stable process in \( \mathbb{R}^d \) with index \( \alpha \in (0,2] \), then for every \( R > 0 \),

\[
\int_{B(0,R)} \Re \left( \frac{1}{\Psi(\xi)} \right) \, d\xi \propto \int_{B(0,R)} \frac{1}{\|\xi\|^\alpha} \, d\xi \propto \int_0^R r^{d-\alpha-1} \, dr
\]

is infinite if and only if \( d > \alpha \). Thus, \( X \) is recurrent iff \( d \leq \alpha \). In particular, \( \alpha \geq 1 \) is the criterion for recurrence in dimension \( d = 1 \). And in dimension \( d = 2 \), only Brownian motion \([\alpha = 2]\) is recurrent. In dimensions three of higher, all isotropic stable processes are transient. \( \square \)

**A partial proof of Theorem 4.** Define

\[
G(r) := \int_{C(2r)} \Re \left( \frac{1}{\Psi(\xi)} \right) \, d\xi \quad \text{for all } r > 0,
\]

\(^1\)Indeed, we can find \( \delta > 0 \) such that \( p := \mathbb{P}\{S_j \in [\delta,1/\delta]\} > 0 \), and note that \( \sum_{j=1}^n S_j \geq \sum_{j=1}^n 1_{[\delta,1/\delta]}(S_j) - np \) as \( n \to \infty \), thanks to the strong law of large numbers.

\(^2\)Port and Stone prove this for \( \{\text{discrete-time}\} \) random walks in 1967. The continuous-time version is proved similarly, in great generality, in 1971.
where \( C(t) := \{ z \in \mathbb{R}^d : \max_{1 \leq j \leq d} |z_j| \leq t \} \) for \( t > 0 \). Because \( B(0, R) \subset C(R) \subset B(0, R\sqrt{d}) \) for all \( R > 0 \), Theorem 4 is equivalent to the statement that \( X \) is recurrent iff \( G(r) < \infty \) for all \( r > 0 \). I will prove half of this, and only make some remarks on the harder half.

Consider probability density functions \( \{ \varphi_r \}_{r>0} \) defined by

\[
\varphi_r(x) := \prod_{j=1}^d \left( 1 - \frac{\cos(2\pi r z_j)}{2\pi r x_j^2} \right) \quad \text{for all } x \in \mathbb{R}^d.
\]

Then, \( \hat{\varphi}_r \) is the normalized Pólya kernel,

\[
\hat{\varphi}_r(\xi) = \prod_{j=1}^d \left( 1 - \frac{|\xi_j|}{2r} \right) \quad \text{for every } r > 0 \ (\xi \in \mathbb{R}^d),
\]

where \( z^+ := \max(z, 0) \), as usual. Since \( 1 - \cos z \geq z^2/4 \) for all \( z \in [-2, 2] \), we conclude that for every \( x, \xi \in \mathbb{R}^d \) and \( r > 0 \),

\[
\varphi_r(x) \geq \left( \frac{r}{2\pi} \right)^d 1_{C(1/r)}(x), \quad \text{and} \quad \hat{\varphi}_r(\xi) \leq 1_{C(2r)}(\xi).
\]

Because \( \varphi_r, \hat{\varphi}_r \in L^1(\mathbb{R}^d) \) and \( \hat{U}_\lambda(\xi) = \text{Re}(1 + \Psi(\xi))^{-1} \), we may apply Parseval’s identity and (4) to find that

\[
U_\lambda(C(2r)) \geq \int_{\mathbb{R}^d} \hat{\varphi}_r \, dU_\lambda = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \varphi_r(\xi) \text{Re} \left( \frac{1}{\lambda + \Psi(\xi)} \right) \, d\xi \geq \frac{r^d}{(2\pi)^{2d}} \int_{C(1/r)} \text{Re} \left( \frac{1}{\lambda + \Psi(\xi)} \right) \, d\xi.
\]

Therefore, Fatou’s lemma implies that \( U_0(C(2r)) \geq \text{const} \cdot r^d G(1/(2r)) \) for all \( r > 0 \). Theorem 2 then tells us that if \( X \) is transient then \( G(t) < \infty \) for all \( t > 0 \). This proves the easier half of the theorem. For the other half we start similarly. By Parseval’s identity,

\[
\int_{\mathbb{R}^d} \varphi_r \, dU_\lambda = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{\varphi}_r(\xi) \text{Re} \left( \frac{1}{\lambda + \Psi(\xi)} \right) \, d\xi \leq \frac{1}{(2\pi)^d} \int_{C(2r)} \text{Re} \left( \frac{1}{\lambda + \Psi(\xi)} \right) \, d\xi.
\]

The left-most term is at least \( (r/(2\pi))^d U_\lambda(C(1/r)) \); see (4). Therefore, we can let \( \lambda \downarrow 0 \) and apply Theorem 2, which tells us that if \( X \) is recurrent, then \( \lim \inf_{\lambda \downarrow 0} \int_{C(2r)} \text{Re}(\lambda + \Psi(\xi))^{-1} \, d\xi = \infty \) for all \( r > 0 \) From here, the remaining difficulty is to prove that one can always “take the limit inside the expectation.” See Port and Stone (1967, 1971) for the [difficult] details in the context of [discrete-time] random walks; the extension to continuous time is performed similarly. \( \square \)
Problems for Lecture 11

1. Describe exactly when $X$ is recurrent when:

   (1) $X$ denote an isotropic stable process in $\mathbb{R}^d$ with index $\alpha \in (0, 2]$;

   (2) $X$ is a nonstandard Brownian motion with exponent $\Psi(\xi) = \frac{1}{2} \|\alpha\xi\|^2$;

   (3) $X$ is a process with stable components; i.e., $\Psi(\xi) = \sum_{j=1}^d |\xi_j|^{\alpha_j}$ for $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_d$ all in $[0, 2]$;

   (4) $X$ is a the Poisson process on the line with intensity $\lambda \in (0, \infty)$.

2. A Lévy process $X$ is said to satisfy the sector condition (Kanda, 1976) if $|\Psi(\xi)| = O(\text{Re}\Psi(\xi))$ as $\|\xi\| \to \infty$.

   (1) Verify that every symmetric Lévy process satisfies the sector condition, and find an example that does not.

   (2) Prove Theorem 4 [completely!] in the case that $X$ satisfies the sector condition.

3. Let $X$ and $X'$ be two i.i.d. Lévy processes on $\mathbb{R}^d$. Prove that if $X$ is recurrent, then $Y_t := X_t - X'_t$ [the so-called “symmetrization of $X$”] is also recurrent. Construct an example that shows that the converse is not always true.

4. Let $B$ denote a $d$-dimensional Brownian motion, and $T$ an independent subordinator with Laplace exponent $\Phi$. Prove that $X \circ T$ is recurrent if and only if $\int_0^1 s^{(d-2)/2} ds/\Phi(s) = \infty$.

5. Suppose $X$ is a transient Lévy process in $\mathbb{R}^d$, and $V : \mathbb{R}^d \to \mathbb{R}_+$ is measurable, bounded, and has compact support. Prove that there exists $\theta > 0$ such that $E(e^{\theta \int_0^X V(s) ds}) < \infty$.

6. Let $B := \{B_t \}_{t \geq 0}$ denote Brownian motion in $\mathbb{R}^d$, where $d \geq 3$. Prove that if $\epsilon > 0$ is fixed but arbitrary, then as $r \to 0$:

\[
\liminf_{r \to 0} \frac{1}{p^{2+\epsilon}} \int_0^\infty 1_{B[0,r]}(B_s) \, ds = \infty, \quad \limsup_{r \to 0} \frac{1}{p^{2-\epsilon}} \int_0^\infty 1_{B[0,r]}(B_s) \, ds = 0.
\]

7. Let $X$ denote a Lévy process on $\mathbb{R}^d$ with exponent $\Psi$. A point $x \in \mathbb{R}^d$ is possible if for all $r > 0$ there exists $t > 0$ such that $P\{X_t \in B(x, r)\} > 0$. Let $\mathcal{P}$ denote the collection of all possible points of $X$. Demonstrate the following assertions:

   (1) $\mathcal{P}$ is a closed additive subsemigroup of $\mathbb{R}^d$;

   (2) $U_\lambda$ is supported on $\mathcal{P}$ for all $\lambda > 0$;

   (3) $X$ is recurrent if and only if $\lim_{t \to \infty} \|X_t - x\| = 0$ for all $x \in \mathcal{P}$.